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FOURTH-RANK GRAVITY AND COSMOLOGY

A.L. Marrakchi *
International Centre for Theoretical Physics, Trieste, Italy

and

Victor Tapia
International Centre for Theoretical Physics, Trieste, Italy.

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* On leave of absence from: Laboratoire de Physique Théorique, Faculté des Sciences, Rabat, Morocco and Département de Physique, Faculté des Sciences, B.P. 1796 – Atlas, Fes, Morocco †.
† Permanent address.

Abstract We consider the consequences of describing the metric properties of space-time through a quartic line element. The associated "metric" is a fourth-rank tensor $G_{\mu\nu\lambda\rho}$. In order to recover a Riemannian behaviour of the geometry it is necessary to have $G_{\mu\nu\lambda\rho}=g_{(\mu\nu}g_{\lambda\rho)}$. We construct a theory for the gravitational field based on the fourth-rank metric $G_{\mu\nu\lambda\rho}$. In the absence of matter the fourth-rank metric becomes separable and the theory coincides with General Relativity. In the presence of matter we can maintain Riemannianicity, but now gravitation couples, as compared to General Relativity, in a different way to matter. We develop a simple cosmological model based on a FRW metric with matter described by a perfect fluid. For the present time the field equations are compatible with $k_{OBS}=0$ and $\Omega_{OBS}<0.18$ and they imply $p/\rho<0.038$ which corresponds to an almost pressureless perfect fluid. Therefore, the flatness problem is solved. However, our approach is valid only for $p/\rho<0.236$. Therefore, we consider an early universe in which the state equation of matter is $p/\rho=0.236$ rather than $p/\rho=1/3$. There is no violation of causality, no horizon problem, for $t>t_{CLAS}\approx 10^{20}t_{PLANCK}\approx 10^{-23}s$. Our final and most important result is the fact that the entropy is an increasing function of time. When interpreted at the light of General Relativity the treatment is shown to be almost equivalent to that of the standard model of cosmology combined with the inflationary scenario.

Short title: Fourth-rank gravity.

"The next case in simplicity includes those manifoldnesses in which the line-element may be expressed as the fourth-root of a quartic differential expression."

B. Riemann, 1854

1. Introduction

Our geometrical conception of the universe is limited by our psychological perception of it. There is in fact a self-consistency in that the physical laws generate the very mathematics necessary to make those laws understandable. In the scale of distances of our daily life, i.e., distances much greater than the Planck lenght, the universe behaves quite smoothly and one hopes that this behaviour might be extrapolated to very large, cosmological, as also to very small, even subnuclear, distances. The previous smooth behaviour allows the universe to be mathematically modellated by a differentiable manifold. Of course, the very concept of a differentiable manifold is possible only because our perception of space allows to conceive it, and one can wonder how our mathematical conceptions are restricted by this kind of anthropic principle.

The problem of determining the geometry realised in nature was already noted by Riemann in his thesis [1] in 1854. He pointed out that this geometry has to be established by empirical means and cannot be decided upon *a priori*. It is a purely experimental and observational issue to decide which one is the correct geometry describing our universe. An important related question is why the structure of space-time is the one that physics teaches us.

From a purely mathematical point of view a differentiable manifold can support many geometrical structures, each one giving rise to a different kind of geometry. The geometrical properties can be roughly divided into affine and metric properties. Affine properties are related to how one moves from one point to a close one, to differentiation and to the curvature properties of the given manifold; all this information is contained in the affine

connection Γ. Metric properties, on the other hand, are related to the way in which distances are to be measured.

Both, affine and metrical properties must be decided upon by observation. The first indirect statements about the metrical properties of our universe can be found in the Euclid axioms and in the Pythagoras theorem. The Pythagoras theorem establishes that distances are to be measured in terms of squares of distances along individual directions. In a modern language this translates into the fact that the metrical properties of our universe are described by Riemannian geometry. This means that the line element is of the form

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu , \tag{1.1}$$

where $g_{\mu\nu}$ is the second-rank metric tensor.

