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ABSTRACT

We consider the "even–time" flow of the Manin–Radul supersymmetric KP hierarchy and show that it possesses bi–Hamiltonian structures by deriving two distinct Gelfand–Dikii brackets corresponding to two successive Hamiltonians of the system. A recursion relation involving them is also obtained. We observe that the first Hamiltonian structure defines a supersymmetric Lie algebra since it is a linear algebra among the super fields appearing in the Lax operator whereas the second Hamiltonian structure is a non–linear algebra and so it does not define a Lie algebra.

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I. Introduction

Apart from being interesting on their own, the integrable models in (1+1) dimensions [1-3] have found many applications in recent years [4-10]. These large class of models, also known as the generalized KdV hierarchy can be obtained from the Kadomtsev-Petviashvili (KP) equation [11], which is a non-linear differential equation in (2+1) dimensions, through a suitable reduction procedure [12]. It is well known that the KP equation is an integrable equation in the sense that it admits a bi-Hamiltonian structure [13,14]. The existence of the bi-Hamiltonian structure in turn implies the existence of a recursion relation among the conserved quantities associated with the non-linear equation. One then uses this relation to construct the infinite number of conserved quantities recursively and shows that these functionally independent conserved quantities are in involution with respect to either of the Hamiltonian structures [3,13]. The KP equation and a whole set of hierarchy associated with it can be represented in terms of a Lax pair. Not only the KP equation but also the whole hierarchy of equations share the same set of conserved quantities which render the KP hierarchy integrable.

One would like to have a supersymmetric version of the KP hierarchy since it is the supersymmetric theory which will ultimately be of physical interest. There exists in the literature different approaches to the supersymmetric generalization of the KP equation. Since $SL(2,R)$ is the underlying Lie algebra of the KP equation, one can take a Lie super algebraic approach to obtain a super KP equation. Recently, we have looked at the zero curvature condition of $OSp(2|1)$ supergroup, but the resulting non-linear equation is a fermionic KP equation which is not supersymmetric [15]. Also, there exists a supersymmetric generalization of Sato-type equation [16] which does not admit a Lax pair representation of the non-linear equation. Thus, the simplest way to supersymmetrize the KP hierarchy would be to construct a super Lax operator by replacing the ordinary derivative in the KP Lax operator by super covariant derivative. This is the Manin-Radul (MR) Lax operator [17] for the super KP hierarchy and by construction it is a fermionic pseudo-differential operator. Recently, super KP hierarchy is also studied starting from a bosonic super Lax operator [18].

Here, we study only the Lax equations corresponding to the "even-time"[17] flow of the MR super Lax operator. The supersymmetric generalized KdV hierarchy are the natural reductions of the "even-time" flow equations of the MR super KP hierarchy. Although, it is well known that the bosonic integrable models have nice local bi-Hamiltonian structure [19], this is not in general true for the supersymmetric integrable systems. For example, the KdV equation has a bi-Hamiltonian structure which are local but the N=1 sKdV equation has bi-Hamiltonian structure where the first one is non-local and the second one is local [20,21]. In general, it is quite nontrivial to show the existence of a bi-Hamiltonian structure associated with a supersymmetric integrable system. On the other hand, the existence of the bi-Hamiltonian structure is very useful in order to understand the geometry of the phase space and to prove the integrability of the system [22].

In this paper, we show that the "even-time" flows of the MR super KP hierarchy admit a bi-Hamiltonian structure following the variational method of Gelfand and Dikii [19]. In Sec. II, we review the essential features of the MR super KP hierarchy. In Sec. III, we show that the super KP hierarchy described by the Lax equation admit two distinct Hamiltonian structures with two successive Hamiltonians of the system. We also write a recursion relation by which one can construct the infinite number of conserved charges. Our conclusions are presented in Sec. IV.

