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DYNAMICAL SYSTEMS OF PROPER CHARACTERISTIC 0

Khader H. Ahmad and Adnan Hamoui

P.8 – Proof – line 8

the region of attraction of the singleton (x). As y is an arbitrary element of $K(x)$, we

should read:

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P.9 – Proposition 4.2 – line 2

compact phase space. Then is of proper characteristic $0^+[0^-]$ whenever is

should read:

compact phase space. Then (X, π) is of proper characteristic $0^+[0^-]$ whenever it is

P.10 – Corollary 4.2 – line 3

(i) (X, π) is of proper characteristic 0^-,

should read:

(i) (X, π) is of proper characteristic 0^+,

P.10 – 2nd & 3rd last lines from bottom

Since x is arbitrary, we conclude that (M_1, π_{M_1}) is of proper characteristic 0^+. Since $x \in D^-_{M_1}(x)$ and $L^-_{M_1}(x) = L^-(x) \cap M_1 = \emptyset$, it follows that $D^-_{M_1}(x) \neq L^-_{M_1}(x)$. Thus (M_1, π_{M_1})

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P.11 – line 3

steps as above, we conclude that $D^+_{M_3}(x) \subseteq L^+_{M_3}(x)$. Hence (M_3, π_{M_3}) is of proper

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P.11– line 6

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September 1991

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DYNAMICAL SYSTEMS OF PROPER CHARACTERISTIC 0

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ABSTRACT

Flows with orbits of proper characteristic 0 exhibit recurrent behaviour, a feature of basic importance in the description of their dynamics. Here, we analyze flows with such orbits relating them with recurrent flows and with flows that exhibit orbital, Poisson or Lagrange stability.

MIRAMARE – TRIESTE
July 1991

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1. Introduction. Continuous flows of characteristics 0^+, 0^- and $0^\pm$ were introduced by Shair Ahmad [1]. Later, Knight [2, 3] studied a related class of flows called dynamical systems of characteristic 0. For these flows the [positive, negative] prolongation set of each [positive, negative] orbit is the same as the closure of the [positive, negative] orbit. The purpose of this paper is to study a class of flows for which the prolongation sets are the same as the limit sets of the orbits, yielding a much more intricate dynamical behaviour: that of nonwandering orbits. These flows will be called dynamical systems of proper characteristics 0^+, 0^-, $0^\pm$ and 0. An analysis of the various relations among these flows and their relations with other flows is given. Also, we characterize these dynamical systems in terms of other known classes of flows that they, or their minimal sets, exhibit some kind of stability: orbital, Poisson or Lagrange.

2. Notations and Definitions. In the usual notations, a triplet $(X, \mathbb{R}, \pi)$ consisting of a metric space X with metric ρ, the set of real numbers $\mathbb{R}$ and a continuous mapping π from the product space $X \times \mathbb{R}$ into X is called a **dynamical system** or a **(continuous) flow** whenever $\pi(x, 0) = x$ and $\pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2)$ for each $x \in X$ and $t_1, t_2 \in \mathbb{R}$. For brevity, $\pi(x, t)$ is denoted by xt and $(X, \mathbb{R}, \pi)$ is written as (X, π).

In this paper we shall adopt the following notations and definitions given in [4], [5] and [6], recalling only the definitions of the 'positively' denoted notions whenever the corresponding 'negatively' denoted notions are similarly defined.

Let $x \in X$. Then $C(x) \overset{\text{def}}{=} x\,\mathbb{R}$ and $C^+(x) \overset{\text{def}}{=} x\,\mathbb{R}^+$ are the **trajectory** (or **orbit**) **of** x and the **positive semi-trajectory of** x, respectively. If $C(x) = x$, the point x is said to be **critical**. The closures of $C(x)$ and $C^+(x)$ will be denoted by $K(x)$ and $K^+(x)$, respectively.