Different geometries are obtained for different choices of the connection and of the metric. A particular choice is a metric connection. In this case the connection and the metric are related by a metricity condition; the natural choice is

$$\nabla_\Gamma g = 0 . \tag{1.2}$$

In this case the connection is the Christoffel symbol of the second kind of the metric g. Therefore one can associate curvature properties to a metric. The possibility that our universe be a curved space is not excluded a priori, therefore it needs experimental verification. The experiment was performed by Gauss [2] in 1826 and it was intended to verify departures from flatness, but as a side result he also verified no departures from Riemannianicity.

The only thing we can try to understand now is the Riemannian nature of the metric. In order to do so let us start by recalling the fundamental definitions concerning the metrical properties of a differentiable manifold. Here we take recourse to the classical argumentation by Riemann [1]. The infinitesimal element of distance ds is a function of the coordinates x's and their differentials dx's

$$ds = f(x, dx) . \tag{1.3}$$

This function must be homogeneous at the first-order in dx's

$$f(x, \lambda\, dx) = \lambda\, f(x, dx) , \tag{1.4a}$$

and must be positive defined

$$ds \geq 0 . \tag{1.4b}$$

Of course the possibilities are infinitely many. Let us restrict our considerations to monomial functions. Then we will have

$$ds = \left[G_{\mu_1 \cdots \mu_r} dx^{\mu_1} \cdots dx^{\mu_r}\right]^{1/r} . \quad (1.5)$$

In order for this quantity to be positive r must be an even number. The simplest choice is $r=2$, which corresponds to Riemannian geometry.

The next possibility is $r=4$. In this case the line element is given by

$$ds^4 = G_{\mu\nu\lambda\rho} dx^\mu \, dx^\nu dx^\lambda \, dx^\rho . \quad (1.6)$$

Of course, at first sight, a space with a line element of the form (1.6) may seem bizarre. However, the possibility of describing the metric properties of our universe by means of a quartic line element is not excluded *a priori* and its exclusion must be done in an educated way.

The existence of a quartic line element was considered seriously by Helmholtz [3]. Helmholtz's answer is satisfactory as long as one accepts the free mobility of rigid bodies in a three-dimensional space as fundamental. The problem was furthermore explored by Weyl [4]; he concluded that the only thing we can still try to understand is- as he called it- the *Pythagorean nature* of the metric. However, in considering this problem, Weyl wanted to characterise the family of his own geometries- the Weyl geometries. This approach received further contributions from Cartan [5]. A different approach was undertaken by Finsler [6]; in order to recover Riemannianicity, at least locally, he constructed an associated quadratic function. It seems that the success of General Relativity, with its underlying Riemannian geometry, made this important problem to be forgotten.

In fourth-rank geometry a central role is played by the tensor $G_{\mu\nu\lambda\rho}$ and since it is related to the metric properties of space-time it is not an error to call it a "metric" [a]. It is clear that at the scale of distances of our daily life this geometry is not realised in nature. Therefore, if a fourth-rank geometry is going to play some role in physics, for large distances a Riemannian behaviour of the geometry must be recovered. For this purpose it is convenient to introduce the concept of *separable spaces* in which the fourth-rank metric is

$$G_{\mu\nu\lambda\rho} = g_{(\mu\nu} \; g_{\lambda\rho)} . \quad (1.7)$$

In this case the line element factorises and one is back to the Riemannian case. This would allow to explain why the universe, if described by a fourth-rank geometry, at large distances looks Riemannian. Separable metrics can also be used as a quality control of later formal developments. In fact, all results and developments one obtains for a generic metric $G_{\mu\nu\lambda\rho}$ must reduce to those for the Riemannian case when applied to a separable metric.

In fourth-rank geometry very few things can be mimicked from Riemannian

geometry; in particular, a metricity condition relating the connection and the metric cannot be imposed consistently. Therefore, for physical applications the connection and the metric must be considered as independent fields.

