IC/92/13

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

q-DEFORMATIONS OF NONCOMPACT LIE (SUPER-) ALGEBRAS: THE EXAMPLES OF q-DEFORMED LORENTZ, WEYL, POINCARE' AND (SUPER-) CONFORMAL ALGEBRAS

V.K. Dobrev

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

MIRAMARE-TRIESTE

IC/92/13

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

q - DEFORMATIONS OF NONCOMPACT LIE (SUPER-) ALGEBRAS: THE EXAMPLES OF q - DEFORMED LORENTZ, WEYL, POINCARE' AND (SUPER-) CONFORMAL ALGEBRAS*

V.K. Dobrev**
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

We review and explain a canonical procedure for the q - deformation of the real forms $\mathcal{G}$ of complex Lie (super-) algebras associated with (generalized) Cartan matrices. Our procedure gives different q - deformations for the non-conjugate Cartan subalgebras of $\mathcal{G}$. We give several in detail the q - deformed Lorentz and conformal (super-) algebras. The q - deformed conformal algebra contains as a subalgebra a q - deformed Poincaré algebra and as Hopf subalgebras two conjugate 11-generator q-deformed Weyl algebras. The q-deformed Lorentz algebra is Hopf subalgebra of both Weyl algebras.

MIRAMARE – TRIESTE
January 1992

* To appear in the Proceedings of the Quantum Groups Workshop of the II Wigner Symposium, (Goslar, Germany, July 1991).

** At ICTP until June 30, 1992; Permanent address: Bulgarian Academy of Sciences, Institute of Nuclear Research and Nuclear Energy, 72 Boul. Trakia, 1784 Sofia, Bulgaria.

1. Introduction

The Lorentz, Poincaré and conformal algebras, and also other non-compact Lie algebras and groups play a very important role in physics. Thus the problem the q - deformation of these and other noncompact algebras is of utmost importance. Actually, the deformation of *compact* simple Lie algebras is used in the physics literature without much explanation assuming implementation of the Weyl unitary trick. In [1] considering the real forms of the matrix quantum groups [1],[2],[3] were introduced the compact matrix quantum groups $SU_q(n)$, (for $n = 2$ first in [3]), $SO_q(n)$, $Sp_q(n)$ and the maximally split real noncompact forms $SL_q(n,\mathbb{R})$, $SO_q(n,n)$, $SO_q(n,n+1)$, $Sp_q(n,\mathbb{R})$. From our point of view it is not accidental that these cases were obtained first since the root systems of these real forms coincide (up to multiple of i in the compact case) with the root systems of their complexifications (cf. the description of our approach below). Besides the above $U_q(su(1,1))$ was considered in [4], $U_q(su(n,1))$ were introduced in [5]. A quantum Lorentz group was introduced and studied in [6] and a seven - dimensional quantum Lorentz algebra was introduced in [7]. The q-deformation of Heisenberg, Galilei and Eucledean algebras in two dimensions were studied in [8].

Thus there is still lacking an universal approach to the q - deformation of real simple algebras. Such an approach was proposed in [9] and is reviewed and explained here. It is well known that the real forms $\mathcal{G}$ of a complex simple Lie algebra $\mathcal{G}^{\mathbb{C}}$ are in 1-to-1 correspondence with the Cartan automorphisms θ of $\mathcal{G}^{\mathbb{C}}$. This allows to study the structure of the real forms and to find their explicit embeddings as real subalgebras of $\mathcal{G}^{\mathbb{C}}$ invariant under θ. This is one basic ingredient of our approach which is enough for the compact case. The other basic ingredient is related to the fact that a real noncompact simple Lie algebra has in general (a finite number of) non-congugate Cartan subalgebras [10]. This is very important since we have to choose which conjugacy class of Cartan subalgebras will correspond to the unique conjugacy class of Cartan subalgebras of $\mathcal{G}^{\mathbb{C}}$ and will be "freezed" under the q - deformation (cf. (1b) below). For each such choice we shall get a different q - deformation. Our approach is easily generalized for the real forms of the basic classical Lie superalgebras and of the corresponding affine Kac-Moody (super-) algebras.

The organization of the paper is as follows. In Section 2 we recall the q - deformation of complex simple Lie algebras. In Section 3 we present our approach. In Sections 4 and 5 we present the q - deformation of the Lorentz algebra $so(3,1)$ and of the conformal algebra $su(2,2)$ (the exposition is much more detailed than [9] and also some misprints there are corrected). In Section 6 we discuss the q - deformed Weyl and Poincaré algebras. In Section 7 we recall the q - deformation of complex Lie superalgebras and present the q - deformation of the conformal superalgebra $su(2,2/N)$.

2. Synopsis on the q - deformation of complex simple Lie algebras

Let $\mathcal{G}_c$ be a complex simple Lie algebra; then the q - deformation $U_q(\mathcal{G}_c)$ of the universal enveloping algebras $U(\mathcal{G}_c)$ is defined [11],[12] as the associative algebra over $\mathbb{C}$ with Chevalley generators $X_j^{\pm}$, H_j , $j = 1,\ldots,\ell = \mathrm{rank}\,\mathcal{G}_c$ and with relations :

$$[H_i\,,\,H_j] \;=\; 0\,,\;\; [H_i\,,\,X_j^{\pm}] \;=\; \pm a_{ij} X_j^{\pm}\,, \tag{1a}$$

$$[X_i^+\,,\,X_j^-] \;=\; \delta_{ij}\frac{q_i^{H_i/2} - q_i^{-H_i/2}}{q_i^{1/2} - q_i^{-1/2}} \;=\; \delta_{ij}\,[H_i]_{q_i}\,,\;\; q_i = q^{(\alpha_i\,,\,\alpha_i)/2}\,, \tag{1b}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}_{q_i} (X_i^{\pm})^k\, X_j^{\pm}\, (X_i^{\pm})^{n-k} \;=\; 0\,,\; i \neq j\,,\; n = 1 - a_{ij}\,, \tag{1c}$$

where $(a_{jk}) = (2(\alpha_j,\alpha_k)/(\alpha_j,\alpha_j))$ is the Cartan matrix of $\mathcal{G}_c$, the scalar product of the roots $(\cdot,\cdot)$ is normalized so that $(\alpha,\alpha) \in 2\mathbb{N}$,

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}\,,\;\; [m]_q! = [m]_q\,[m-1]_q\ldots[1]_q\,, \tag{2a}$$

$$[m]_q \;=\; \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}} \;=\; \frac{sh(mh/2)}{sh(h/2)} \;=\; \frac{sin(\pi m\tau)}{sin(\pi\tau)}\,,\;\; q = e^h = e^{2\pi i\tau},\;\; h,\tau \in \mathbb{C}, \tag{2b}$$

$$q_i^{a_{ij}} = q^{(\alpha_i\,,\,\alpha_j)} = q_j^{a_{ji}}. \tag{2c}$$

The elements H_j span the Cartan subalgebra $\mathcal{H}_c$ of $\mathcal{G}_c$, while the elements $X_j^{\pm}$ generate the subalgebras $\mathcal{G}_c^{\pm} = \underset{\beta\in\Delta^{\pm}}{\oplus} \mathcal{G}_{c\beta}$, where $\Delta = \Delta^+ \cup \Delta^-$ is the root system of $\mathcal{G}_c$, Δ^+ , Δ^- are the sets of positive, negative, roots, respectively. Thus one has the standard decomposition

$$\mathcal{G}_c \;=\; \mathcal{G}_c^+ \oplus \mathcal{H}_c \oplus \mathcal{G}_c^-\,. \tag{3}$$

We recall that H_j correspond to the simple roots α_j of $\mathcal{G}_c$, and if $\beta^\vee = \sum_j n_j \alpha_j^\vee$, $\beta^\vee \equiv 2\beta/(\beta,\beta)$, then to β corresponds $H_\beta = \sum_j n_j H_j$. The elements of $\mathcal{G}_c$ which span $\mathcal{G}_{c\beta}$, ($\dim \mathcal{G}_{c\beta} = 1$), are denoted by X_β . These Cartan–Weyl generators [12],[14] are normalized so that

$$[X_\beta\,,\,X_{-\beta}] \;=\; [H_\beta]_{q_\beta}\,,\;\; [H_\beta\,,\,X_{\pm\beta'}] \;=\; \pm(\beta^\vee,\beta')X_{\pm\beta'}\,,\;\; \beta,\beta' \in \Delta^+\,,\;\; q_\beta \;\equiv\; q^{(\beta,\beta)/2}\,. \tag{4}$$

The algebra $U_q(\mathcal{G}_c)$ is a Hopf algebra [15] with co-multiplication δ, co-unit ε (homomorphisms) and antipode γ (antihomomorphism) defined on the generators of $U_q(\mathcal{G}_c)$ as follows [11],[12]:

$$\delta(H_j) \;=\; H_j \otimes 1 \;+\; 1 \otimes H_j\,,\;\; \delta(X_j^{\pm}) \;=\; X_j^{\pm} \otimes q_j^{H_j/4} \;+\; q_j^{-H_j/4} \otimes X_j^{\pm}\,, \tag{5a}$$

$$\varepsilon(H_j) \;=\; \varepsilon(X_j^{\pm}) \;=\; 0\,, \tag{5b}$$

$$\gamma(H_j) \;=\; -H_j\,,\;\; \gamma(X_j^{\pm}) \;=\; -q_j^{\pm\hat{\rho}/2}\, X_j^{\pm}\, q_j^{\mp\hat{\rho}/2} \;=\; -q_j^{\pm 1/2}\, X_j^{\pm}\,, \tag{5c}$$

where $\hat{\rho} \in \mathcal{H}_c$ corresponds to $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$, $\hat{\rho} = \frac{1}{2}\sum_{\alpha\in\Delta^+} H_\alpha$. The action of δ, ε, γ on the Cartan–Weyl generators H_β, X_β is obtained easily from (5) since H_β (see above) and X_β (cf. [12],[14] and, e.g., formulae (24) below) are given algebraically in terms of the Chevalley generators. (Of course, (5b) holds for all Cartan–Weyl generators.)