A set $M \subseteq X$ is **invariant** if $C(M) = M$, and **positively invariant** if $C^+(M) = M$. A nonempty closed invariant set having no proper subset having these three properties is said to be **minimal**. The **positive limit set** of $x \in X$ is $L^+(x) \overset{\text{def}}{=} \bigcap_{t \in \mathbb{R}^+} K^+(xt)$ and **the limit set of x** is $L(x) \overset{\text{def}}{=} L^-(x) \cup L^+(x)$. The **positive prolongation of** $x \in X$ is $D^+(x) \overset{\text{def}}{=} \bigcap_{M \in \eta(x)} K^+(M)$, $\eta(x)$ being the neighbourhood filter of x. The **positive prolongational limit set of x** is $J^+(x) \overset{\text{def}}{=} \bigcap_{t \in \mathbb{R}^+} D^+(xt)$ and the **prolongational limit set of x** is $J(x) \overset{\text{def}}{=} J^+(x) \cup J^-(x)$.

The point $x \in X$ is said to be **[positively] Lagrange stable** if $[K^+(x)]K(x)$ is compact. Also, x is said to be **positively Poisson stable** whenever $x \in L^+(x)$. It is said to be **Poisson stable** whenever $x \in L^+(x) \cap L^-(x)$. The point $x \in X$ is called **recurrent** if for each $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $C(x) \subseteq S(x[t - T, t + T], \varepsilon)$ for all $t \in \mathbb{R}$. The point x is said to be **non-wandering** if $x \in J^+(x)$.

A subset $M \subseteq X$ is said to be **positively stable** if every neighbourhood U of M contains a neighbourhood V of M such that $V\mathbb{R}^+ \subseteq U$. If $\mathbb{R}^+$ is replaced by $\mathbb{R}$ in the latter inclusion, then M is called **stable**. A flow (X, π) will be called **positively orbitally stable** whenever $K^+(x)$ is positively stable for each $x \in X$. The flow is said to be **(bilaterally) orbitally stable** whenever it is positively and negatively orbitally stable.

The set $A^+_\omega(M) \overset{\text{def}}{=} \{x \in X : L^+(x) \cap M \neq \emptyset\}$ is called **the region of positive weak attraction of the set** $M \subseteq X$. If $A^+_\omega(M)$ is a neighbourhood of M, then M is called a **positive weak attractor**.

Following [1] and [2], a dynamical system (X, π) is said to be **of characteristic 0^+ or 0** if for each $x \in X$, $D^+(x) = K^+(x)$ or $D(x) = K(x)$, respectively. The system is **of characteristic $0^\pm$** if it is both of characteristic 0^+ and 0^-.

In this paper, a point $x \in X$ will be called **a point of proper characteristic** 0^+ or 0 if $D^+(x) = L^+(x)$ or $D(x) = L(x)$, respectively. It will be called a **point of proper**

characteristic $0^\pm$ if it is both of proper characteristic 0^+ and 0^-. It is clear that if a point is of proper characteristic $0^\pm$, then it is of proper characteristic 0.

A dynamical system (X, π) is said to have a **property pointwise** if every point in the phase space X possesses the corresponding property.

Each of the classes of flows of proper characteristic 0^+, 0^-, $0^\pm$ and 0 is nonempty. To verify this fact consider the planar flow defined by the differential equations

$$\frac{dx}{dt} = y\,, \quad \frac{dy}{dt} = -x.$$

The trajectories consist of the origin, which is a rest point, and of circles with centre at 0. Since $D^+(x, y) = D^-(x, y) = D(x, y) = L^+(x, y) = L^-(x, y) = L(x, y) = C(x, y)$ for each $(x, y) \in \mathbb{R}^2$, the flow is of proper characteristic 0^+, 0^-, $0^\pm$ and 0.

Flows of proper characteristic 0^+, 0^-, $0^\pm$ and 0 are not necessarily periodic as in the above example. To see this consider the flow defined on a 2-torus X by the system of differential equations

$$\frac{d\theta}{dt} = \sqrt{2}\,, \quad \frac{d\varphi}{dt} = 1,$$

where θ and φ denote respectively the latitude and longitude of an arbitrary point $x \in X$. This flow is known as an irrational flow on the torus [11]. Each of its orbits is dense and so $L^+(x) = L^-(x) = X$ for each $x \in X$. Since $L^+(x) \subseteq D^+(x)$ and $L^-(x) \subseteq D^-(x)$ for each $x \in X$, it follows that $D^+(x) = L^+(x)$ and $D^-(x) = L^-(x)$ for every $x \in X$. Consequently, this non-periodic flow is of proper characteristic 0^+ and 0^-; and hence of proper characteristics $0^\pm$ and 0 as well.