II. Essentials of the Manin-Radul Super KP hierarchy :

The Manin-Radul super KP hierarchy [17] is described in terms of a Lax equation with the Lax pair (L, L^i_-) as

$$\frac{\partial L}{\partial t_i} = -[L^i_-, L] \qquad i = 1, 2, \ldots\ldots \tag{2.1}$$

where t_i are an infinite set of even and odd variables for even and odd i respectively. The supersymmetric fermionic Lax operator L is given by

$$L = D + \sum_{p=0}^{\infty} u_{p+1}\, D^{-p} \tag{2.2}$$

where the infinite set of superfields u_{p+1} are functions of the variables t_i. D is the super covariant derivative defined as $D = \partial_\theta + \theta\, \partial_x$ which satisfies the relation $D^2 = \partial_x$ and θ is the odd Grassmann variable with $\theta^2 = 0$. Formally, the inverse of D is given by $D^{-1} = \theta + \partial_\theta\, \partial_x^{-1}$. In (2.1), L^i_- simply means that it is the negative super differential part of the ith power of L. Also, note that the grading of the super fields u_p is given by $|u_p| = p \ (mod\ 2)$. One can define the even and odd time derivatives respectively as (for $i = 1, 2, \ldots.$)

$$\begin{aligned} D_{2i} &= \frac{\partial}{\partial t_{2i}} \\ D_{2i-1} &= \frac{\partial}{\partial t_{2i-1}} + \sum_{j=1}^{\infty} t_{2j-1} \frac{\partial}{\partial t_{2i+2j-2}} \end{aligned} \tag{2.3}$$

satisfying the Lie super algebra

$$\begin{aligned} [D_{2i}, D_{2j}] &= 0 \\ [D_{2i}, D_{2j-1}] &= 0 \\ [D_{2i-1}, D_{2j-1}] &= 2\, D_{2i+2j-2} \end{aligned} \tag{2.4}$$

The graded commutator is defined as $[A, B] = A\,B - (-)^{|A||B|}\, B\,A$. Also, note that because $[L^{2i}, L] = 0$ and $[L^{2i-1}, L] = 2L^{2i}$, the even and the odd time flow of the Lax equation (2.1) can be rewritten as

$$D_{2i} L = -[L^{2i}_-, L] = [L^{2i}_+, L] \tag{2.5}$$

and

$$D_{2i-1} L = -[L^{2i-1}_-, L] = [L^{2i-1}_+, L] - 2L^{2i} \tag{2.6}$$

We will consider only the part of the Lax equation corresponding to the even-time flow i.e. (2.5). Now in order to write down the Lax equation explicitly in terms of the superfields, we make use of the generalized Leibniz rule for the super covariant derivative given by,

$$D^k f = \sum_{j=0}^{\infty} \begin{bmatrix} k \\ j \end{bmatrix} (-)^{|f|(k+j)}\, f^{[j]}\, D^{k-j} \tag{2.7}$$

for k both positive and negative integers. Here, $f^{[j]}$ simply stands for $(D^j f)$. The super binomial coefficients $\begin{bmatrix} k \\ j \end{bmatrix}$ are defined, for $k \geq 0$, as

$$\begin{bmatrix} k \\ j \end{bmatrix} = \begin{cases} \binom{[k/2]}{[j/2]} & \text{for } k \geq j \text{ and } (k, j) \neq (0, 1) \bmod 2; \\ 0 & \text{otherwise} \end{cases} \tag{2.8a}$$

and similarly for $k < 0$,

$$\begin{bmatrix} k \\ j \end{bmatrix} = \begin{cases} \binom{[k/2]}{[j/2]} & \text{if } (k,j) \neq (0,1) \bmod 2; \\ 0, & \text{otherwise.} \end{cases} \tag{2.8b}$$

Here, $[k/2]$ denotes the integral part of $k/2$ and we define that

$$\binom{-n}{m} = (-)^m \binom{n+m-1}{m} \tag{2.9}$$

The Lax equation (2.1) implies that the superfields u_1 and u_2 satisfy the relation

$$\frac{\partial}{\partial t_i}(u_1^{[1]} + 2\,u_2) = 0 \tag{2.10}$$

for all $i = 1, 2, \ldots$. So, we can set $u_1^{[1]} + 2u_2 = 0$ where $u_1^{[1]} = (Du_1)$. Then it is straight forward to calculate L_+^2, L_+^4 and L_+^6 and are given as [15]

$$\begin{aligned} L_+^2 &= D^2 \\ L_+^4 &= D^4 + 2V_3 D + 2V_4 \\ L_+^6 &= D^6 + 3V_3 D^3 + 3V_4 D^2 + V_5 D + V_6 \end{aligned} \tag{2.11}$$

where the functions V_3, V_4, V_5, V_6 are given as follows:

$$\begin{aligned} V_3 &= u_2^{[1]} \\ V_4 &= (u_3^{[1]} + 2u_4 - u_2^2) \\ V_5 &= 3(u_4^{[1]} - u_2 u_2^{[1]} + \frac{\partial u_2^{[1]}}{\partial x}) \\ V_6 &= 3(u_5^{[1]} + 2u_6 - u_2\frac{\partial u_2}{\partial x} + u_2 u_3^{[1]} + 2u_3 u_2^{[1]} + \frac{\partial u_3^{[1]}}{\partial x} + 2\frac{\partial u_4}{\partial x}) \end{aligned} \tag{2.12}$$

Defining $t_2 \equiv x$; $t_4 \equiv y$ and $t_6 \equiv t$, we can write down the Lax equation from (2.5) as

$$\frac{\partial u_{p+1}}{\partial t_2} = \frac{\partial u_{p+1}}{\partial x} \qquad p = 0, 1, 2, \ldots\ldots \tag{2.13}$$

which is the consistency equation. Similarly the y and the t equations are:

$$\begin{aligned} \frac{\partial u_{p+2}}{\partial t_4} \equiv \frac{\partial u_{p+2}}{\partial y} &= \frac{\partial^2 u_{p+2}}{\partial x^2} + 2\frac{\partial u_{p+2}}{\partial x} + 2V_3 u_{p+2}^{[1]} \\ &+ 2u_{p+3}V_3 + 2V_4 u_{p+2} - 2\sum_{n=0}^{p} \begin{bmatrix} n-p-1 \\ n \end{bmatrix} u_{p-n+2} V_4^{[n]} \\ &- 2\sum_{n=0}^{p} (-)^p \begin{bmatrix} n-p-2 \\ n \end{bmatrix} u_{p-n+3} V_3^{[n]} - 2\begin{bmatrix} -1 \\ p+1 \end{bmatrix} (-1)^p u_2 V_3^{[p+1]} \end{aligned} \tag{2.14}$$

$$\begin{aligned} \frac{\partial u_{p+2}}{\partial t_6} \equiv \frac{\partial u_{p+2}}{\partial t} &= \frac{\partial^3 u_{p+2}}{\partial x^3} + 3\frac{\partial^2 u_{p+4}}{\partial x^2} + 3\frac{\partial u_{p+6}}{\partial x} + V_6 u_{p+2} + 3V_3 u_{p+4}^{[1]} + 3V_3 \frac{\partial u_{p+2}^{[1]}}{\partial x} \\ &+ 3u_{p+5}V_3 + 3\frac{\partial u_{p+3}}{\partial x}V_3 + 3V_4\frac{\partial u_{p+2}}{\partial x} + 3V_4 u_{p+4} + V_5 u_{p+2}^{[1]} + u_{p+3}V_5 \\ &- 3\begin{bmatrix} -1 \\ p+3 \end{bmatrix}(-)^p u_2 \frac{\partial V_3}{\partial x}^{[p+1]} - 3\begin{bmatrix} -2 \\ p+2 \end{bmatrix}(-)^p u_3 \frac{\partial V_3}{\partial x}^{[p]} - 3\begin{bmatrix} -3 \\ p+1 \end{bmatrix}(-)^p u_4 V_3^{[p+1]} \\ &- 3\begin{bmatrix} -1 \\ p+2 \end{bmatrix} u_2 \frac{\partial V_4}{\partial x}^{[p]} - 3\begin{bmatrix} -2 \\ p+1 \end{bmatrix} u_3 V_4^{[p+1]} - (-)^p \begin{bmatrix} -1 \\ p+1 \end{bmatrix} u_2 V_5^{[p+1]} \\ &- \sum_{n=o}^{p} \begin{bmatrix} n-p-1 \\ n \end{bmatrix} u_{p-n+2} V_6^{[n]} - \sum_{n=0}^{p} \begin{bmatrix} n-p-4 \\ n \end{bmatrix} u_{p-n+5} V_3^{[n]} \\ &- 3\sum_{n=0}^{p} \begin{bmatrix} n-p-3 \\ n \end{bmatrix} u_{p-n+4} V_4^{[n]} - \sum_{n=0}^{p} (-)^p \begin{bmatrix} n-p-3 \\ n \end{bmatrix} u_{p-n+3} V_5^{[n]} \end{aligned} \tag{2.15}$$