3. q - deformation of real semisimple Lie algebras

3.1. Synopsis on real semisimple Lie algebras

Let $\mathcal{G}$ be a real semisimple Lie algebra, θ be the Cartan involution in $\mathcal{G}$, and $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ be the Cartan decomposition of $\mathcal{G}$, so that $\theta X = X$, $X \in \mathcal{K}$, $\theta X = -X$, $X \in \mathcal{P}$; $\mathcal{K}$ is the maximal compact subalgebra of $\mathcal{G}$. Let $\mathcal{A}$ be the maximal subspace of $\mathcal{P}$ which is an abelian subalgebra of $\mathcal{G}$; $r = \dim \mathcal{A}$ is the *real* (or *split*) rank of $\mathcal{G}$, $0 \leq r \leq \ell = \mathrm{rank}\,\mathcal{G}$.

Let Δ_R be the root system of the pair $(\mathcal{G},\mathcal{A})$, also called ($\mathcal{A}$ -) *restricted root system*:

$$\Delta_R \;=\; \{\lambda \in \mathcal{A}^* \mid \lambda \neq 0, \mathcal{G}_\lambda \neq 0\}\,,\;\; \mathcal{G}_\lambda = \{X \in \mathcal{G} \mid [Y,X] = \lambda(Y)X\,,\;\; \forall Y \in \mathcal{A}\}\,. \tag{6}$$

The elements of $\Delta_R = \Delta_R^+ \cup \Delta_R^-$ are called ($\mathcal{A}$ -) *restricted roots* ; if $\lambda \in \Delta_R$, $\mathcal{G}_\lambda$ are called ($\mathcal{A}$-) *restricted root spaces* , $\dim_R \mathcal{G}_\lambda \geq 1$. Now we can introduce the subalgebras corresponding to the positive (Δ_R^+) and negative (Δ_R^-) restricted roots:

$$\tilde{\mathcal{N}} \;=\; \underset{\lambda\in\Delta_R^+}{\oplus} \mathcal{G}_\lambda \;=\; \tilde{\mathcal{N}}^1 \oplus \tilde{\mathcal{N}}^2\,,\quad \mathcal{N} \;=\; \underset{\lambda\in\Delta_R^-}{\oplus} \mathcal{G}_\lambda \;=\; \mathcal{N}^1 \oplus \mathcal{N}^2 \;=\; \theta\tilde{\mathcal{N}}\,, \tag{7}$$

where $\tilde{\mathcal{N}}^1$, $\tilde{\mathcal{N}}^2$, resp., is the direct sum of $\mathcal{G}_\lambda$ with $\dim_R \mathcal{G}_\lambda = 1$, $\dim_R \mathcal{G}_\lambda > 1$, resp., and analogously for $\mathcal{N}^a = \theta\tilde{\mathcal{N}}^a$. Then we have the (Bruhat) decompositions which we shall use for our q-deformation :

$$\mathcal{G} \;=\; \tilde{\mathcal{N}} \oplus \mathcal{A} \oplus \mathcal{M} \oplus \mathcal{N} \;=\; \tilde{\mathcal{N}}^1 \oplus \tilde{\mathcal{N}}^2 \oplus \mathcal{A} \oplus \mathcal{M} \oplus \mathcal{N}^1 \oplus \mathcal{N}^2\,, \tag{8}$$

where $\mathcal{M}$ is the centralizer of $\mathcal{A}$ in $\mathcal{K}$, i.e., $\mathcal{M} = \{X \in \mathcal{K} \mid [X,Y] = 0,\; \forall Y \in \mathcal{A}\}$. In general $\mathcal{M}$ is a compact reductive Lie algebra, and we shall write $\mathcal{M} = \mathcal{M}_s \oplus \mathcal{Z}_m$, where $\mathcal{M}_s = [\mathcal{M},\mathcal{M}]$ is the semisimple part of $\mathcal{M}$, and $\mathcal{Z}_m$ is the centre of $\mathcal{M}$.

Further let $\mathcal{H}_m$ be the Cartan subalgebra of $\mathcal{M}$, i.e., $\mathcal{H}_m = \mathcal{H}_m^s \oplus \mathcal{Z}_m$, where $\mathcal{H}_m^s$ is the Cartan subalgebra of $\mathcal{M}_s$. Then $\mathcal{H}_0 \equiv \mathcal{H}_m \oplus \mathcal{A}$ is a Cartan subalgebra of $\mathcal{G}$, the most noncompact one. Let $\mathcal{H}^{\mathbb{C}}$ be the Cartan subalgebra of the complexification $\mathcal{G}^{\mathbb{C}}$ of $\mathcal{G}$. Of course ℓ = rank $\mathcal{G}^{\mathbb{C}} = \dim_{\mathbb{C}} \mathcal{H}^{\mathbb{C}} = \dim_{\mathbb{R}} \mathcal{H} = \dim_{\mathbb{R}} \mathcal{H}_m^s + \dim_{\mathbb{R}} \mathcal{Z}_m + r$.

Next it is natural to choose the basis in $\mathcal{H}^{\mathbb{C}}$ so that the elements of Δ take real values on $i\mathcal{H}_m \oplus \mathcal{A}$, namely, if H, resp., H', is an element of the basis of $\mathcal{H}_m$, resp., $\mathcal{A}$, then we shall take iH, resp., H', as an element of the basis of $\mathcal{H}^{\mathbb{C}}$.

It is important for our procedure to choose consistently the basis of the rest of $\mathcal{G}$ and $\mathcal{G}^{\mathbb{C}}$. For this we use the classification of the roots from Δ with respect to $\mathcal{H}$. The set $\Delta_r \equiv \{\alpha \in \Delta \mid \alpha \mid_{\mathcal{H}_m} = 0\}$ is called the set of *real roots*, $\Delta_i \equiv \{\alpha \in \Delta \mid \alpha \mid_{\mathcal{A}} = 0\}$ - the set of *imaginary roots*, $\Delta_c \equiv \Delta \backslash (\Delta_r \cup \Delta_i)$ - the set of *complex roots* [10]. Thus $\Delta = \Delta_r \cup \Delta_i \cup \Delta_c$. Further, let $\alpha \in \Delta^+$, let $\mathcal{L}_\alpha^c$ be the complex linear span of H_α, X_α, $X_{-\alpha}$, and let $\mathcal{L}_\alpha = \mathcal{L}_\alpha^c \cap \mathcal{G}$. Then $\dim_{\mathbb{R}} \mathcal{L}_\alpha = 3$ iff the $\alpha \in \Delta_r \cup \Delta_i$ [10]. If $\alpha \in \Delta_r$ then $\mathcal{L}_\alpha$ is noncompact. For $\alpha \in \Delta_i$ the root α is called *singular*, $\alpha \in \Delta_s$, if $\mathcal{L}_\alpha$ is noncompact, and α is called *compact*, $\alpha \in \Delta_k$, if $\mathcal{L}_\alpha$ is compact. Thus $\Delta_i = \Delta_s \cup \Delta_k$. Let $\mathcal{H} = \mathcal{H}_0$, then $\Delta_s = \emptyset$ and the algebras $\mathcal{L}_\alpha$ are given by :

$$\mathcal{L}_\alpha = r.l.s.\{H_\alpha , X_\alpha , X_{-\alpha}\} , \quad \alpha \in \Delta_r^+ , \tag{9a}$$

$$\mathcal{L}_\alpha = r.l.s.\{iH_\alpha , X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha})\} , \quad \alpha \in \Delta_k^+ , \tag{9b}$$

where r.l.s. stands for real linear span. Note that $X_\alpha \in \mathcal{P}^{\mathbb{C}}$ if $\alpha \in \Delta_r$, $X_\alpha \in \mathcal{K}^{\mathbb{C}}$ if $\alpha \in \Delta_i$. If $\mathcal{H} \not\cong \mathcal{H}_0$ we have to consider also the singular roots. Explicitly, for $\alpha \in \Delta_s$ we have:

$$\mathcal{L}_\alpha = r.l.s.\{iH_\alpha , i(X_\alpha - X_{-\alpha}), X_\alpha + X_{-\alpha}\} , \quad \alpha \in \Delta_s^+ . \tag{9c}$$

All notions above are easily generalized for the real forms of the basic classical Lie superalgebras [13].

3.2. q - deformation of real semisimple Lie algebras

Let $\mathcal{G}$ be a real semisimple Lie algebra. We shall use the standard deformation from section 2. to the simple components of the complexification $\mathcal{G}^{\mathbb{C}}$ of $\mathcal{G}$.

The first step in our procedure is the choice of Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$. First we shall give the q-deformation using the most non-compact Cartan subalgebra $\mathcal{H}_0$. We consider the basis elements in (9) also as basis elements of the q - deformation $U_q(\mathcal{G})$ with commutation relations and Hopf algebra structure inherited from $U_q(\mathcal{G}^{\mathbb{C}})$.

For the real roots, $\alpha \in \Delta_r^+$, the generators in (9a) obey (4), if $\alpha \in \Delta_r^+ \cap \Delta_S$ also (5), and otherwise as explained after (5). Thus formulae (9a) determine completely a q - deformation of any

maximally split real form (or normal real form), when all roots are real, $\mathcal{M} = 0$, and $\mathcal{H}_0 = \mathcal{A}$, and (3) is reduced to

$$\mathcal{G} = \tilde{\mathcal{N}} \oplus \mathcal{A} \oplus \mathcal{N} , \tag{10}$$

i.e., this is the restriction to $\mathbb{R}$ of the standard decomposition $\mathcal{G}^{\mathbb{C}} = \mathcal{G}_+^{\mathbb{C}} \oplus \mathcal{H}^{\mathbb{C}} \oplus \mathcal{G}_-^{\mathbb{C}}$, and hence $U_q(\mathcal{G})$ is just the restriction of $U_q(\mathcal{G}^{\mathbb{C}})$ to $\mathbb{R}$. For the classical complex Lie algebras these forms are $sl(n, \mathbb{R})$, $so(n,n)$, $so(n+1, n)$, $sp(n, \mathbb{R})$, which are dual to the matrix quantum groups $SL_q(n, \mathbb{R})$, $SO_q(n,n)$, $SO_q(n, n+1)$, $Sp_q(n, \mathbb{R})$, introduced in [1] from another point of view than ours.