Let us now turn to consider how the geometry realised in nature can be determined. Apart from the direct determinations on how distances must be measured at our daily life scale we must explore the consequences at scales of distances to which we have not direct experimental access. This is the small distance, high-energy regime. The geometrical properties of space-time can be explored through field theory. In fact, according to our contemporary conceptions interactions are to be described through field theory. According to our modern conception of the gravitational interaction the gravitational field is the responsible for the geometrical properties of space-time. Therefore, the objects characterising geometry can be identified with the gravitational field. This is not enough however since one must give a dynamics to this field. The first step is to construct a geometrical invariant to be used as a Lagrangian density. The field equations one obtains must then be confronted with observation. This procedure should select the correct theory for the gravitational interaction and its underlying geometry.

In General Relativity space-time is conceived as a Riemannian space in which the fundamental field, the gravitational field, is identified with the metric. The dynamics of the gravitational field is described by the Hilbert Lagrangian

$$\mathcal{L}_{EH} = k_E \, R(g) \, (-g)^{1/2} \quad . \quad (1.8)$$

where k_E is the Einstein gravitational constant. The full Lagrangian must incorporate also the contributions of matter. One obtains then the Einstein field equations

$$\kappa_E \, G_{\mu\nu} = T_{\ell,\mu\nu} \; , \quad (1.9)$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\ell,\mu\nu}$ is the energy-momentum tensor of matter.

Einstein field equations have been applied to different physical situations. The simplest case is in the absence of matter. A particular application is to the motion of planets, considered as test particles, in the solar gravitational field. The solution, the Schwarzschild metric, gives account of the anomalous shift of the perihelion of inner planets to an accuracy of 1 per cent or better. In this case one is describing the effects of pure gravity. The next step is the coupling of gravity to matter.

The coupling of gravity to matter is achieved, for instance, when considering the large scale structure of the universe where gravity becomes coupled to a perfect fluid. The solution, the standard model of cosmology, gives qualitatively good results concerning the evolution of the universe from an initial singularity. However, the field equations predict relations between the cosmological parameters in disagreement with the observed values. Therefore General Relativity describes well the dynamics of the gravitational field alone, but it fails when coupled to matter.

Furthermore, when one tries to quantise General Relativity one faces with irremovable ultraviolet divergences. This pathological behaviour is associated with very small distances which in a field theoretical language corresponds to a high-energy regime. The common view is that at high energies the geometrical properties of space-time are completely different from those corresponding to a purely metric Riemannian space. One must therefore construct a field theory for a more general geometry. The field theory one constructs for this new geometry must produce, in the absence of matter, a purely metric Riemannian geometry. Furthermore, gravitation, in the form of General Relativity must be recovered.

The departures from purely metric Riemannian geometry might be of several kinds and usually the considerations have been restricted to modifications of the affine structure of space-time. Till now no one of the proposed theories can claim success. Modifications of the metric structure of space-time are, if not difficult to implement, less popular and even inexistent. This is due to the previously mentioned fact that the Riemannian perception of the universe is so strongly rooted in our minds that it is difficult to conceive any departure from it. This fact is in part due to our psychological limitations in the same way in which we cannot imagine higher-dimensional spaces, we can conceive them mathematically, but not imagine them at all. Due to the previous reasons all physics which have been developed to explore the high-energy behaviour of space-time keeps using Riemannian geometry. This is not even assumed or postulated, it is just taken as a default principle of nature. It is due to this fact that everything has been tried less the possibility that the metric properties of space-time could be non-Riemannian. Our aim is to explore this possibility.

Let us suppose that the geometry of space-time is fourth-rank. Since we observe a Riemannian behaviour we expect that in the low-energy regime of a field theory describing the dynamics of the gravitational field the fourth-rank metric becomes separable. Departures from Riemannianicity will appear, as suggested previously, only for the high-energy regime. This do not contradict the classical argumentation by Helmholtz which is no longer

acceptable once one considers the geometric structure of space-time at very small distances, a regime to which we have not direct experimental access as to verify the rigidity of bodies and in which quantum mehcanical effects dominate.