The first few equations can be written down as follows:

$$\begin{aligned} 3\frac{\partial V_3}{\partial y} &= -3\frac{\partial^2 V_3}{\partial x^2} + \frac{\partial V_5}{\partial x} \\ 3\frac{\partial V_4}{\partial y} &= -3\frac{\partial^2 V_4}{\partial x^2} + 6V_3\frac{\partial V_3}{\partial x} - 4V_3 V_5 + 2\frac{\partial V_6}{\partial x} \\ \frac{\partial V_5}{\partial y} - 2\frac{\partial V_3}{\partial t} &= \frac{\partial^2 V_5}{\partial x^2} - 2\frac{\partial^3 V_3}{\partial x^3} - 6V_3\frac{\partial V_3^{[1]}}{\partial x} - 6\frac{\partial (V_3 V_4)}{\partial x} - 2(V_3 V_5)^{[1]} \\ \frac{\partial V_6}{\partial y} - 2\frac{\partial V_4}{\partial t} &= \frac{\partial^2 V_6}{\partial x^2} + 2V_3 V_6^{[1]} - 2\frac{\partial^3 V_4}{\partial x^3} - 6V_3\frac{\partial V_4^{[1]}}{\partial x} - 6V_4\frac{\partial V_4}{\partial x} + 2V_4^{[1]}V_5 \end{aligned} \tag{2.16}$$

The super KP equation can be obtained from (2.16) by setting $V_4 = V_6 = 0$ and expanding the superfield V_3 as,

$$V_3(x,y,t) = \phi(x,y,t) + \theta u(x,y,t) \tag{2.17}$$

where $\phi(x,y,t)$ and $u(x,y,t)$ are fermionic and bosonic fields respectively. The resulting super KP equation is of the form:

$$\begin{aligned} \frac{3}{4}\frac{\partial^2 u}{\partial y^2} &= -\frac{\partial}{\partial x}[\frac{\partial u}{\partial t} + \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + \frac{3}{2}\frac{\partial^2 \phi}{\partial x^2}\phi + 3u\frac{\partial u}{\partial x}] \\ \frac{3}{4}\frac{\partial^2 \phi}{\partial y^2} &= -\frac{\partial}{\partial x}[\frac{\partial \phi}{\partial t} + \frac{1}{4}\frac{\partial^3 \phi}{\partial x^3} + \frac{3}{2}\frac{\partial (u\phi)}{\partial x}] \end{aligned} \tag{2.18}$$

We recognize (2.18) to be the supersymmetric generalization of KP equation because by setting $\phi = 0$, we recover the KP equation [15].

III. Hamiltonian Structures of MR sKP Hierarchy:

The super KP equation for the Manin-Radul fermionic super Lax operator given in (2.2) implies the existence of an infinite number of conserved densities [17] (Hamiltonian densities) associated with the system. These Hamiltonian densities are given as

$$H_i = \frac{1}{i} \, Res \, L^i \tag{3.1}$$

Here, the Res of a formal pseudo-superdifferential operator is defined as the coefficient of the D^{-1} term placed to the right. Also, note that when i is even H_i is a total derivative, since in that case we have

$$H_{2i} = \frac{1}{2i} Res \, L^{2i} = \frac{1}{4i} Res[L^{2i-1}, \, L] = \frac{1}{4i}[DF(L^{2i-1}, L)] \tag{3.2}$$

where F is some function of the fields in L. The last equality in (3.2) follows from the fact that for any two pseudo-superdifferential operators P and Q, one can show that,

$$Res[P, \, Q] = [DF(P, Q)] \tag{3.3}$$

This can be proved by considering a general term in P and Q and using the Leibniz rule as given in (2.7) [17].