For the compact roots, $\alpha \in \Delta_k^+$, we have :

$$[C_\alpha^+ , C_\alpha^-] = \frac{sinh(\tilde{H}_\alpha h_\alpha/2)}{sin(h_\alpha/2)} , \ [\tilde{H}_\alpha, C_\alpha^\pm] = \pm C_\alpha^\mp , \ q_\alpha = q^{(\alpha,\alpha)/2} = e^{-ih_\alpha} , \tag{11a}$$

$$C_\alpha^+ \equiv (i/\sqrt{2})(X_\alpha + X_{-\alpha}) , \ C_\alpha^- \equiv (1/\sqrt{2})(X_\alpha - X_{-\alpha}) . \ \tilde{H}_\alpha \equiv -iH_\alpha , \tag{11b}$$

$$\delta(C_\alpha^\pm) = C_\alpha^\pm \otimes e^{\tilde{H}_\alpha h_\alpha/4} + e^{-\tilde{H}_\alpha h_\alpha/4} \otimes C_\alpha^\pm , \ \alpha \in \Delta_k^+ \cap \Delta_S , \tag{11c}$$

where Δ_S is the simple root system, and for the non-simple roots as explained after (5). Note that formulae (11) (with $h_\alpha \in \mathbb{R}$) determine completely the unique q - deformation of any compact semisimple Lie algebra [11] (when all roots of Δ are imaginary). Note that in this case the q - deformation inherited from $U_q(\mathcal{G}^{\mathbb{C}})$ is often used in the physics literature without the basis change (11).

Note that there is a 1-to-1 correspondence between the real roots $\alpha \in \Delta$ and the restricted roots $\lambda \in \Delta_R$ with $\dim_{\mathbb{R}} \mathcal{G}_\lambda = 1$ and naturally this correspondence is realized by the restriction : $\lambda = \alpha \mid_{\mathcal{A}}$. Further note that the set of the imaginary roots in Δ may be identified with the root system of $\mathcal{M}_s^{\mathbb{C}}$.

Thus so far we have chosen consistently the generators of $\tilde{\mathcal{N}}^1 \oplus \mathcal{A} \oplus \mathcal{M} \oplus \mathcal{N}^1$ (cf. (8)) as linear combinations of the generators of $\mathcal{H}_0 \oplus \oplus_{\alpha \in \Delta_r \cup \Delta_i} \mathcal{G}_\alpha$. Now it remains to choose consistently the generators of $\tilde{\mathcal{N}}^2 \oplus \mathcal{N}^2$ as linear combinations of the generators of the rest of $\mathcal{G}^{\mathbb{C}}$, i.e., $\oplus_{\alpha \in \Delta_c} \mathcal{G}_\alpha$. Indeed, if $\alpha \in \Delta_c$, $\lambda = \alpha \mid_{\mathcal{A}}$, then $\dim_{\mathbb{R}} \mathcal{G}_\lambda > 1$. Let $\Delta_\lambda = \{\alpha \in \Delta \mid \alpha \mid_{\mathcal{A}} = \lambda\}$. If $\alpha \in \Delta_c$, then we have $X_\alpha = Y_\alpha + Z_\alpha$, where $Y_\alpha \in \mathcal{P}^{\mathbb{C}}$, $Z_\alpha \in \mathcal{K}^{\mathbb{C}}$. Now we can see that $\mathcal{G}_\lambda$ = r.l.s. $\{\tilde{X}_\alpha = Y_\alpha + iZ_\alpha , \ \forall \alpha \in \Delta_\lambda\}$. The actual choice of basis in $\mathcal{G}_\lambda$ is a matter of convenience, cf. the examples below.

A general property of the deformation $U_q(\mathcal{G})$ obtained by the above procedure is that $U_q(\mathcal{M})$, $U_q(\mathcal{P}_0)$, $U_q(\tilde{\mathcal{P}}_0)$ are Hopf subalgebras of $U_q(\mathcal{G})$, where $\mathcal{P}_0 \equiv \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, $\tilde{\mathcal{P}}_0 \equiv \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$.

We recall that a real noncompact simple Lie algebra has in general (a finite number of) non-congugate Cartan subalgebras. The conjugacy classes may be represented by Cartan subalgebras $\mathcal{H}' = \mathcal{H}'_k \oplus \mathcal{A}'$, where $\mathcal{H}_m \subseteq \mathcal{H}'_k \subseteq \mathcal{H}_k$, $\mathcal{H}_k$ being the Cartan subalgebra of $\mathcal{K}$, $\mathcal{A}' \subseteq \mathcal{A}$. The Cartan subalgebras with maximal dimension of $\mathcal{A}'$ are conjugate to $\mathcal{H}$; also those with minimal dimension of $\mathcal{A}'$ are conjugate to each other. Thus if we choose a Cartan subalgebra non-conjugate to $\mathcal{H}$, we can apply the same scheme as above, changing the basis of $\mathcal{H}^{\mathbb{C}}$ the classification of the roots, etc. The results of the q-deformation will be different. In particular, for $\mathcal{H} \not\cong \mathcal{H}_0$ we have to consider also the singular roots. Explicitly, for $\alpha \in \Delta_s$, using (9c) we have instead of (11):

$$[S^+_\alpha , S^-_\alpha] = \frac{sinh(\tilde{H}_\alpha h_\alpha/2)}{sin(h_\alpha/2)} , \ [\tilde{H}_\alpha, S^\pm_\alpha] = \mp S^\mp_\alpha , \ q_\alpha = q^{(\alpha,\alpha)/2} = e^{-ih_\alpha} , \qquad (12a)$$

$$S^+_\alpha \equiv (1/\sqrt{2})(X_\alpha + X_{-\alpha}) , \ S^-_\alpha \equiv (i/\sqrt{2})(X_\alpha - X_{-\alpha}) , \ \tilde{H}_\alpha \equiv -iH_\alpha , \qquad (12b)$$

$$\delta(S^\pm_\alpha) = S^\pm_\alpha \otimes e^{\tilde{H}_\alpha h_\alpha/4} + e^{-\tilde{H}_\alpha h_\alpha/4} \otimes S^\pm_\alpha , \ \alpha \in \Delta^+_s \cap \Delta_S . \qquad (12c)$$

Thus our scheme provides a different q - deformation for each conjugacy class of Cartan subalgebras.

3.3. q - deformations with other parabolic subalgebras and q - deformation of reductive Lie algebras and superalgebras

With the notation of Section 2. we recall that $\mathcal{P}_0 \equiv \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ ($\cong \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$) is the minimal parabolic subalgebra of $\mathcal{G}$. A standard parabolic subalgebra is any subalgebra $\mathcal{P}'$ of $\mathcal{G}$ such that $\mathcal{P}_0 \subseteq \mathcal{P}'$. The number of standard parabolic subalgebras, including $\mathcal{P}_0$ and $\mathcal{G}$, is 2^r. They are all of the form $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, $\mathcal{M}' \supseteq \mathcal{M}$, $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{N}' \subseteq \mathcal{N}$; $\mathcal{M}'$ is the centralizer of $\mathcal{A}'$ in $\mathcal{G}(mod\ \mathcal{A}')$; $\mathcal{N}'$ (resp. $\tilde{\mathcal{N}}' = \theta\mathcal{N}'$) is comprised from the negative (resp. positive) root spaces of the restricted root system of $(\mathcal{G}, \mathcal{A}')$. One also has the analogue of (8) :

$$\mathcal{G} = \tilde{\mathcal{N}}' \oplus \mathcal{A}' \oplus \mathcal{M}' \oplus \mathcal{N}' . \qquad (13)$$

We would like to have a deformation of $\mathcal{G}$ which is compatible with this decomposition. In the scheme desribed in subsection 3.2. we have used the fact that the deformation of $\mathcal{M}_s$ is inherited from the deformation of $\mathcal{M}^{\mathbb{C}}_s$. However, in general $\mathcal{M}'$ is a *noncompact* reductive Lie algebra.

Thus we need to extend our scheme to reductive Lie algebras. Let $\hat{\mathcal{G}} = \mathcal{G} \oplus \mathcal{Z} = \hat{\mathcal{K}} \oplus \hat{\mathcal{P}}$ be a real reductive Lie algebra, where $\mathcal{G}$ is the semisimple part of $\hat{\mathcal{G}}$, $\mathcal{Z}$ is the centre of $\hat{\mathcal{G}}$; $\hat{\mathcal{K}}, \hat{\mathcal{P}}$ are the $+1, -1$ eigenspaces of the Cartan involution $\hat{\theta}$; $\hat{\mathcal{A}} = \mathcal{A} \oplus \mathcal{Z}_p$, is the analogue of $\mathcal{A}$, $\mathcal{Z}_p = \mathcal{Z} \cap \hat{\mathcal{P}}$. The root system of the pair $(\hat{\mathcal{G}}, \hat{\mathcal{A}})$ coincides with Δ_R and the subalgebras $\tilde{\mathcal{N}}$ and $\mathcal{N}$ are inherited from $\mathcal{G}$. The decomposition (8) then is:

$$\hat{\mathcal{G}} = \tilde{\mathcal{N}} \oplus \hat{\mathcal{A}} \oplus \hat{\mathcal{M}} \oplus \mathcal{N} , \qquad (14)$$

where $\hat{\mathcal{M}} = \hat{\mathcal{M}}_s \oplus \hat{\mathcal{Z}}_m$, $\hat{\mathcal{M}}_s = \mathcal{M}_s$, $\hat{\mathcal{Z}}_m = \mathcal{Z}_m \oplus \mathcal{Z} \cap \hat{\mathcal{K}}$. Finally we use the q - deformation of $\hat{\mathcal{G}}^{\mathbb{C}}$ which is the direct sum $U_q(\mathcal{G}^{\mathbb{C}}) \oplus \mathcal{Z}^{\mathbb{C}}$.