As for Riemannian geometry we need to construct a geometrical invariant to describe the dynamics of the gravitational field. The simplest Lagrangian which can be constructed is

$$\mathcal{L}_{GRAV} = K_E \langle R^2 \rangle^{1/2} \; G^{1/4} \, , \qquad (1.10)$$

where

$$\langle R^2 \rangle = G^{\mu\nu\lambda\rho} R_{\mu\nu}(\Gamma) \, R_{\lambda\rho}(\Gamma) \, . \qquad (1.11)$$

As before the full Lagrangian must also consider the contributions of matter.

Now we must apply a Palatini-like variational principle in which the connection and the metric are varied independently. In all known cases of physical interest the matter Lagrangian does not depend on the affine connection. In this case the variation of the gravitational Lagrangian with respect to the connection leads to a metricity condition for which the solution is

$$\Gamma^{\lambda}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}(\gamma) \, . \qquad (1.12)$$

I.e., the connection is the Christoffel symbol of the second kind of the tensor $\gamma^{\mu\nu}$ given by

$$\gamma^{\mu\nu} = \langle R^2 \rangle^{-1/2} \; G^{\mu\nu\lambda\rho} R_{\lambda\rho} \, , \qquad (1.13)$$

which we have assumed to be regular. Equations (1.12) and (1.13) are a metricity condition since they give the relation between $\Gamma^{\lambda}_{\mu\nu}$ and $G_{\lambda\mu\nu\rho}$. Therefore

$$R_{\mu\nu}(\Gamma) = R_{\mu\nu}(\gamma). \qquad (1.14)$$

One easily verifies then that

$$\langle R^2 \rangle^{1/2} = R(\gamma) = \gamma^{\mu\nu} R_{\mu\nu}(\gamma) \, . \qquad (1.15)$$

Variation of the Lagrangian with respect to the metric $G_{\mu\nu\lambda\rho}$ gives

$$\frac{1}{2} K_E R^{-1} [R_{(\mu\nu} R_{\lambda\rho)} - \frac{1}{2} R^2 G_{\mu\nu\lambda\rho}] = T_{\mu\nu\lambda\rho} \, , \qquad (1.16)$$

where $T_{\mu\nu\lambda\rho}$ is the energy-momentum tensor of matter, to be defined below, and use has already been made of eq. (1.15).

The field equations (1.16) exhibit three energy regimes: low, medium, and high. In the low-energy regime there is no matter and therefore the fourth-rank metric is separable, $G_{\mu\nu\lambda\rho} = g_{(\mu\nu} g_{\lambda\rho)}$, as can be read from (1.16). In this case the field equations reduce to

$$R_{\mu\nu}(g) = 0 \ , \tag{1.17}$$

which are in fact Einstein field equations in vacuum. This result will be proved in full detail in the main text.

In the medium-energy regime the geometry is still Riemannian, $G_{\mu\nu\lambda\rho} = g_{(\mu\nu}g_{\lambda\rho)}$, but there is matter involved. This possibility is not excluded as a closer analysis of eqs. (1.16) reveals. In this case the gravitational field couples in a different way, as compared to General Relativity, to matter. Lastly, we have the true high-energy regime in which there is matter and the geometry is truly fourth-rank.

Since in vacuum the previous field equations coincide with those of General Relativity we will have the same predictions concerning the Schwarzschild metric and its applications. Departures from General Relativity and from Riemannianicity will appear only in the presence of matter, which is precisely the regime in which General Relativity fails in its predictions.

The large scale geometry of the universe seems to be Riemannian and since there is matter present we are in the medium-energy regime previously mentioned. In this context we develope a cosmological model based on the Friedman-Robertson-Walker metric but which couples in a different way to cosmic matter.