It is, therefore, clear that the even Hamiltonians will not generate any flow. So, the Hamiltonians which are of relevance for the flow equations are

$$H_{2i+1} = \frac{1}{2i+1} Res L^{2i+1} \qquad i = 0, 1, 2, \ldots\ldots\ldots \tag{3.4}$$

Let us also note that if we consider some even and odd powers of the Lax operator L, they can be expanded as follows:

$$L^{2i} = \sum_{m=0}^{2i} D^m a_{m+1}(2i) + \sum_{m=0}^{\infty} D^{-m-1} a_{-m}(2i) \tag{3.5}$$

and similarly

$$L^{2i+1} = \sum_{m=0}^{2i+1} D^m a_{m+1}(2i+1) + \sum_{m=0}^{\infty} D^{-m-1} a_{-m}(2i+1) \qquad i = 0, 1, 2, \ldots\ldots\ldots \tag{3.6}$$

where $a_m(2i)$ and $a_m(2i+1)$ are some functions of the fields present in L. The grading of these objects are

$$\begin{aligned} |a_m(2i)| &= m+1 \ \ (mod \ 2) \\ |a_m(2i+1)| &= m \ \ (mod \ 2) \end{aligned} \tag{3.7}$$

Notice that $a_0(2i+1)$ are the residue of L^{2i+1} and hence are the relevant Hamiltonians as discussed earlier. Next, we show how the different coefficients in the expansion (3.5) can be obtained in terms of $a_0(2i+1)$. Consider the variation of $a_0(2i+1)$, i.e.

$$\begin{aligned} \delta a_0(2i+1) &= (2i+1) \, Res \, \delta L \, L^{2i} \qquad (mod \, (DF)) \\ &= (2i+1) \, Res \, [\delta L_+ \, L_-^{2i} + \delta L_- \, L_+^{2i}] \\ &= (2i+1) \, (\sum_{m=0}^{2i} (-)^{m+1} \, \delta u_{m+1} \, a_m(2i)) \end{aligned} \tag{3.8}$$

Therefore, for $m = 0, 1, 2, \ldots, 2i$ we have

$$a_m(2i) = \frac{(-)^{m+1}}{2i+1} \frac{\delta a_0(2i+1)}{\delta u_{m+1}} = (-)^{m+1} \frac{\delta H_{2i+1}}{\delta u_{m+1}} \tag{3.9}$$

One can now write down the Lax equation (2.5) in terms of these Hamiltonians and the corresponding Hamiltonian structures. Since

$$\frac{\partial L}{\partial t_{2i}} = [L_+^{2i}, L] = -[L_-^{2i}, L]$$

we need to keep only the zeroth and the negative powers of D in the right hand side or in other words,

$$\frac{\partial L}{\partial t_{2i}} = [L_+^{2i}, L]_{\leq 0} \tag{3.10}$$

Using the Leibniz rule and the expansions in (3.5), for $p = 0, 1, 2, \ldots$and $i = 1, 2, \ldots$ we obtain that

$$\frac{\partial u_{p+2}}{\partial t_{2i}} = \sum_{m=0}^{2i} K_{pm}^{(1)} \frac{\delta H_{2i+1}}{\delta u_{m+2}} \tag{3.11}$$

where

$$K_{pm}^{(1)} = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} (-)^{j(p+j)} \, D^j u_{p+m-j+2} - \sum_{j=0}^{p} \begin{bmatrix} -j-1 \\ p-j \end{bmatrix} (-)^{mp} \, u_{j+m+2} D^{p-j} \tag{3.12}$$

and the variational derivative is defined as

$$\frac{\delta P}{\delta u} = \sum_{k=0}^{\infty} (-)^{|u|k+k(k+1)/2} \left(\frac{\partial P}{\partial u^{[k]}}\right)^{[k]} \tag{3.13}$$

Thus (3.12) defines the first Hamiltonian structure in the sense of Gelfand and Dikii [19]. It is very tedious but straightforward to check that (3.11) reproduces the correct Lax equations (2.13 - 2.15) for $i = 1, 2, 3$. Also, we note that since the Hamiltonian structure (3.12) is derived from the commutator in (3.10), it is automatically skew-adjoint and defines a linear superalgebra.