The above scheme can be immediately applied in the case when $\mathcal{G}$ is a real form of a basic classical Lie superalgebra. This is illustrated in Section 7.

4. q - deformed Lorentz algebra $U_q(so(3,1))$

4.1. so(p,r)

Let $\mathcal{G} = so(p,r)$, with $p \geq r \geq 2$ or $p > r \geq 1$ with generators : $M_{AB} = -M_{BA}$, $A, B = 1, \ldots, p+r$, $\eta_{AB} = \text{diag}\,(- \cdots - + \cdots +)$, ($p$ times minus, r times plus) which obey :

$$[M_{AB}, M_{CD}] = i(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC}) . \qquad (15)$$

Besides the "physical" generators M_{AB} we shall also use the "mathematical" generators $Y_{AB} = -iM_{AB}$. One has: $\mathcal{K} \cong so(p) \oplus so(r)$ if $r \geq 2$ and $\mathcal{K} \cong so(p)$ if $r = 1$. The generators of $\mathcal{K}$ are M_{AB} with $1 \leq A < B \leq p$ and $p+1 \leq A < B \leq p+r$. The split rank is equal to r ; $\mathcal{M} \cong so(p-r)$, if $p - r \geq 2$ and $\mathcal{M} = 0$ if $p - r = 0, 1$, $\dim \tilde{\mathcal{N}} = \dim \mathcal{N} = r(p-1)$. Furthermore the dimensions of the roots in the root system Δ of $so(p+r, \mathbb{C})$, and in Δ_R depending on the parity κ of $p+r$ are given by:

$$|\Delta^\pm_r| = r(r - 1 + \kappa) \qquad (16a)$$

$$|\Delta^\pm_i| = (p - r - \kappa)(p - r - 2 + \kappa)/2 \qquad (16b)$$

$$|\Delta^\pm_c| = r(p - r - \kappa) \qquad (16c)$$

$$|\Delta^\pm_R| = r(r + \kappa) \qquad (16d)$$

$$\kappa = \begin{cases} 0 & \text{for } p+r \text{ even} \\ 1 & \text{for } p+r \text{ odd} \end{cases}$$

Note that the algebra $so(2n+1, 1)$ has only one conjugacy class of Cartan subalgebras. Thus in these cases our q - deformation is unique in our procedure. The algebra $so(2n, 1)$ has two conjugacy classes of Cartan subalgebras and in these cases there are two q - deformations.

4.2. $U_q(so(3,1))$

With $A, B = 1, 2, 3, 0$, $(- - - +)$, choose $\tilde{D} = M_{30}$ for the generator of $\mathcal{A}$ and $H = M_{12}$ for the generator of $\mathcal{M}$. All roots of the complexification $\mathcal{G}^{\mathbb{C}} = so(4, \mathbb{C}) \cong so(3, \mathbb{C}) \ni$

$so(3,\mathbb{C})$ are complex as is verified by a simple calculation. It is convenient to use the generators $M^{\pm} \equiv -M_{23} \pm iM_{13} \in \mathcal{K}^{\mathbb{C}}$, $N^{\pm} \equiv -M_{10} \mp iM_{20} \in \mathcal{P}^{\mathbb{C}}$. In terms of these generators the Lorentz algebra is given by :

$$[H, M^{\pm}] = \pm M^{\pm}, \ [M^+, M^-] = 2H, \tag{18a}$$

$$[H, N^{\pm}] = \pm N^{\pm}, \ [N^+, N^-] = -2H, \tag{18b}$$

$$[\tilde{D}, M^{\pm}] = \pm N^{\pm}, \ [\tilde{D}, N^{\pm}] = \mp M^{\pm}, \tag{18c}$$

$$[M^{\pm}, N^{\mp}] = \pm 2\tilde{D}, \tag{18d}$$

the rotation subalgebra being given by (18a). Using $so(4,\mathbb{C}) \cong so(3,\mathbb{C}) \oplus so(3,\mathbb{C})$ we denote the generators of the two commuting $so(3,\mathbb{C})$ algebras by $X_1^{\pm}, H_1$ and $X_2^{\pm}, H_2$, and then we have:

$$X_1^{\pm} = (1/2)(M^{\pm} - iN^{\pm}), \ H_1 = H - i\tilde{D}, \tag{19a}$$

$$X_2^{\pm} = (1/2)(M^{\pm} + iN^{\pm}), \ H_2 = H + i\tilde{D}. \tag{19b}$$

The we use $U_q(so(4,\mathbb{C})) = U_q(so(3,\mathbb{C})) \oplus U_q(so(3,\mathbb{C}))$ given by:

$$[X_a^+, X_a^-] = [H_a], \ [H_a, X_a^{\pm}] = \pm 2X_a^{\pm}, \ a = 1,2, \tag{20}$$

and with Hopf algebra structure given by (5) replacing X_i with X_a. This q - Lorentz algebra obtained in [9] as an application of our procedure was actually first proposed in [16] as the quantum group of Liouville theory in the strong coupling regime.

Thus we obtain the following $U_q(so(3,1))$ relations with $q = e^h \in \mathbb{R}$:

$$[H, M^{\pm}] = \pm M^{\pm}, \ [M^+, M^-] = 2[H]cos(\tilde{D}h/2), \tag{21a}$$

$$[H, N^{\pm}] = \pm N^{\pm}, \ [N^+, N^-] = -2[H]cos(\tilde{D}h/2), \tag{21b}$$

$$[\tilde{D}, M^{\pm}] = \pm N^{\pm}, \ [\tilde{D}, N^{\pm}] = \mp M^{\pm}, \tag{21c}$$

$$[M^{\pm}, N^{\mp}] = \pm 2\frac{cosh(Hh/2)\,sin(\tilde{D}h/2)}{sinh(h/2)}, \tag{21d}$$

$$\delta(M^{\pm}) = M^{\pm} \otimes e^{Hh/4} cos(\tilde{D}h/4) - N^{\pm} \otimes e^{Hh/4} sin(\tilde{D}h/4) + + e^{-Hh/4} cos(\tilde{D}h/4) \otimes M^{\pm} + e^{-Hh/4} sin(\tilde{D}h/4) \otimes N^{\pm}, \tag{22a}$$

$$\delta(N^{\pm}) = N^{\pm} \otimes e^{Hh/4} cos(\tilde{D}h/4) + M^{\pm} \otimes e^{Hh/4} sin(\tilde{D}h/4) + + e^{-Hh/4} cos(\tilde{D}h/4) \otimes N^{\pm} - e^{-Hh/4} sin(\tilde{D}h/4) \otimes M^{\pm}. \tag{22b}$$

$$\delta(H) = H \otimes 1 + 1 \otimes H, \ \delta(\tilde{D}) = \tilde{D} \otimes 1 + 1 \otimes \tilde{D}, \tag{22c}$$

$$\gamma(H) = -H, \ \gamma(M^{\pm}) = -q^{\pm 1/2} M^{\pm}, \tag{23a}$$

$$\gamma(\tilde{D}) = -\tilde{D}, \ \gamma(N^{\pm}) = -q^{\pm 1/2} N^{\pm}. \tag{23b}$$

5. q - deformed conformal algebra $U_q(su(2,2))$

5.1. $U_q(sl(4,\mathbb{C}))$

The root system of the complexification $sl(4,\mathbb{C})$ of $su(2,2)$ is given by $\Delta^{\pm} = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_{12}, \pm\alpha_{23}, \pm\alpha_{13}\}$; the simple roots are $\alpha_1, \alpha_2, \alpha_3$, while $\alpha_{12} = \alpha_1 + \alpha_2$, $\alpha_{23} = \alpha_2 + \alpha_3$, $\alpha_{13} = \alpha_1 + \alpha_2 + \alpha_3$; all roots are of length 2 and the non-zero products between the simple roots are: $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = -1$. The roots $\pm\alpha_1$, $\pm\alpha_3$ are real, while the other roots are complex. The Cartan-Weyl basis for the non-simple roots is given by (cf. [12],[17]):

$$X_{jk}^{\pm} = \pm q^{\mp 1/4}(q^{1/4} X_j^{\pm} X_k^{\pm} - q^{-1/4} X_k^{\pm} X_j^{\pm}), \ (jk) = (12),(23), \tag{24a}$$

$$\begin{aligned} X_{13}^{\pm} &= \pm q^{\mp 1/4}(q^{1/4} X_1^{\pm} X_{23}^{\pm} - q^{-1/4} X_{23}^{\pm} X_1^{\pm}) = \\ &= \pm q^{\mp 1/4}(q^{1/4} X_{12}^{\pm} X_3^{\pm} - q^{-1/4} X_3^{\pm} X_{12}^{\pm}). \end{aligned} \tag{24b}$$

All other commutation relations follow from these definitions. Besides those in (4) we have ($X_{aa}^{\pm} \equiv X_a^{\pm}$):

$$[X_a^+, X_{ab}^-] = -q^{H_a/2} X_{a+1b}^-, \ [X_b^+, X_{ab}^-] = X_{ab-1}^- q^{-H_b/2}, \ 1 \le a < b \le 3, \tag{25a}$$

$$[X_a^-, X_{ab}^+] = X_{a+1b}^+ q^{-H_a/2}, \ [X_b^-, X_{ab}^+] = -q^{H_b/2} X_{ab-1}^+, \ 1 \le a < b \le 3, \tag{25b}$$

$$X_a^{\pm} X_{ab}^{\pm} = q^{1/2} X_{ab}^{\pm} X_a^{\pm}, \ 1 \le a < b \le 3, \tag{25c}$$

$$[X_2^{\pm}, X_{13}^{\pm}] = 0, \ [X_2^{\pm}, X_{13}^{\mp}] = 0, \tag{25d}$$

$$[X_{12}^+, X_{13}^-] = -q^{H_1+H_2} X_3^-, \ [X_{12}^-, X_{13}^+] = X_3^+ q^{-H_1-H_2}, \tag{25e}$$

$$[X_{23}^+, X_{13}^-] = X_1^- q^{-H_2-H_3}, \ [X_{23}^-, X_{13}^+] = -q^{H_2+H_3} X_1^+ \tag{25f}$$

$$[X_{12}^{\pm}, X_{23}^{\pm}] = \tilde{\lambda} X_2^{\pm} X_{13}^{\pm}, \ [X_{12}^{\pm}, X_{23}^{\mp}] = -\tilde{\lambda} q^{\pm H_2/2} X_1^{\pm} X_3^{\mp}, \tag{25g}$$

$$\tilde{\lambda} \equiv q^{1/2} - q^{-1/2} . \tag{26}$$

Note that for $q \longrightarrow 1$ the RHS of eqs. (25g) vanishes.