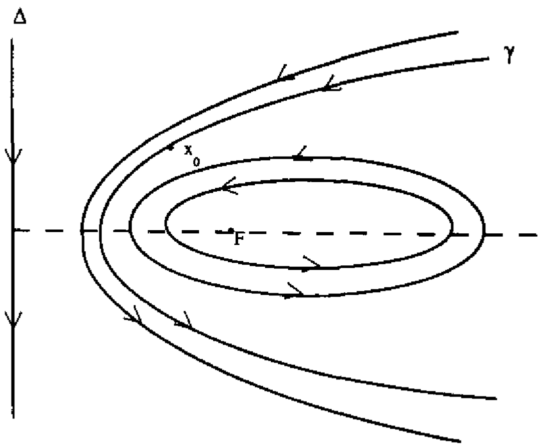
3. Points [flows] of Proper Characteristic [0^+, 0^-, $0^\pm$] 0. To begin with, we introduce the following proposition, the proof of which is straightforward and is thus omitted.

Proposition 3.1. If a point [flow] is of proper characteristic [0^+, 0^-, $0^\pm$] 0, then it is of characteristic [0^+, 0^-, $0^\pm$] 0.

The converse of this proposition does not hold as it is shown in the next two examples.

Example 3.1. Consider the flow on the real line defined by the differential equation $\frac{dx}{dt} = -x$. Then $x = x_0 e^{-t}$, x_0 being a real constant. If we take $x_0 > 0$, then $D^+(x_0) = K^+(x_0) = [0, x_0]$, $L^+(x_0) = \{0\}$ and $L^-(x_0) = \emptyset$. Hence the point x_0 is of characteristic 0^+ but not of proper characteristic 0^+.

Example 3.2. Consider the flow in $\mathbb{R}^2$ depicted schematically in the following figure. In this example, take the open trajectory γ and a point x_0 on it. Since in this case $L(x_0) = \emptyset$ and $D^+(x_0) = D^-(x_0) = K(x_0) = \gamma$, the point x_0 is of characteristic 0, but not of proper characteristic 0 (or proper characteristic 0^+, 0^-).



The following proposition is rather obvious.

Proposition 3.2. If a point [flow] is of proper characteristic [0^+, 0^-] $0^\pm$, then it is [positively, negatively] Poisson stable.

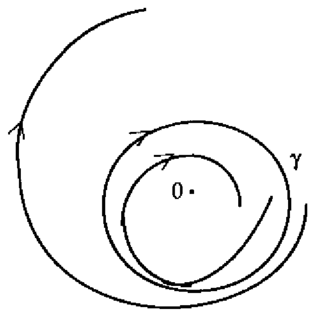
Corollary 3.1. If a point [flow] is of proper characteristic 0^+ or 0^-, then it is nonwandering.

The following example shows that the converse of Proposition 3.2 does not hold.

Example 3.3. Consider the flow in $\mathbb{R}^2$ defined by the differential equations:

$$\frac{dr}{dt} = r(r-1),\ \frac{d\theta}{dt} = -1\ ,$$

whose trajectories are depicted in the following figure. Let $x \in \gamma$. Then x



is Poisson stable. To show that x is not of proper characteristic 0^+, let $\{x_n\}$ be a sequence of points in the interior of the disc bounded by γ such that $x_n \to x$. For each $n \in \mathbb{Z}^+$ choose $t_n \in \mathbb{R}^+$ such that $x_n t_n$ is the closed disc d' with centre in 0 and radius $\frac{1}{2}$. Since d' is compact, it contains a convergent subsequence $\{x_{n_k} t_{n_k}\}$. Let $x_{n_k} t_{n_k} \to y \in d'$. Then, since $\{t_{n_k}\} \subseteq \mathbb{R}^+$ and $x_{n_k} \to x$, we have $y \in D^+(x)$. As $y \notin \gamma = L^+(x)$, we have $D^+(x) \neq L^+(x)$. Thus x is not a of proper characteristic 0^+. Similarly, $D^-(x) \neq L^-(x)$. Hence x is neither of proper characteristic 0^+ nor 0^-.

Example 3.3 shows that a periodic point needs not to be of proper characteristic 0^+, 0^-, $0^\pm$ or 0. However, it can be easily shown that if a point $x \in X$ is periodic and of characteristic [0^+, 0^-, $0^\pm$] 0, then it is of proper characteristic [0^+, 0^-, $0^\pm$] 0.