To obtain the second Hamiltonian structure, we rewrite the Lax equation (3.10) as

$$\begin{aligned} \frac{\partial L}{\partial t_{2i}} &= [(L^2\, L^{2i-2})_+ L \;-\; L(L^{2i-2}\, L^2)_+]_{\leq 0} \\ &= [(D^2 L_-^{2i-2})_+ \;-\; L(L_-^{2i-2} D^2)_+ \;+\; (L^2\, L_+^{2i-2})_+ L \;-\; L(L_+^{2i-2}\, L^2)_+]_{\leq 0} \\ &= [(L^2\, L_+^{2i-2})_+ L \;-\; L(L_+^{2i-2}\, L^2)_+ \;+\; [Da_0(2i-2), L] \;+\; [a_{-1}(2i-2), L]\,]_{\leq 0} \end{aligned} \tag{3.14}$$

We note that in the above equation, except the last term all other terms can be written in terms of the Hamiltonian H_{2i-1}, because they involve the functions $a_m(2i-2)$ with $m \geq 0$. However, as it is clear from (3.9), the last term which involves $a_{-1}(2i-2)$ can not be expressed in terms of H_{2i-1}. This difficulty can be surpassed if we redefine the Lax operator as

$$L = u_0 D + \sum_{p=0}^{\infty} u_{p+1} D^{-p} \tag{3.15}$$

Let us first note from the Lax equation (2.5) that for all $i = 1, 2, 3, \ldots$ we have

$$\frac{\partial u_0}{\partial t_{2i}} = 0 \tag{3.16}$$

Thus u_0 does not have any dynamics. Also (3.15) with $u_0 = 1$ is our original Lax operator. Notice that the first Hamiltonian structure (3.12) is unaffected with the new Lax operator. Also the Hamiltonians, though formally change, will be the same once we use the constraint that $u_0 = 1$. Therefore, $u_0 = 1$ can be treated as a constraint in our system. We would like to mention that such a situation also arises in defining the second Hamiltonian structure of the bosonic KP hierarchy. In ref.[13], it was treated differently. The advantage of

redefining the Lax operator as in the above is that using this we find

$$a_{-1}(2i-2) = \frac{\delta H_{2i-1}}{\delta u_0}\Big|_{u_0=1} \tag{3.17}$$

Thus, now we are in a position to find the second Hamiltonian structure by writing the Lax equation as

$$\frac{\partial u_{p+2}}{\partial t_{2i}} = \Big[\sum_{m=0}^{2i-2} K_{pm}^{(2)} \frac{\delta H_{2i-1}}{\delta u_{m+2}} + K_{p,-1}^{(2)} \frac{\delta H_{2i-1}}{\delta u_1} + K_{p,-2}^{(2)} \frac{\delta H_{2i-1}}{\delta u_0}\Big]\Big|_{u_0=1} \tag{3.18}$$

where $K_{pn}^{(2)}$ are the Gelfand-Dikii brackets and are given by

$$\begin{aligned} K_{pm}^{(2)} &= \sum_{l=0}^{m+2} (-)^{(p+1)l} \begin{bmatrix} m+2 \\ l \end{bmatrix} D^l u_{p+m-l+4} + \sum_{n=0}^{m-1} \sum_{l=0}^{m-n-1} (-)^{(p+1)(n+l+1)} \begin{bmatrix} m-n-1 \\ l \end{bmatrix} \\ &\times (u_{n+2}^{[1]} + u_{n+3} + (-)^{(n+1)} u_{n+3} + u_1 u_{n+2}) D^l u_{p+m-n-l+1} + \sum_{n=0}^{m-1} \sum_{j=0}^{m-n-1} \sum_{l=0}^{m-n-j-1} \\ &(-)^{p(n+j+l+1)+l} \begin{bmatrix} -1-n \\ j \end{bmatrix} \begin{bmatrix} m-n-j-1 \\ l \end{bmatrix} u_{n+2} u_1^{[j]} D^l u_{p+m-n-j-l+1} + \sum_{n=0}^{m-2} \sum_{k=0}^{m-n-2} \\ &\sum_{j=0}^{m-n-k-2} \sum_{l=0}^{m-n-k-j-2} (-)^{k(n+j+1)+(p+1)(n+k+j+l)} \begin{bmatrix} m-n-k-j-2 \\ l \end{bmatrix} \\ &\times \begin{bmatrix} -1-n \\ j \end{bmatrix} u_{n+2} u_{k+2}^{[j]} D^l u_{p+m-n-k-j-l} - \sum_{j=0}^{m} \sum_{r=0}^{p+m-j+3} (-)^{(p+m)(m+j)+m} \begin{bmatrix} -r \\ p+m-j-r+3 \end{bmatrix} \\ &\times \begin{bmatrix} m \\ j \end{bmatrix} u_{r+1} D^{p+m-r+3} - \sum_{n=0}^{m-1} \sum_{j=0}^{m-n-1} \sum_{r=0}^{p+m-n-j} (-)^{(m+n+1)(p+j)+jp} \begin{bmatrix} -r \\ p+m-n-j-r \end{bmatrix} \\ &\times \begin{bmatrix} m \\ j \end{bmatrix} u_{r+1} D^{p+m-n-r} (u_{n+2}^{[1]} + u_{n+3} + (-)^{n+1} u_{n+3} + u_1 u_{n+2}) - \sum_{n=0}^{m-1} \sum_{l=0}^{m-n-1} \sum_{j=0}^{m-n-l-1} \\ &\sum_{r=0}^{p+m-n-l-j} (-)^{m+(p+1)(m+n+l+1)} \begin{bmatrix} -1-n \\ l \end{bmatrix} \begin{bmatrix} -r \\ p+m-n-l-j-r \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} u_{r+1} \\ &D^{p+m-n-l-r} u_{n+2} u_1^{[l]} - \sum_{n=0}^{m-2} \sum_{k=0}^{m-n-2} \sum_{l=0}^{m-n-k-2} \sum_{j=0}^{m-n-k-l-2} \sum_{r=0}^{p+m-n-k-l-j-1} \begin{bmatrix} -1-n \\ l \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \\ &(-)^{p(m+n+k+l)+k(n+l+1)} \begin{bmatrix} -r \\ p+m-n-k-l-j-r-1 \end{bmatrix} u_{r+1} D^{p+m-n-k-l-r-1} u_{n+2} u_{k+2}^{[l]} \end{aligned} \tag{3.19a}$$