5.2. $U_q(su(2,2))$

Let $\mathcal{G} = su(2,2) \cong so(4,2)$. It has three non-conjugate classes of Cartan subalgebras represented, say, by $\mathcal{H}^a$, $a = 0,1,2$ with a non-compact generators. We shall work with the most noncompact Cartan subalgebra $\mathcal{H} = \mathcal{H}_0 = \mathcal{H}^2$. Using the notation from subsection 4.1. with $A, B = 1,2,3,5,6,0$, $(- - - - ++)$, choose Y_{30} and Y_{56} as generators of $\mathcal{A}$ and Y_{12} for the generator of $\mathcal{M}$. Since $su(2,2)$ is the conformal algebra of 4 - dimensional Minkowski space - time we would like to deform it consistently with the subalgebra structure relevant for the physical applications. These subalgebras are : the *Lorentz* subalgebra $\mathcal{M}' \cong so(3,1)$ generated by $Y_{\mu\nu}$, $\mu,\nu = 1,2,3,0$, the subalgebra $\tilde{\mathcal{N}}'$ of *translations* generated by $P_\mu = Y_{\mu 5} + Y_{\mu 6}$, the subalgebra $\mathcal{N}'$ of *special conformal transformations* generated by $K_\mu = Y_{\mu 5} - Y_{\mu 6}$, the *dilatations* subalgebra $\mathcal{A}'$ generated by $D = Y_{56}$. The commutation relations besides those for the Lorentz subalgebra are:

$$[D, Y_{\mu\nu}] = 0, \quad [D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu . \tag{27a}$$

$$[Y_{\mu\nu}, P_\lambda] = \eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu, \quad [Y_{\mu\nu}, K_\lambda] = \eta_{\nu\lambda} K_\mu - \eta_{\mu\lambda} K_\nu , \tag{27b}$$

$$[P_\mu, K_\nu] = 2 Y_{\mu\nu} + 2\eta_{\mu\nu} D , \tag{27c}$$

From formulae (27a) we see that the Lorentz subalgebra $\mathcal{M}'$ is a maximal subalgebra of $\mathcal{G}$ commuting with $\mathcal{A}'$, and that $\tilde{\mathcal{N}}'$, resp., $\mathcal{N}'$, is a 4 - dimensional root vector space of the restricted root system $\Delta'_R = \{\pm\lambda ; \lambda(D) = 1\}$ of the pair $(\mathcal{G}, \mathcal{A}')$, corresponding to the restricted root λ, resp., $-\lambda$. In short, the algebra $\mathcal{P}_{max} = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$ (or equivalently $\tilde{\mathcal{P}}_{max} = \mathcal{M}' \oplus \mathcal{A}' \oplus \tilde{\mathcal{N}}'$) is the so called *maximal parabolic subalgebra* of $\mathcal{G}$, where $\tilde{\mathcal{N}}'$, resp., $\mathcal{N}'$, is the root vector space of the restricted root system $\Delta'_R = \{\pm\lambda ; \lambda(D) = 1\}$ of $(\mathcal{G}, \mathcal{A}')$, corresponding to λ, resp., $-\lambda$, (cf. subsection 3.3.).

For the Lorentz algebra generators we have the following expressions:

$$H = (1/2)(H_1 + H_3), \quad \tilde{D} = (i/2)(H_1 - H_3), \tag{28a}$$

$$M^\pm = X_3^\pm + X_1^\pm, \quad N^\pm = i(X_3^\pm - X_1^\pm), \tag{28b}$$

For the dilatations, translations and special conformal transformations we have:

$$D = (1/2)(H_1 + H_3) + H_2 , \tag{29}$$

$$P_0 = i(X_{12}^+ + X_{23}^+), \quad P_1 = i(X_{13}^+ + X_2^+), \quad P_2 = X_{13}^+ - X_2^+, \quad P_3 = i(X_{23}^+ - X_{12}^+), \tag{30}$$

$$K_0 = -i(X_{12}^- + X_{23}^-), \quad K_1 = i(X_{13}^- + X_2^-), \quad K_2 = X_2^- - X_{13}^-, \quad K_3 = i(X_{23}^- - X_{12}^-). \tag{31}$$

Now we can derive the relations in $U_q(su(2,2))$:

1) According to our general scheme the deformed Lorentz subalgebra is a Hopf subalgebra. This is seen also directly since formulae (28) are just the inverse of (19), only $X_2^\pm$, H_2 should be replaced by $X_3^\pm$, H_3. The two commuting subalgebras here are generated by $X_1^\pm$, H_1 and $X_3^\pm$, H_3 and they commute because $(\alpha_1, \alpha_3) = 0$. Thus the deformation of the Lorentz subalgebra is described by formulae (21),(22),(23).

2) Formulae (27a) are not deformed since $D \in \mathcal{H}$.

3) The deformation of the translation subalgebra is given by:

$$P_a(P_0 \pm P_3) = q^{\pm 1/2}(P_0 \pm P_3)P_a, \quad a = 1,2, \tag{32a}$$

$$[P_1, P_2] = 0, \quad [P_0 - P_3, P_0 + P_3] = \tilde{\lambda}(P_1^2 + P_2^2) \tag{32b}$$

4) For the subalgebra of special conformal transformations we have:

$$K_a(K_0 \pm K_3) = q^{\mp 1/2}(K_0 \pm K_3)K_a, \quad a = 1,2, \tag{33a}$$

$$[K_1, K_2] = 0, \quad [K_0 - K_3, K_0 + K_3] = \tilde{\lambda}(K_1^2 + K_2^2) \tag{33b}$$

5.1) Formulae (27b) are not deformed for the Cartan generators (28a) :

$$[H, P_1 \pm iP_2] = \pm(P_1 \pm iP_2), \quad [H, P_3 \pm P_0] = 0, \tag{34a}$$

$$[\tilde{D}, P_3 \pm P_0] = \pm i(P_3 \pm P_0), \quad [\tilde{D}, P_1 \pm iP_2] = 0, \tag{34b}$$

$$[H, K_1 \pm iK_2] = \pm(K_1 \pm iK_2), \quad [H, K_3 \pm K_0] = 0, \tag{35a}$$

$$[\tilde{D}, K_3 \pm K_0] = \pm i(K_3 \pm K_0), \quad [\tilde{D}, K_1 \pm iK_2] = 0, \tag{35b}$$

5.2) For the commutation relations between the Lorentz generators and the translation generators instead of those in (27b) we have:

$$M^+(P_3 + P_0) - q^{-1/2}(P_3 + P_0)M^+ = P_1 + iP_2 , \tag{36a}$$

$$M^+(P_3 - P_0) - q^{1/2}(P_3 - P_0)M^+ = q^{1/2}(P_1 + iP_2) , \tag{36b}$$

$$M^+(P_1 + iP_2) - \frac{[2]}{2}(P_1 + iP_2)M^+ = \frac{i\tilde{\lambda}}{2}(P_1 + iP_2)N^+ , \tag{36c}$$

$$M^+(P_1 - iP_2) - \frac{[2]}{2}(P_1 - iP_2)M^+ = \frac{-i\tilde{\lambda}}{2}(P_1 - iP_2)N^+ + (P_0 - P_3) - q^{1/2}(P_3 + P_0) , \tag{36d}$$

$$[M^- , P_3 + P_0] = -q^{(i\bar{D}+H)/2}(P_1 - iP_2) , \tag{37a}$$
$$[M^- , P_0 - P_3] = (P_1 - iP_2)q^{(i\bar{D}-H)/2} , \tag{37b}$$
$$[M^- , P_1 + iP_2] = (P_3 + P_0)q^{(i\bar{D}-H)/2} + q^{(i\bar{D}+H)/2}(P_3 - P_0) , \tag{37c}$$
$$[M^- , P_1 - iP_2] = 0 . \tag{37d}$$

$$N^+(P_0 + P_3) - q^{-1/2}(P_0 + P_3)N^+ = -i(P_1 + iP_2) , \tag{38a}$$
$$N^+(P_3 - P_0) - q^{1/2}(P_3 - P_0)N^+ = iq^{1/2}(P_1 + iP_2) , \tag{38b}$$
$$N^+(P_1 + iP_2) - \frac{[2]}{2}(P_1 + iP_2)N^+ = \frac{-i\tilde{\lambda}}{2}(P_1 + iP_2)M^+ , \tag{38c}$$
$$N^+(P_1 - iP_2) - \frac{[2]}{2}(P_1 - iP_2)N^+ = \frac{i\tilde{\lambda}}{2}(P_1 - iP_2)M^+ + i(P_3 - P_0) - iq^{1/2}(P_0 + P_3) \tag{38d}$$

$$[N^- , P_0 + P_3] = -iq^{(i\bar{D}+H)/2}(P_1 - iP_2) , \tag{39a}$$
$$[N^- , P_3 - P_0] = i(P_1 - iP_2)q^{(i\bar{D}-H)/2} , \tag{39b}$$
$$[N^- , P_1 + iP_2] = iq^{(i\bar{D}+H)/2}(P_3 - P_0) - i(P_0 + P_3)q^{(i\bar{D}-H)/2} , \tag{39c}$$
$$[N^- , P_1 - iP_2] = 0 . \tag{39d}$$