Moreover, the following proposition holds.

Proposition 3.3. If a point [flow] is [positively, negatively] Poisson stable and of characteristic [0^+, 0^-] 0, then it is of proper characteristic [0^+, 0^-] 0.

We are now in the position of proving the following propositions:

Proposition 3.4. **Let (X, π) be a Lagrange stable flow. Then (X, π) is of proper characteristic 0^+ if and only if it is of proper characteristic 0^-.**

Proof. Let (X, π) be of proper characteristic 0^+. Then, by Corollary 3.1, (X, π) is nonwandering and hence $D^+(x) = D^-(x)$ for each $x \in X$ [7]. Since $L^-(x) \neq \emptyset$ for all $x \in X$, it follows from Proposition 3.1 and from Lemma 4.5 of [8] that $L^-(x) = L^+(x)$ for each $x \in X$. Consequently, we have $L^-(x) = L^+(x) = D^+(x) = D^-(x)$ for all $x \in X$. Thus (X, π) is of proper characteristic 0^-. The converse can be proved quite similarly.

Proposition 3.5. Consider the following sets:

$$\mathcal{L}^+ = \{(x, y) \in X \times X \ : \ y \in L^+(x)\} ,$$

$$\mathcal{L}^- = \{(x, y) \in X \times X \ : \ y \in L^-(x)\} ,$$

$$\mathcal{L}^\pm = \{(x, y) \in X \times X \ : \ y \in L^+(x) \text{ and } y \in L^-(x)\} ,$$

$$\mathcal{L} \ = \{(x, y) \in X \times X \ : \ y \in L(x)\} .$$

Then (X, π) is minimal and of proper characteristic 0^+, 0^-, $0^\pm$ or 0 if and only if $\mathcal{L}^+$, $\mathcal{L}^-$, $\mathcal{L}^\pm$ or $\mathcal{L}$ equals $X \times X$, respectively.

Proof. We shall carry out the proof for the first case only as the second and fourth cases can be proven quite similarly; also, the third one is implied by the first two. Let $\mathcal{L}^+ = X \times X$. In this case, if $x \in X$ is a fixed element, then $y \in L^+(x)$ for all $y \in X$. Hence $X = L^+(x)$. To prove that X is minimal assume that Y is a nonempty closed invariant subset of X. Then if $x \in Y$, we have $X = L^+(x) \subseteq K(x) \subseteq Y \subseteq X$. Hence $Y = X$ and X is minimal. Also, since $X = L^+(x) \subseteq D^+(x)$ for each $x \in X$, it follows that $D^+(x) = L^+(x)$ for every $x \in X$. Thus (X, π) is of proper characteristic 0^+.

Conversely, let X be minimal and of proper characteristic 0^+. Then $L^+(x)$ is a nonempty closed invariant subset of the minimal set X and hence $L^+(x) = X$ for each $x \in X$. This clearly implies that $\mathcal{L}^+ = X \times X$.

The following proposition, which was proved by Knight [10], provides a characterization of flows of proper characteristic $[0^+, 0^-]\ 0^\pm$.

Proposition 3.6. **For a flow (X, π) to be of proper characteristic $[0^+, 0^-]\ 0^\pm$, it is necessary and sufficient that (X, π) be nonwandering and of characteristic $[0^+, 0^-]\ 0^\pm$.**

We now proceed to show that invariant subspaces inherit the property of being of proper characteristic $[0^+, 0^-, 0^\pm]\ 0$ from the whole space.

Proposition 3.7. **Let (X, π) be a flow of proper characteristic $[0^+, 0^-, 0^\pm]\ 0$. Let M be an invariant subspace of X and π_M be the restriction of π to $M \times R$. Then (M, π_M) is of proper characteristic $[0^+, 0^-, 0^\pm]\ 0$.**

Proof. We carry out the proof for flows of proper characteristic 0. A similar argument applies to flows of proper characteristics 0^+, 0^- and $0^\pm$.

