IC/92/57

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

MIDISUPERSPACE-INDUCED CORRECTIONS TO THE WHEELER DE WITT EQUATION

Francisco D. Mazzitelli

INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

MIRAMARE-TRIESTE

IC/92/57

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

MIDISUPERSPACE–INDUCED CORRECTIONS TO THE WHEELER DE WITT EQUATION *

Francisco D. Mazzitelli
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

We consider the midisuperspace of four dimensional spherically symmetric metrics and the Kantowski–Sachs minisuperspace contained in it. We discuss the quantization of the midisuperspace using the fact that the dimensionally reduced Einstein Hilbert action becomes a scalar–tensor theory of gravity in two dimensions. We show that the covariant regularization procedure in the midisuperspace induces modifications into the minisuperspace Wheeler DeWitt equation.

MIRAMARE – TRIESTE
April 1992

* Essay submitted to the Gravity Research Foundation.

Proposed by DeWitt and Misner [1] more than twenty years ago, the minisuperspace approximation has been almost the only practical way of dealing with the difficulties of quantizing General Relativity.

Although it was clear from the very beginning that the minisuperspace approximation could be considered only as a crude model of quantum gravity, the question of the validity of the approximation has been addressed quantitatively only in the last years. Kuchar and Ryan [2] have investigated a hierarchy of minisuperspace models trying to find a criteria to decide when the quantum minisuperspace results are meaningful. Alternatively, Hu and collaborators [3], have developed a formalism to obtain an effective Wheeler DeWitt equation in the minisuperspace which takes into account the backreaction of the ignored modes. In any case, and because a complete theory of quantum gravity is still lacking, it seems that the only way to learn what are we missing when we use the minisuperspace approximation is to embed the minisuperspace into a larger one, analyze the enlarged theory and compare the results with the ones obtained in the original minisuperspace.

In the present work, we will follow this procedure. We will study the midisuperspace of spherically symmetric four dimensional metrics and the Kantowski-Sachs minisuperspace contained in it. We will see how the gauge fixing and covariant regularization procedures in the midisuperspace induce important modifications into the minisuperspace-Wheeler DeWitt equation.

Let us consider the spherically symmetric metrics

$$ds^2 = g_{ab}(x^0, x^1)dx^a dx^b + 2G\Psi(x^0, x^1)(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

where $x^0 = t, x^1 = r, \theta$ and φ are coordinates adapted to the spherical symmetry. The Newton constant G is inserted in order to make Ψ dimensionless. We will assume that the radial coordinate is compact.

The four dimensional Einstein-Hilbert action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{g^{(4)}}\left(R^{(4)} + \Lambda\right) \tag{2}$$

reads, after inserting the ansatz (1),

$$S = \frac{1}{2}\int d^2x\sqrt{g}\left(R\Psi + \frac{1}{G} + \Lambda\Psi + \frac{g^{ab}}{2\Psi}\Psi_{,a}\Psi_{,b}\right) \quad . \tag{3}$$

It can be explicitly checked that the Euler-Lagrange equations associated with the action (2) coincide with the Einstein equations for spherically symmetric metrics.[4]

We can interpret the dimensionally reduced action (3) as a scalar-tensor theory of gravity in two dimensions. The action can be further simplified by adopting the conformal gauge for the two dimensional metric g_{ab}, that is $g_{ab} = \hat{g}_{ab}\exp\lambda$. In this gauge we have

$$S = \int d^2x\sqrt{\hat{g}}\left(\frac{1}{2}G_{ij}(X)\hat{g}^{ab}\partial_a X^i\partial_b X^j + \frac{1}{2}\hat{R}D(X) + T(X)\right) \tag{4}$$

where

$$X^i = \begin{pmatrix}\lambda\\ \Psi\end{pmatrix}$$

$$G_{ij}(X) = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & \Psi^{-1}\end{pmatrix}$$

$$D(X) = \Psi$$

$$T(X) = \frac{1}{2}(\frac{1}{G} + \Lambda\Psi)\exp\lambda$$

We recognize that Eq. (4) corresponds to a two dimensional non linear sigma-model with target space metric $G_{ij}(X)$, dilaton field $D(X)$ and tachyon field $T(X)$.