5.3) For the commutation relations between the Lorentz generators and the generators of special conformal transformations instead of those in (27b) we have:

$$[M^+ , K_0 + K_3] = q^{(H-i\bar{D})/2}(K_1 + iK_2) , \tag{40a}$$
$$[M^+ , K_3 - K_0] = (K_1 + iK_2)q^{-(i\bar{D}+H)/2} , \tag{40b}$$
$$[M^+ , K_1 - iK_2] = q^{(H-i\bar{D})/2}(K_0 - K_3) - (K_0 + K_3)q^{-(i\bar{D}+H)/2} , \tag{40c}$$
$$[M^+ , K_1 + iK_2] = 0 . \tag{40d}$$

$$M^-(K_0 + K_3) - q^{1/2}(K_0 + K_3)M^- = -(K_1 - iK_2) , \tag{41a}$$
$$M^-(K_0 - K_3) - q^{-1/2}(K_0 - K_3)M^- = q^{-1/2}(K_1 - iK_2) , \tag{41b}$$
$$M^-(K_1 - iK_2) - \frac{[2]}{2}(K_1 - iK_2)M^- = \frac{i\tilde{\lambda}}{2}(K_1 - iK_2)N^- , \tag{41c}$$
$$M^-(K_1 + iK_2) - \frac{[2]}{2}(K_1 + iK_2)M^- = \frac{-i\tilde{\lambda}}{2}(K_1 + iK_2)N^- + K_3 - K_0 + q^{-1/2}(K_0 + K_3) \tag{41d}$$

$$[N^+ , K_0 + K_3] = -iq^{(H-i\bar{D})/2}(K_1 + iK_2) , \tag{42a}$$
$$[N^+ , K_3 - K_0] = i(K_1 + iK_2)q^{-(i\bar{D}+H)/2} , \tag{42b}$$
$$[N^+ , K_1 - iK_2] = iq^{(H-i\bar{D})/2}(K_3 - K_0) - i(K_0 + K_3)q^{-(i\bar{D}+H)/2} , \tag{42c}$$
$$[N^+ , K_1 + iK_2] = 0 . \tag{42d}$$

$$N^-(K_0 + K_3) - q^{1/2}(K_0 + K_3)N^- = -i(K_1 - iK_2) , \tag{43a}$$
$$N^-(K_3 - K_0) - q^{-1/2}(K_3 - K_0)N^- = -iq^{-1/2}(K_1 - iK_2) , \tag{43b}$$
$$N^-(K_1 - iK_2) - \frac{[2]}{2}(K_1 - iK_2)N^- = \frac{-i\tilde{\lambda}}{2}(K_1 - iK_2)M^- , \tag{43c}$$
$$N^-(K_1 + iK_2) - \frac{[2]}{2}(K_1 + iK_2)N^- = \frac{i\tilde{\lambda}}{2}(K_1 + iK_2)M^- + i(K_3 - K_0) - iq^{-1/2}(K_0 + K_3) \tag{43d}$$

The commutation relations between $M^\pm$ and K_μ may be obtained from those between $M^\pm$ and P_μ by the following changes: $M^\pm \mapsto M^\mp$, $N^+ \mapsto -N^-$, $H \mapsto -H$, $\bar{D} \mapsto \bar{D}$, $P_\mu \mapsto (-1)^{\delta_{\mu 1}} K_\mu$, $q^{1/2} \mapsto q^{-1/2}$. These follow from the automorphism of $U_q(\mathcal{G}^{\mathcal{C}})$: $X_1^\pm \longleftrightarrow X_3^\mp$, $H_1 \longleftrightarrow -H_3$, $X_2^\pm \mapsto -X_2^\mp$, $H_2 \mapsto -H_2$, $q^{1/2} \mapsto q^{-1/2}$, (then $X_{12}^\pm \longleftrightarrow -X_{23}^\mp$, $X_{13}^\pm \mapsto -X_{13}^\mp$). The commutation relations between $N^\pm$ and P_μ, K_μ may be obtained from those between $M^\pm$ and P_μ by the changes $M^\pm \longleftrightarrow iN^\pm$, $P_0 \longleftrightarrow P_3$, $P_1 \longleftrightarrow iP_2$ and from those between $M^\pm$ and K_μ by the changes $M^\pm \longleftrightarrow -iN^\pm$, $K_0 \longleftrightarrow K_3$, $K_1 \longleftrightarrow -iK_2$.

6) For the commutation relations between translations and special conformal transformations we have:

$$[P_3 \pm P_0 , K_3 \pm K_0] = \pm\tilde{\lambda}\, q^{\pm(H-D)/2} (M^+ \mp iN^+)(M^- \pm iN^-) , \tag{44a}$$
$$[P_3 \pm P_0 , K_3 \mp K_0] = 4[\mp i\bar{D} - D] \tag{44b}$$
$$[P_1 \pm iP_2 , K_1 \mp iK_2] = 4 [\mp H - D] , \tag{44c}$$
$$[P_1 \pm iP_2 , K_1 \pm iK_2] = 0 , \tag{44d}$$
$$[P_3 + P_0 , K_1 + iK_2] = 2 (M^+ - iN^+) q^{(H-D)/2} , \tag{44e}$$
$$[P_3 + P_0 , K_1 - iK_2] = -2 (M^- + iN^-) q^{-(D+i\bar{D})/2} , \tag{44f}$$
$$[P_3 - P_0 , K_1 + iK_2] = 2 q^{(D-H)/2} (M^+ + iN^+) , \tag{44g}$$
$$[P_3 - P_0 , K_1 - iK_2] = -2 q^{(D-i\bar{D})/2} (M^- - iN^-) , \tag{44h}$$
$$[P_1 - iP_2 , K_3 + K_0] = 2 (M^- + iN^-) q^{(H-D)/2} , \tag{44i}$$

$$[P_1 + iP_2\,, K_3 + K_0] \;=\; -2\,(M^+ - iN^+)\,q^{(i\tilde{D}-D)/2}\,, \tag{44j}$$

$$[P_1 - iP_2\,, K_3 - K_0] \;=\; 2\,q^{(D-H)/2}\,(M^- - iN^-)\,, \tag{44k}$$

$$[P_1 + iP_2\,, K_3 - K_0] \;=\; -2\,q^{(D+i\tilde{D})/2}\,(M^+ + iN^+)\,. \tag{44l}$$

The comultiplication for the Lorentz subalgebra is given by (22), for the translations, special conformal transformations and dilataions we have:

$$\begin{aligned} \delta(T^\pm) &= T^\pm \otimes q^{(D\pm i\tilde{D})/4} + q^{-(D\pm i\tilde{D})/4} \otimes T^\pm + \delta_1(T^\pm)\,, \\ &\quad T^\pm = P_3 \pm P_0\,,\; K_3 \mp K_0\,, \\ \delta_1(T^\pm) &= \frac{\tilde{\lambda}}{2}\,(M^\pm \mp iN^\pm)\,q^{(H-D)/4} \otimes q^{(H\pm i\tilde{D})/4}\,\tilde{T}^\pm\,, \\ &\quad T^+ = P_3 + P_0,\, T^- = K_3 + K_0,\, \tilde{T}^+ = P_1 - iP_2,\, \tilde{T}^- = K_1 + iK_2 \\ \delta_1(T^\pm) &= -\frac{\tilde{\lambda}}{2}\,\tilde{T}^\mp\,q^{(-H\mp i\tilde{D})/4} \otimes q^{(D-H)/4}\,(M^\mp \pm iN^\mp)\,, \\ &\quad T^+ = K_3 - K_0\,,\; T^- = P_3 - P_0 \;; \end{aligned} \tag{45a}$$

$$\begin{aligned} \delta(T^\pm) &= T^\pm \otimes q^{(D\pm H)/4} + q^{-(D\pm H)/4} \otimes T^\pm + \delta_1(T^\pm)\,, \\ &\quad T^\pm = P_1 \pm iP_2\,,\; K_1 \mp iK_2 \\ \delta_1(T^\pm) &= \frac{\tilde{\lambda}}{2}\,\tilde{T}'^\pm\,q^{(-H\pm i\tilde{D})/4} \otimes q^{(D\pm i\tilde{D})/4}\,(M^\pm \pm iN^\pm) - \\ &\quad - \frac{\tilde{\lambda}}{2}\,(M^\pm \mp iN^\pm)\,q^{(-D\pm i\tilde{D})/4} \otimes q^{(H\pm i\tilde{D})/4}\,\tilde{T}''^\pm\,, \\ &\quad T^+ = P_1 + iP_2\,,\; T^- = K_1 - iK_2\,,\; \tilde{T}'^+ = P_3 + P_0\,, \\ &\quad \tilde{T}'^- = K_3 + K_0\,,\; \tilde{T}''^+ = P_3 - P_0\,,\; \tilde{T}''^- = K_3 - K_0\,, \\ \delta_1(\tilde{T}^\pm) &= 0\;; \end{aligned} \tag{45b}$$

$$\delta(D) \;=\; D \otimes 1 + 1 \otimes D\,. \tag{45c}$$

The antipode for the Lorentz subalgebra is given by (23), for the translations, special conformal transformations and dilataions we have:

$$\begin{aligned} \gamma(P_0 \pm P_3) &= -q^{1\pm 1/2}(P_0 \pm P_3) + \frac{q^{1/4\pm 1/4}(q-1)}{2}(P_1 - iP_2)(M^+ \mp iN^+) \\ \gamma(P_1 + iP_2) &= -q^{3/2}(P_1 + iP_2) + \frac{q(q-1)}{2}(P_0 + P_3)(M^+ + iN^+) + \\ &\quad + \frac{q^{1/2}(q-1)}{2}(P_0 - P_3)(M^+ - iN^+) - \\ &\quad - \frac{(q-1)^2}{4}(P_1 - iP_2)((M^+)^2 + (N^+)^2)\,, \\ \gamma(P_1 - iP_2) &= -q^{1/2}(P_1 - iP_2)\;; \end{aligned} \tag{46a}$$

$$\begin{aligned} \gamma(K_0 \pm K_3) &= -q^{-1\mp 1/2}(K_0 \pm K_3) - \frac{q^{-1/4\mp 1/4}(q^{-1}-1)}{2}(K_1 + iK_2)(M^- \pm iN^-) \\ \gamma(K_1 - iK_2) &= -q^{-3/2}(K_1 - iK_2) - \frac{q^{-1/2}(q^{-1}-1)}{2}(K_0 - K_3)(M^- + iN^-) - \\ &\quad - \frac{q^{-1}(q^{-1}-1)}{2}(K_0 + K_3)(M^- - iN^-) - \\ &\quad - \frac{(q^{-1}-1)^2}{4}(K_1 + iK_2)((M^-)^2 + (N^-)^2)\,, \\ \gamma(K_1 + iK_2) &= -q^{-1/2}(K_1 + iK_2)\;; \end{aligned} \tag{46b}$$

$$\gamma(D) \;=\; -D\,. \tag{46c}$$

Consistently with the general scheme formulae (45),(46) tell us that the deformed subalgebras of translations and special conformal transformations are not Hopf subalgebras of $\mathcal{G}$, rather the algebras $U_q(\mathcal{P}_{max})$, $U_q(\tilde{\mathcal{P}}_{max})$ are Hopf subalgebras of $U_q(\mathcal{G})$.

6. q - deformed Poincaré and Weyl algebras

The Poincaré algebra is not a semisimple (or reductive) Lie algebra and our procedure is not directly applicable. One may try to use the fact that it is a subalgebra of the conformal algebra. Indeed, there is a q - deformed Poincaré algebra with generators $M^\pm$, $N^\pm$, H, $\hat{D} = i\tilde{D}$, P_μ, and with relations given by (21), (32), (34), (36) - (39), (45) (restricted to P_μ), (46a). This q - deformed Poincaré is a (commutation) subalgebra of the deformed conformal algebra (cf. (21), (32), (34), (36) - (39)), however, from formulae (45) follows that it is not a Hopf subalgebra of $U_q(su(2,2))$. On the other hand the deformation $U_q(\tilde{\mathcal{P}}_{max})$ of the 11 - generator Weyl subalgebra = Poincaré & dilatations = $\tilde{\mathcal{P}}_{max}$ - is a Hopf subalgebra of $U_q(su(2,2))$. Another Weyl algebra conjugate to this is $U_q(\mathcal{P}_{max})$ with generators $M^\pm$, $N^\pm$, H, $\hat{D} = i\tilde{D}$, K_μ, D, and with commutations relations given by (21), (33), (35), (40) - (43), (45) (restricted to K_μ), (46b) [9]. Consistently with our procedure one may choose other relations instead of (28) - (30), as is done in [18], and obtain a Weyl algebra which is seemingly different from $U_q(\tilde{\mathcal{P}}_{max})$.

Other deformed Poincaré algebras may be obtained by contraction of $U_q(so(4,1))$ and $U_q(so(3,2))$. Let us denote by U^k_{41}, $k = 1, 2$, U^k_{32}, $k = 0, 1, 2$, respectively, the deformations of $U(so(4,1))$, $U(so(3,2))$, respectively, with Cartan subalgebras with k compact generators. As explained in [9], one cannot obtain in this way a Poincaré algebra with a deformed Lorentz algebra as a subalgebra, since one has to use contractions which involve Cartan generators. This may be a noncompact generator which is possible for U^1_{41} and U^a_{32}, $a = 0, 1$, or a compact generator which is possible for U^a_{41}, $a = 1, 2$, and U^2_{32}. (The last case was studied in [19].) The resulting deformed

Poincaré algebras will have a noncompact Hopf subalgebra in the case U^0_{32} and in one of the U^1_{41} cases and a compact Hopf subalgebra in the other four cases.

7. q - deformed conformal superalgebras $U_q(su(2,2/N))$

7.1. q - deformed complex superalgebras

Let $\mathcal{G}_c$ be a complex Lie superalgebra with a symmetrizable Cartan matrix $A = (a_{jk}) = A^d A^s$, where $A^s = (a^s_{jk})$ is a symmetric matrix, and $A^d = \text{diag}\,(d_1, ..., d_n)$, $d_j > 0$. Then the q - deformation $U_q(\mathcal{G}_c)$ of the universal enveloping algebras $U(\mathcal{G}_c)$ is defined [14],[20] as the associative algebra over $\mathbb{C}$ with generators $X^\pm_j$, H_j , $j \in J = \{1, \ldots, \ell\}$ and with relations :

(i) (1a) with a_{jk} replaced by a^s_{jk} and [,] being the supercommutator : $[Y, Z] \equiv YZ - (-1)^{\deg Y \deg Z} ZY$, $\deg H_j = \bar{0}$, $j \in J$, $\deg X^\pm_j = \bar{0}$, $j \notin \mathcal{T}$, $\deg X^\pm_j = \bar{1}$, $j \in \mathcal{T}$, $\mathcal{T} \subset J$ enumerates the set of *odd* simple roots, $J \backslash \mathcal{T}$ - the set of *even* simple roots;

(ii)

$$(\text{ad}_{q^\kappa} X^\pm_j)^{n_{jk}} (X^\pm_k) = 0 \ , \text{ for } j \neq k \ , \ \kappa = \pm ; \tag{47a}$$

(iii) [21-24], for every three simple roots, say, α_j , $\alpha_{j\pm1}$, such that $(\alpha_j, \alpha_j) = 0$, $(\alpha_{j\pm1}, \alpha_{j\pm1}) \neq 0$, $(\alpha_{j+1}, \alpha_{j-1}) = 0$, $(\alpha_j, \alpha_{j+1} + \alpha_{j-1}) = 0$, also holds:

$$[[X^\pm_j \ , X^\pm_{j-1}]_{q^\kappa} \ , [X^\pm_j, X^\pm_j]_{q^\kappa}]_{q^\kappa} = 0 \ , \tag{47b}$$

where :

$$n_{jk} = \begin{cases} 1 & \text{if } a^s_{jj} = a^s_{jk} = 0 \\ 2 & \text{if } a^s_{jj} = 0, \ a^s_{jk} \neq 0 \\ 1 - 2a^s_{jk}/a^s_{jj} & \text{if } a^s_{jj} \neq 0 \end{cases} \tag{48}$$

in (47a,b) one uses the deformed supercommutator:

$$(\text{ad}_{q^\kappa} X^\pm_j)\,(X^\pm_k) = [X^\pm_j, X^\pm_k]_{q^\kappa} \equiv$$
$$\equiv X^\pm_j X^\pm_k - (-1)^{\deg X^\pm_j \deg X^\pm_k} q^{\kappa(\alpha_j,\alpha_k)/2} X^\pm_k X^\pm_j \ . \tag{49}$$

When $\mathcal{T} = \emptyset$ relations (47a) for $\kappa = 1$ are the same as for $\kappa = -1$ and coincide with (1c). The necessity of the extra relations (iii) was communicated to the author in May 1991 independently by M. Scheunert [21], V.G. Kac [22], and D.A. Leites [23]. These relations were written first in [21] for $U_q(sl(M/N;\mathbb{C}))$; here they are given as in [24].

The Hopf algebra structure is given by formulae (5), however, with $\rho = \rho_0 - \rho_1$, $\rho_s = \frac{1}{2}\sum_{\alpha\in\Delta^+_{(s)}} \alpha$, $\Delta^+_{(0)}$, $\Delta^-_{(1)}$, resp., is the set of even, odd, resp., positive roots.

Let $\mathcal{G}_c = \mathcal{G}^{\mathbb{C}} = sl(M/N;\mathbb{C})$, $\ell = M + N - 1$. We choose a Cartan matrix with elements: $a_{jj} = a^s_{jj} = 2(1 - \delta_{jM})$, $a_{jj\pm1} = a^s_{jj\pm1} = -1$ except for $a_{jj+1} = 1$, all other elements are zero; $d_j = 1$, $j \leq M$, $d_j = -1$, $j > M$. Consistently the products between the simple roots are: $(\alpha_j, \alpha_j) = 2, \ 0, \ -2$ for $j < M$, $j = 0$, $j > M$, resp., $(\alpha_j, \alpha_{j+1}) = -1, \ 1$ for $j \leq M$, $j > M$, respectively, all other products are zero. The root system is given by: $\Delta^\pm = \{\pm\alpha_{jk} = \pm(\alpha_j + \alpha_{j+1} + \cdots + \alpha_k) \mid 1 \leq j \leq k \leq \ell, \alpha_{jj} = \alpha_j\}$. The roots $\pm\alpha_{jk}$ with $1 \leq j \leq M$, $M < k \leq \ell$ are odd, the rest are even. The Cartan-Weyl generators corresponding to nonsimple roots are defined inductively in analogy to (24) (cf. also [14]):

$$X^+_{jk} \equiv X^+_j X^+_{j+1k} - (-1)^{\deg X^+_j \deg X^+_{j+1k}} q^{(\alpha_j,\alpha_{j+1k})/2} X^+_{j+1k} X^+_j \ , \ j < k \ , \tag{50a}$$

$$X^-_{jk} \equiv X^-_{j+1k} X^-_j - (-1)^{\deg X^-_j \deg X^-_{j+1k}} q^{-(\alpha_j,\alpha_{j+1k})/2} X^-_j X^-_{j+1k} \ , \ j < k \ . \tag{50b}$$

Note that $sl(M/M;\mathbb{C})$ is a reductive Lie superalgebra with centre generated by $Z_M \equiv H_1 - H_{2M-1} + 2(H_2 - H_{2M-2}) + \cdots + (M-1)(H_{M-1} - H_{M+1}) + MH_M$.