Let $x \in M$ and y be an element of M such that $y \in D_M(x)$, where $D_M(x)$ denotes the prolongation of x relative to M. Then $y \in D^+_M(x)$ or $y \in D^-_M(x)$. If $y \in D^+_M(x)$, then there is a sequence $\{x_n\}$ in M and a sequence $\{t_n\}$ in R^+ such that $\rho_M(x_n, x) \to 0$ and $\rho_M(\pi_M(x_n,t_n), y) \to 0$ where $\rho_M = \rho|_{M \times M}$. Since x_n, x and $y \in M$, and since the invariance of M implies that $\pi_M(x_n, t_n) \in M$, we conclude that $\rho(x_n, x) = \rho_M(x_n, x) \to 0$ and $\rho(x_n t_n, y) = \rho_M(\pi_M(x_n, t_n), y) \to 0$. Hence, $y \in D^+(x)$. It follows that $y \in D^+(x) \cap M$. But if $y \in D^-_M(x)$, then it can be shown similarly that $y \in D^-(x) \cap M$. Thus $y \in (D^+(x) \cap M) \cup (D^-(x) \cap M) = D(x) \cap M$. Since y is an arbitrary element of $D_M(x)$, it follows that $D_M(x) \subseteq D(x) \cap M$. Analogously we conclude that $L_M(x) = L(x) \cap M$, where $L_M(x)$ denotes the limit set of $x \in M$ relative to M. Consequently, $D_M(x) \subseteq D(x) \cap M = L(x) \cap M = L_M(x)$. Thus (M, π_M) is of proper characteristic 0.

Proposition 3.8. **If (X, π) is a flow of proper characteristic 0, then $J(x)$ is minimal for every $x \in X$.**

Proof. Let $x \in X$ and let y be any element of the nonempty set $J(x)$. Then $x \in J(y)$ and $J(x) \subseteq D(x) = L(x) \subseteq K(x) \subset J(y) \subseteq D(y) = L(y) \subseteq K(y) \subseteq J(x)$. Thus $K(y) = J(x)$ and hence $J(x)$ is minimal.

Corollary 3.2. **Let (X, π) be a flow of proper characteristic 0 with compact phase space. Then (X, π) is recurrent.**

Proof. By Corollary 3.1 we conclude that (X, π) is nonwandering, therefore $x \in J(x)$ for each $x \in X$. As $J(x)$ is closed, it is compact. Thus the trajectory of each $x \in X$ belongs to the compact minimal set $J(x)$ and hence (X, π) is recurrent.

Corollary 3.2. implies that if a flow with compact phase space is of proper characteristic $0^{\pm}$, then it is recurrent. The following proposition extends this result to more general spaces, namely locally compact metric spaces.

Proposition 3.9. Let (X, π) be a flow of proper characteristic $0^{\pm}$ with locally compact phase space. Then (X, π) is recurrent.

Proof. Since every flow of proper characteristic $0^{\pm}$ is of proper characteristic 0, it follows from Proposition 3.8 that $J(x)$ is minimal for each $x \in X$. As (X, π) is nonwandering according to Corollary 3.1, it follows that $x \in J(x)$ for each $x \in X$. Thus $K(x) = J(x)$ and hence $K(x)$ is minimal for every $x \in X$. Now, let x be any fixed element of X and let $y \in K(x)$. Then $L^+(y) \subsetneq K(x)$. The set $L^+(y)$ is not empty due to the fact that $y \in D^+(y) = L^+(y)$. Since in addition $L^+(y)$ is closed invariant set and $K(x)$ is minimal, we conclude that $L^+(y) = K(x)$. Hence $x \in L^+(y)$. This implies that $y \in A^+_\omega(\{x\})$, i.e. y lies in the region of attraction of the singleton $\{x\}$. As y is an arbitrary element of $K(x)$, we conclude that $K(x) \subsetneq A^+_\omega(\{x\})$. Bhatia and Hajek (see 4.6 of [9]) showed that for such a point x, $K(x)$ is not only minimal, but also compact. Thus the trajectory of any $x \in X$ lies in the compact minimal set $K(x)$ and hence (X, π) is recurrent.

Proposition 3.10. If (X, π) is a flow of characteristic 0, and if, for some $x \in X$, either $L^+(x)$ or $L^-(x)$ is nonempty and compact, then x is of proper characteristic $0^{\pm}$.

Proof. Suppose that $L^+(x)$ is nonempty and compact for some $x \in X$. Then it can be proved that $D^+(x) = L^+(x)$. Since $x \in D^+(x) = L^+(x)$ and since $L^+(x)$ is a compact invariant set, it follows that x is Lagrange stable. Hence $L^-(x)$ is nonempty and compact. Then it can be proved as above that $D^-(x) = L^-(x)$. Thus x is of proper characteristic $0^{\pm}$.