As we have mentioned, the spherically symmetric midisuperspace in Eq. (1) contains as a particular case the Kantowski-Sachs minisuperspace

$$ds^2 = a^2(t)\left(-dt^2 + dr^2\right) + b^2(t)(d\theta^2 + \sin^2\theta d\varphi^2) \quad , \tag{5}$$

which describes a $S^1 \times S^2$ universe. The Wheeler DeWitt equation (WDWE) associated with this minisuperspace can be obtained from (4) assuming that the fields X^i are only functions of t and setting

$$\hat{g}_{ab} = \begin{pmatrix}-N^2(t) & 0\\ 0 & 1\end{pmatrix} \quad .$$

Here N is the lapse function. After standard manipulations we find,

$$\left[-\frac{1}{2}\nabla^2 + T(X)\right]\psi(X) = 0 \qquad , \tag{6}$$

where ∇^2 is the Laplacian of the supermetric G_{ij}. We have chosen a covariant factor ordering in minisuperspace.

Now comes the crucial point of this essay. Before assuming the spatial homogeneity of X^i, the quantum theory associated to the midisuperspace (1) is a two dimensional *quantum field theory.* The quantum fields are $\lambda(r,t)$ and $\Psi(r,t)$. As in any field theory, a regularization is needed, and in our case this regularization must be covariant with respect to the full metric $g_{ab} = \hat{g}_{ab}\exp\lambda$. As a consequence, the theory is complicated by the fact there is an additional λ-dependence in the path-integral measure $[d\lambda d\Psi]_{\hat{g}\exp\lambda}$. This is a very well known problem in $2d$ gravity coupled to conformally invariant matter. There, it was conjectured by David, Distler and Kawai [5] (DDK), that the λ-dependence of the measure can be taken into account as follows

$$\int [d\lambda]_{\hat{g}\exp\lambda} \exp(-S_L) = \int [d\lambda]_{\hat{g}} \exp(-\bar{S}_L)$$

where S_L is the Liouville action and $\bar{S}_L$ is an action of the same form but with different coupling constants. The new coefficients in $\bar{S}_L$ are fixed by requiring the quantum theory to be invariant under the transformation

$$\hat{g}_{ab} \to \hat{g}_{ab}\exp\tau(x) \qquad \lambda \to \lambda - \tau(x) \qquad . \tag{7}$$

This is an obvious requirement since the *full* metric $g_{ab} = \hat{g}_{ab}\exp\lambda$ is left unchanged and it is only g_{ab} that enters in both the original theory and the regulator.

In our case, the classical action is a non linear sigma-model in two dimensions. Power counting implies that, at the quantum level, the theory is renormalizable in a generalized sense. One must allow for a change in the form of the metric, the dilaton and the tachyon fields. A DDK-like argument then suggests [6] that the λ dependence of the measure can be taken into account by considering, instead of the classical action (4), an action $\bar{S}$ of the same form but with general couplings $\bar{G}_{ij}(X)$, $\bar{D}(X)$ and $\bar{T}(X)$. As before, the couplings must be such that the 'split' transformation Eq. (7) is an exact quantum symmetry. As the new measure $[d\lambda d\Psi]_{\hat{g}}$ is translationally invariant, transformation (7) implies Weyl invariance with respect to $\hat{g}_{ab}$. As a consequence, the beta functions of the couplings should vanish, that is [7]

$$\begin{aligned} \beta_{\bar{G}} &= 0 = \bar{R}_{ij} + 2\bar{\nabla}_i\bar{\nabla}_j\bar{D} \\ \beta_{\bar{D}} &= 0 = -c - \frac{1}{2}\bar{\nabla}^2\bar{D} + \bar{\nabla}\bar{D}.\bar{\nabla}\bar{D} \\ \beta_{\bar{T}} &= 0 = -\frac{1}{2}\bar{\nabla}^2\bar{T} + \bar{\nabla}\bar{T}.\bar{\nabla}\bar{D} - 2\bar{T} \end{aligned} \tag{8}$$

where we omited quadratic terms in the tachyon field. The constant c in the second equation is the Weyl anomaly coefficient, given by $c = \frac{1}{6}(26-d)$, where d is the dimension of the target space. In our case we have $c = 4$. We stress that the Ricci tensor and the covariant derivatives in Eqs. (8) are formed with the target space metric $\bar{G}_{ij}$.