$$K^{(2)}_{p,-1} = (-)^{p+1}Du_{p+2} + u_{p+3} - \sum_{n=0}^{p}(-)^p \begin{bmatrix} -1-n \\ p-n \end{bmatrix} u_{n+3}D^{p-n} \quad (3.19b)$$

$$K^{(2)}_{p,-2} = u_{p+2} - \sum_{n=0}^{p} \begin{bmatrix} -1-n \\ p-n \end{bmatrix} u_{n+2}D^{p-n} \quad (3.19c)$$

It is again very tedious but straightforward to check that (3.18) indeed reproduces the Lax equation of motion. Therefore, (3.19) is the second Hamiltonian structure associated with the MR super KP hierarchy. The Hamiltonian structure in this case is non-linear and hence does not define a supersymmetric Lie algebra. Since we have obtained both the Hamiltonian structures from the same equation of motion, it is clear that the Hamiltonians of this system obey the following recursion relation:

$$\sum_{m=0}^{2i} K^{(1)}_{pm}\frac{\delta H_{2i+1}}{\delta u_{m+2}} = \sum_{m=0}^{2i-2} K^{(2)}_{pm}\frac{\delta H_{2i-1}}{\delta u_{m+2}} + K^{(2)}_{p,-1}\frac{\delta H_{2i-1}}{\delta u_1} + K^{(2)}_{p,-2}\frac{\delta H_{2i-1}}{\delta u_0} \quad (3.20)$$

where $i = 1,2,3......$ and $p = 0,1,2,......$ The above recursion relation will be useful to understand the geometry of the phase-space and to show the integrability of the system.

IV. Conclusion

We have shown that the "even-time" flow of the MR super KP hierarchy is a bi-Hamiltonian system. Starting from the Lax equation we have been able to show that the Lax equation can be written as Hamilton's equation of motion in terms of two distinct Hamiltonian structures and two successive odd Hamiltonians. We have also given the recursion relation among the conserved quantities. It is well known that the bosonic KP hierarchy is a bi-Hamiltonian system where the first Hamiltonian structure is isomorphic to $W_{1+\infty}$ algebra and the second Hamiltonian structure is an universal W_∞ algebra [23,14] of the generalized KdV hierarchy. One would expect a corresponding situation for the super KP hierarchy. Recently, the structure of super W_∞ algebra has been studied generalizing the superconformal algebra in (1+1) dimensions [24]. Also it has been shown that these super W_∞ generators describe the symmetry of the MR super KP hierarchy [25]. It will

be interesting to see if there is any relation between the Hamiltonian structures we have obtained with the super W_∞ algebra.

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