A similar argument applies in the case where $L^-(x)$ is nonempty and compact.

4. *Flows with Locally Compact Phase Space.*

Proposition 4.1. Let (X, π) be a flow with locally compact phase space. Then (X, π) is positively [negatively] orbitally stable whenever it is of proper characteristic $0^+[0^-]$.

Proof. We limit our demonstration to the case where (X, π) is of proper characteristic 0^+. To show that (X, π) is positively orbitally stable, we need only to show that $D^+(K^+(x)) = K^+(x)$ for every $x \in X$. Given $x \in X$ and any $y \in D^+(K^+(x))$, there is $z \in K^+(x) = C^+(x) \cup L^+(x)$ such that $y \in D^+(z)$. If $z \in C^+(x)$, there is $t \in \mathbb{R}^+$ such that $z = xt$. Hence $y \in D^+(xt) \subsetneq D^+(x) = L^+(x) \subsetneq K^+(x)$. But if $z \in L^+(x)$, then $y \in D^+(z) \subsetneq D^+(L^+(x)) = \bigcup_{u \in L^+(x)} D^+(u) = \bigcup_{u \in L^+(x)} L^+(u) \subsetneq L^+(x) \subsetneq K^+(x)$. As $y \in D^+(K^+(x))$ is arbitrary, we have $D^+(K^+(x)) \subsetneq K^+(x)$ for each $x \in X$; but clearly $K^+(x) \subsetneq D^+(K^+(x))$. Hence the result.

The following is a partial converse of the previous proposition.

Proposition 4.2. Let (X, π) be a Lagrange stable flow with locally compact phase space. Then is of proper characteristic $0^+[0^-]$ whenever is positively [negatively] orbitally stable.

Proof. We shall carry out the proof assuming that (X, π) is positively orbitally stable. A similar argument applies when the flow is negatively orbitally stable. Let x be any element in X. Since $x \in K^+(x)$, we have $D^+(x) \subsetneq D^+(K^+(x)) = K^+(x)$. Hence $D^+(x) = K^+(x)$ for any $x \in X$. This proves that (X, π) is of characteristic 0^+. Now, since the flow is Lagrange stable, we have $L^-(x) \neq \emptyset$. This leads to [8]: $L^-(x) = L^+(x) = K(x)$. Hence $x \in L^+(x) \subsetneq J^+(x)$. Thus (X, π) is nonwandering. Using Proposition 3.6 we conclude that (X, π) is of proper characteristic 0^+.

Corollary 4.1. Let (X, π) be a Lagrange stable flow with locally compact phase space. Then (X, π) is of proper characteristic $0^+[0^-]$ if and only if (X, π) is positively [negatively] orbitally stable.

Using Corollary 4.1 and Propositions 3.4 and 3.10, we obtain the following result.

Corollary 4.2. Let (X, π) be a flow with compact phase space. Then the following statements are pairwise equivalent:

(i) (X, π) is of proper characteristic 0^-,
(ii) (X, π) is of proper characteristic 0^-,
(iii) (X, π) is of proper characteristic $0^\pm$,
(iv) (X, π) is of proper characteristic 0,
(v) (X, π) is of positively orbitally stable,
(vi) (X, π) is negatively orbitally stable,
(vii) (X, π) is (bilaterally) orbitally stable.

Proposition 4.3. Let (X, π) be a flow of proper characteristic 0, and let

$$M_1 = \{x \in X : L^+(x) \neq \emptyset \text{ and } L^-(x) = \emptyset\},$$

$$M_2 = \{x \in X : L^+(x) = \emptyset \text{ and } L^-(x) \neq \emptyset\},$$

$$M_3 = \{x \in X : L^+(x) \neq \emptyset \text{ and } L^-(x) \neq \emptyset\} .$$

Then $\{M_1, M_2, M_3\}$ is a partition of X. Furthermore, we have the following statements:

(i) The restriction of (X, π) to M_1 is of proper characteristics 0 and 0^+, but not of proper characteristic 0^-.

(ii) The restriction of (X, π) to M_2 is of proper characteristics 0 and 0^-, but not of proper characteristic 0^+.

(iii) The restriction of (X, π) to M_3 is of proper characteristics 0 and $0^\pm$.