In principle one can go ahead and quantize the theory based on the new action using the λ-independent measure $[d\lambda d\Psi]_{\hat{g}}$. The physical states can be obtained from the BRST cohomology. This is beyond the scope of this essay. We will simply implement the minisuperspace approximation in the 'effective' action $\bar{S}$, that is, after the λ dependence of the measure has been removed [8]. The physical states must then satisfy the modified WDWE

$$\left[-\frac{1}{2}\bar{\nabla}^2 + \bar{T}(X)\right]\psi(X) = 0 \tag{9}$$

where $\bar{\nabla}^2$ is the Laplacian of $\bar{G}_{ij}$. Both $\bar{G}_{ij}$ and $\bar{T}$ must satisfy Eqs. (8). This is our main result.

It is easy to find solutions to the equations $\beta_{\bar{G}} = 0$ and $\beta_{\bar{D}} = 0$ which coincide with the classical G_{ij} and D when $c = 0$. In fact, one can show that

$$\begin{aligned} \bar{G}_{ij}(X) &= \frac{1}{2}\begin{pmatrix} c/2 & 1 \\ 1 & \Psi^{-1} \end{pmatrix} \\ \bar{D}(X) &= \Psi + \frac{c}{2}\lambda \end{aligned} \tag{10}$$

satisfy both equations. With these $\bar{G}_{ij}$ and $\bar{D}$ one can solve the tachyon equation $\beta_{\bar{T}} = 0$ using the classical T as a boundary condition. We conclude that the minisuperspace WDWE has been modified in two ways. On the one hand, the Weyl anomaly $c \neq 0$

modifies the structure of the differential operator. As $\bar{G}_{\lambda\lambda} \neq 0$, the operator $\bar{\nabla}^2$ will contain a term with second derivatives with respect to Ψ. This term is absent in ∇^2. On the other hand, the new superpotential $\bar{T}$ will differ from the original one.

Summarizing, we arrived at the following picture. In the minisuperspace approximation the Wheeler DeWitt equation (6) is constructed with the classical supermetric G_{ij} and superpotential T. After taking into account that the midisuperspace quantum field theory must be covariantly regularized, the classical supermetric and superpotential must be replaced by the quantum couplings $\bar{G}_{ij}$ and $\bar{T}$. We expect this to be a general feature, not restricted to the Kantowski-Sachs example considered here. That is, similar modifications should take place in other minisuperspace examples.

For simplicity we considered a minisuperspace approximation to the modified action $\bar{S}$. A more complete treatment of the problem should include the backreaction of the neglected inhomogeneous modes on the minisuperspace variables [3]. We expect this backreaction to produce additional non local terms in the modified Wheeler DeWitt equation.

Acknowledgements: I would like to thank N. Mohammedi for his helpful comments and Prof. Abdus Salam, the International Atomic Energy Agency and UNESCO for financial support.

REFERENCES

1. B. DeWitt, Phys. Rev. 160, 1113 (1967); C.W. Misner, Phys. Rev. 186, 1319 (1969)
2. K. Kuchar and M.P. Ryan, Phys. Rev. D40, 3982 (1989)
3. B.L. Hu, Lectures delivered at the Seventh Latin American International Symposium on General Relativity and Gravitation, Mexico, 1990, and references therein.
4. P. Thomi, B. Isaak and P. Hajicek, Phys. Rev. D30, 1169 (1984)
5. F. David, Mod. Phys. Lett A3, 1651 (1988); J. Distler and H. Kawai, Nucl. Phys. B321, 509 (1989)
6. A.A. Tseytlin, Int. J. Mod. Phys. A5, 1833 (1990); J.G. Russo and A.A. Tseytlin, preprint SU-ITP 92-2 (1992); A. Cooper, L. Susskind and L. Thorlacius, Nucl.Phys. B363, 132 (1991)
7. C.G. Callan, D. Friedan, E. Martinec and M.J. Perry, Nucl. Phys. B262, 593 (1985)
8. A similar approximation in $2d$ gravity has been done in J. Polchinski, Nucl. Phys. B324, 123 (1989)