7.2. $U_q(su(2,2/N))$

The Lie superalgebra $\mathcal{G}^S \equiv su(2,2/N)$ [13] is a real noncompact form of $\mathcal{G}^{\mathbb{C}} = sl(4/N;\mathbb{C})$ with Cartan decomposition and splitting into even and odd parts: $\mathcal{G}^S = \mathcal{K}^S + \mathcal{P}^S = \mathcal{G}^S_{(0)} + \mathcal{G}^S_{(1)}$ such that $\mathcal{G}^S_{(0)} \cong su(2,2) \oplus u(1) \oplus su(N)$, $\mathcal{K}^S_{(0)} \cong u(2) \oplus u(2) \oplus su(N)$, $\dim_R \mathcal{P}^S_{(0)} = 8$, $\dim_R \mathcal{K}^S_{(1)} = \dim_R \mathcal{P}^S_{(1)} = 4N$.

The parabolic subalgebras of $\mathcal{G}^S$ are determined by the parabolic subalgebras of the non-compact subalgebra $su(2,2)$ of the even part $\mathcal{G}^S_{(0)}$. As for $su(2,2)$ in the present paper we consider only q - deformation of $\mathcal{G}$ consistent with the maximal parabolic subalgebra $\mathcal{P}^S_{max} = \mathcal{M}^S \oplus \mathcal{A}^S \oplus \mathcal{N}^S$, where

$$\mathcal{A}^S = \mathcal{A}^S_{(0)} = l.s.\{D\} \cong \mathcal{A}', \tag{51a}$$

$$\mathcal{M}^S = \mathcal{M}^S_{(0)} \cong \mathcal{M}' \oplus u(1) \oplus su(N) \ , \ \mathcal{M}' \cong so(3,1) \ , \tag{51b}$$

$$\mathcal{N}^S = \mathcal{G}^-_1 \oplus \mathcal{G}^-_2 \ , \ \mathcal{G}^-_k \equiv \mathcal{G}_{-\lambda_k} \ , \ \mathcal{N}^S_{(0)} = \mathcal{G}^-_2 \cong \mathcal{N}' \ , \tag{51c}$$

$$\tilde{\mathcal{N}}^S = \mathcal{G}^+_1 \oplus \mathcal{G}^+_2 \ , \ \mathcal{G}^+_k \equiv \mathcal{G}_{\lambda_k} = \theta\mathcal{G}^-_k \ , \ \tilde{\mathcal{N}}^S_{(0)} = \mathcal{G}^+_2 \cong \tilde{\mathcal{N}}', \tag{51d}$$

$$\lambda_1(D) = 1/2 \ , \ \lambda_2 = 2\lambda_1 \ , \ \dim \mathcal{G}^\pm_1 = 4N \ , \ \dim \mathcal{G}^\pm_2 = 4 \ ,$$

where the primed objects are $su(2,2)$ subalgebras. The Cartan subalgebra $\mathcal{H}^S \subset \mathcal{G}^S_{(0)}$ is chosen as follows:

$$\mathcal{H}^S = \mathcal{H} \oplus u(1) \oplus \mathcal{H}_N , \qquad (52)$$

where $\mathcal{H}$ is the Cartan subalgebra of $su(2,2)$, $\mathcal{H}_N$ is the Cartan subalgebra of $su(N)$.

We express the generators of $U_q(\mathcal{G}^S)$ in terms of those of $U_q(\mathcal{G}^S)$. For $U_q(su(2,2))$ we use formulae (28) - (31), and for $U_q(su(N))$ formulae (11). For the latter we note that $\{\pm i\alpha_{jk} \mid 5 \leq j \leq k \leq N+3\}$ form the root system of $su(N)$. For the generator of the $u(1)$ subalgebra in $\mathcal{G}^S_{(0)}$, $\mathcal{M}^S_{(0)}$ and $\mathcal{H}^S$ we have:

$$e_N = \sum_{k=1}^{4} kH_k + \frac{4}{N}\sum_{k=5}^{N+3}(k-4-N)H_k . \qquad (53)$$

Note that e_4 coincides with Z_4 described above. Next we have to express the $8N$ generators of $\mathcal{G}^S_{(1)}$. Let us denote the generators of $\tilde{\mathcal{N}}^S_{(1)} = \mathcal{G}^+_1$ by $P^\pm_{ak}$, and of $\mathcal{N}^S_{(1)} = \mathcal{G}^-_1$ by $K^\pm_{ak}$. Then we have

$$P^+_{ak} = iX^+_{a,k+4} - X^+_{a+2,k+4},\ P^-_{ak} = X^+_{a,k+4} - iX^+_{a+2,k+4},\ a = 1,2,\ k = 1,\ldots,N \qquad (54a)$$

$$K^+_{ak} = iX^-_{a+2,k+4} - X^-_{a,k+4},\ K^-_{ak} = X^-_{a+2,k+4} - iX^-_{a,k+4},\ a = 1,2,\ k = 1,\ldots,N \qquad (54b)$$

The commutation and Hopf algebra relations of $U_q(su(2,2/N))$ can be explicitly written now using formulae (28) - (31), (11), (53), (54), (50), (1), (47), (5). These formulae are omitted here for the lack of space.

Acknowledgments

The author would like to thank Professor Abdus Salam for hospitality and financial support at the ICTP. He would like to thank L. Castellani for noticing a misprint in [9]. This work was partially supported by the Bulgarian National Foundation for Science, Grant $\Phi - 11$.

References

[1] L.D. Faddeev, N. Yu. Reshetikhin and L.A. Takhtajan, LOMI Leningrad preprint E-14-87 (1987) & Leningrad Math. J. **1** (1990) 193-225.

[2] Yu.I. Manin, Ann. Inst. Fourier **37** (1987) 191-205; Montreal University preprint, CRM-1561 (1988).

[3] S.L. Woronowicz, Comm.Math.Phys. **111** (1987) 613; Lett.Math.Phys. **21** (1991) 35.

[4] M. Chaichian, D. Ellinas and P. Kulsih, Phys.Rev.Lett. **65** (1990) 980; T. Curtright and C. Zachos, Phys.Lett. **243B** (1990) 237; P.P. Kulish and E.V. Damashinsky, J.Phys.A: **23** (1990) L415.

[5] A. Chakrabarti, J. Math. Phys. **32** (1991) and these Proceedings; L.L. Vaksman and L.I. Korogodsky, Funkts. Anal. Prilozh. **25** (1991) No. 1, 60-62.

[6] P. Podles and S.L. Woronowicz, Comm. Math. Phys. **130** (1990) 381-431; U. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura, Zeit. f. Physik **C48** (1990) 159-166.

[7] W.B. Schmidke, J. Wess and B. Zumino, preprint MPI-PAE/PTh 15/91 (1991).

[8] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, J. Math. Phys. **31** (1990) 2548; J. Math. Phys. **32**, 1155 & 1159 (1991).

[9] V.K. Dobrev, Goettingen Univ. preprint, (July 1991).

[10] N. Bourbaki, *Groupes at algèbres de Lie, Chapitres 4,5 et 6*, (Hermann, Paris, 1968).

[11] V.G. Drinfeld, Soviet.Math.Dokl. **32** (1985) 2548; Proceedings ICM (1986) 798.

[12] M. Jimbo, Lett. Math. Phys. **10** (1985) 63-69; Lett. Math. Phys. **11** (1986) 247-252.

[13] V.G. Kac, Adv. Math. **26** (1977) 8-96.

[14] V.N. Tolstoy, Quantum Groups Workshop (Clausthal, 1989) Proceedings, Eds. H.D. Doebner and J.D. Hennig, Lect. Notes in Physics, Vol. 370 (Springer-Verlag 1990) p. 118.

[15] E. Abe, *Hopf Agebras* (Cambridge Univ. Press, 1980).

[16] C. Gómez and G. Sierra, Phys. Lett. **255B** (1991) 51-60.

[17] V.K. Dobrev, Proceedings of the Quantum Groups Workshop, (Clausthal, 1989) Eds. H.-D. Doebner and J.-D. Hennig, Lecture Notes in Physics, Vol. 370 (Springer-Verlag, Berlin, 1990) p. 107; ICTP Trieste internal report IC/89/142 (June 1989) & Proceedings of the International Group Theory Conference (St. Andrews, 1989) Eds. C.M. Campbell and E.F. Robertson, Vol. 1, London Math. Soc. Lect. Note Ser. 159 (1991) p. 87.

[18] J. Lukierski and A. Nowicki, Wroclaw University preprint ITP UWr 787/91 (October 1991); J. Lukierski, A. Nowicki and H. Ruegg, these Proceedings.

[19] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. **264B** (1991) 331-338.

[20] M. Chaichian and P.P. Kulish, Phys. Lett. **234B** (1990) 72-80.

[21] M. Scheunert, Bonn University preprint BONN-HE-91-10 (May 1991) and these Proceedings.

[22] V.G. Kac, private communication (May 1991).

[23] D.A. Leites and V. Serganova, private communication (May 1991).

[24] S.M. Khoroshkin and V.N. Tolstoy, these Proceedings.