The astronomical observations indicate that our universe is isotropic and homogeneous. These observations lead to the use of Friedman-Robertson-Walker (FRW) metrics. This geometry is characterised by a function a(t) which is roughly the radius of the universe and a parameter k taking the values +1, 0, or -1, for a closed, a spatially flat or an open universe, respectively. On the other hand this geometry can be parametrised in terms of the cosmological parameters H, the Hubble constant, and q, the deacceleration parameter. These functions are important because they are observable.

The observed redshift of galaxies shows that the universe has evolved from a highly dense phase to a present day diluted universe. Since matter cannot be compressed beyond the Planck density the universe has evolved from an initial hot ball with radius a_o which we have estimated to be $\sim 10^{60} L_{PLANCK}$. At large scales matter can be described as a perfect fluid, characterised by the energy density ρ and the pressure p. Associated to the FRW geometry there is a critical energy density ρ_c which sets the scale of energy densities. This allows to define the energy density parameter Ω which is also an observable.

The values of the cosmological parameters H, q, and Ω, are quite difficult to determine with accuracy. In fact, the reported values have very big dispersions. In our work we are going to use the more conservative values

reported in [7]. Most relevant for our later applications is the energy density parameter for which the reported value is $\Omega_{OBS} \approx 0.1$-0.3, with an upper safety bound $\Omega_{SAFE} \approx 0.18$.

Further observations show that to a very big extent our space is quite flat. Therefore, the parameter k characterising the FRW geometries must be zero, $k_{OBS}=0$. The last important observation is the fact that the universe is quite isotropic and homogeneous at scales by far larger than the light horizon. This is an indication that matter was in causal contact in the very remote past. This condition roughly translates into $r_H > a$, where r_H is the horizon radius.

The next task is to develop a gravitational theory fitting the previous observations. The first candidate we have is General Relativity.

When General Relativity is applied to cosmology one obtains field equations which are in conflict with the observed values of the cosmological parameters. In this case the field equations are

$$q = \frac{1}{2}\left(1 + 3\frac{p}{\rho}\right)\Omega \ , \tag{1.18a}$$

$$\Omega = 1 + \frac{k}{\dot{a}^2} \ . \tag{1.18b}$$

The first equation shows that q>0 and from here one is used to conceive the universe as expanding from an initial singularity. This fact is in agreement with observation. However, the second equation is incompatible with the reported values for Ω_{OBS} and k_{OBS}. If one insists in keeping $k_{OBS}=0$ one has $\Omega_{PRED}=1$. This is the missing mass problem. In order to solve this problem one considers inflation [8, 9, 10]. Inflation is intended to solve not only the flatness or entropy problem, k=0, but also the horizon or causality problem. These are the most intriguing puzzles of the standard model of cosmology. However, inflationary cosmology will be sound only if later observations will show that $\Omega_{OBS}=1$.

Therefore, from a cosmological point of view our conclusion is that General Relativity is an incomplete theory. In fact, it describes well, to a very high accuracy, the effects of the gravitational field alone: the shift of the perihelion of inner planets, the bending of light on strong gravitational fields, etc., however it fails to describe the coupling of the gravitational field to matter, for example, the standard model of cosmology.

We must look therefore for an improved theory for the gravitational field coinciding with General Relativity in the vacuum case and with a different way of coupling the gravitational field to matter. The theory of fourth-rank gravity we have developed satisfies these requirements.

When fourth-rank gravity is applied to cosmology we obtain field equations in which the gravitational field couples, as compared to General Relativity, in a different way to cosmic matter. For positive pressure the flatness and the causality problems can be solved provided that $q<0$. This does not contradict the observed expansion of the universe from an initial hot ball. For $k=0$ the field equations give

$$\Omega = \frac{4y}{1-4y-y^2} , \tag{1.19}$$

where $y=p/\rho$. For Ω_{SAFE} we obtain $y_{SAFE}=0.038$ which corresponds to an almost pressureless perfect fluid.