Proof. The proof that $\{M_1, M_2, M_3\}$ is a partition of X is straightforward. According to Proposition 3.7 each one of the three restrictions is of proper characteristic 0.

Let $\pi_1 = \pi|_{M_1} \times \mathbb{R}$ and let $x \in M_1$. Then $L^+(x) \neq \emptyset$. If $y \in L^+(x)$, then $y \in J^+(x)$ and $x \in J^-(y) \subseteq D(y) = L(y) \subseteq L^+(x)$. Thus $L(x) \subseteq L^+(x)$ and hence $L(x) = L^+(x)$. Consequently, we have $D^+_{M_1}(x) \subseteq D^+(x) \cap M_1 \subseteq D(x) \cap M_1 = L(x) \cap M_1 = L^+(x) \cap M_1 = L^+_{M_1}(x)$. Since x is arbitrary, we conclude that (M_1, π_{M_1}) is of proper characteristic 0^+. Since $x \in D^-_{M_1}(x)$ and $L^-_{M_1}(x) = L^-(x) \cap M_1 = \emptyset$, it follows that $D^-_{M_1}(x) \neq L^-_{M_1}(x)$. Thus (M_1, π_{M_1}) is not of proper characteristic 0^-.

The proof of (ii) is quite similar.

Finally, let $\pi_3 = \pi|_{M_3} \times \mathbb{R}$ and let $x \in M_3$. Since $L^+(x) \neq \emptyset$, then, following the same steps as above, we conclude that $D^+_{M_3}(x) \subseteq L^+_{M_3}(x)$. Hence (M_3, π_{M_3}) is of proper characteristic 0^+. Also, $L^-(x) \neq \emptyset$ implies that $L(x) = L^-(x)$. Consequently we have $D^-_{M_3}(x) \subseteq D^-(x) \cap M_3 \subseteq D(x) \cap M_3 = L(x) \cap M_3 = L^-(x) \cap M_3 = L^-_{M_3}(x)$. Since x is arbitrary, we conclude that (M_3, π_{M_3}) is of proper characteristic 0^-. This completes the proof.

Corollary 4.3. Let (X, π) be a flow of proper characteristic 0. Then (X, π) is of proper characteristic $0^+[0^-]$ if and only if $M_2 = \emptyset$ $(M_1 = \emptyset)$. Also, (X, π) is of proper characteristic $0^\pm$ if and only if $X = M_3$.

The following proposition gives a characterization of flows having proper characteristic 0, and it is analogous to Theorem 7 of [3].

Proposition 4.4. Let (X, π) be a flow with locally compact phase space. Then for (X, π) to be of proper characteristic 0 it is necessary and sufficient that:

(i) Each compact minimal set is positively and negatively stable.

(ii) For each x not in a compact minimal set, $J(x) = L(x)$ and x is either positively or negatively Poisson stable.

Proof. The necessity of the condition (i) can be established using Proposition 3.1, while that of the condition (ii) follows directly from the equality $D(x) = L(x)$. To prove the sufficiency, assume that conditions (i) and (ii) are satisfied and let $x \in X$. If there is a compact minimal set H such that $x \in H$, then (see [5]: III, 3.4, p. 37) we have $L(x) = H$. Thus $x \in L(x)$ and hence $D(x) \subseteq D(L(x))$. But then the condition (i) implies that $D(x) \subseteq L(x)$. On the other hand, if there is no compact minimal set that contains x, then according to (ii), we have $x \in L(x)$ and $J(x) = L(x)$. This clearly implies that $D(x) \subseteq L(x)$. Hence (X, π) is of proper characteristic 0. This completes the proof.

Corollary 4.4. Let (X, π) be of proper characteristic 0 with locally compact phase space. Then each compact minimal set in X has a neighbourhood of Lagrange stable points.

Proof. If H is a compact minimal set in X, then, since X is locally compact, there exists a neighbourhood U of H such that $\overline{U}$ is compact. As H is positively and negatively stable according to (i) of Proposition 4.4, there exists a neighbourhood V of M such that $V\mathbb{R} \subseteq U$. It follows that if x is a point of V, then $K(x) = \overline{x\mathbb{R}} \subseteq \overline{U}$. Thus K(x) is compact. Hence x is Lagrange stable.

Acknowledgments

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics. Thanks are due to the University of Kuwait for support.

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