As seen from (1.19) y cannot take a value greater than $y=\sqrt{5}-2\simeq 0.236$. This means that a relativistic fluid, $y=1/3$, cannot be described within our approach. Therefore, we consider a model for the early universe in which matter is governed by the state equation $y=0.236$ rather than that for pure radiation, $y=1/3$. We find that causality is not violated for an age of the universe larger than $t_{CLAS} \approx 10^{20} t_{PLANCK} \approx 10^{-23}$ s. Before this classical time quantum gravity effects dominate and the very concept of causality is lost. The model predicts an increasing total entropy such that the expansion of the universe is an adiabatic non-isentropic process. Therefore, the evolution of the universe in the framework of fourth-rank cosmology is, as expected, an irreversible process.

Therefore, our model agrees with the observed values of the cosmological parameters and is able to solve at least the flatness and horizon problems which are usually approached by inflationary cosmological models.

One is used to think of cosmology in terms of General Relativity. In this case $q>0$. Our field equations will be compatible with this assumption and the observed flatness of the universe only if $p<0$. In this case we obtain

$$\Omega = -\frac{4y}{1-4y-y^2} . \tag{1.20}$$

For Ω_{SAFE} we obtain $y_{SAFE}=-0.055$.

Our field equations can be reinterpreted at the light of General Relativity. In fact, they can be rewritten as Einstein field equations coupled to a perfect fluid with an extra contribution which behaves as a cosmological constant.

Our terminology concerning the evolution of the universe is as follows. We consider the very early universe to be the regime in which quantum gravity effects dominate; the early universe is therefore that for $t>t_{CLAS}$ in which a classical treatment is sufficient; the present universe is that in which matter is almost pressureless.

The paper is organised as follows. In section 2 we start by giving some mathematical considerations. In section 3 we recall the fundamentals of General Relativity. In section 4 we develop the fundamentals of fourth-rank gravity. In section 5 we collect the observational results concerning our universe. In section 6 we briefly show that the observed values of the cosmological parameters do not fit into the Einstein field equations. In section 7 we apply fourth-rank gravity to cosmology. Section 8 is dedicated to some concluding remarks.

To our regret, due to the nature of this approach, in sections 3 and 6 we must bore the reader by exhibiting some standard and well known results in order to illustrate where the new approach departs from the standard one.

2. Mathematical Preliminaries. Differentiable Manifolds

Here we consider some elementary results for differentiable manifolds. Let M be a d-dimensional differentiable manifold, and let x^μ, $\mu=0,\dots,d-1$, be local coordinates. The geometric properties of a differentiable manifold can be roughly divided into affine properties and metric properties.

Affine properties are related to how one moves from one point to a close one. These properties are mathematically described by the connection $\Gamma^\lambda{}_{\mu\nu}$. In terms of the connection one can define the curvature or Riemann tensor

$$R^\lambda{}_{\rho\mu\nu} = \partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\nu \Gamma^\lambda{}_{\mu\rho} + \Gamma^\lambda{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\rho} - \Gamma^\lambda{}_{\nu\alpha}\Gamma^\alpha{}_{\mu\rho} . \tag{2.1}$$

From here we can define the Ricci tensor, or contracted Riemann tensor, given by

$$R_{\rho\nu} = R^\lambda{}_{\rho\lambda\nu} = \partial_\lambda \Gamma^\lambda{}_{\nu\rho} - \partial_\nu \Gamma^\lambda{}_{\lambda\rho} + \Gamma^\lambda{}_{\lambda\alpha}\Gamma^\alpha{}_{\nu\rho} - \Gamma^\lambda{}_{\nu\alpha}\Gamma^\alpha{}_{\lambda\rho} . \tag{2.2}$$

This is the only sensible contraction of the Riemann tensor.

The metric properties of a differentiable manifold are related with the way in which one measures distances. Let us remember the fundamental definitions concerning the metrical properties of our universe. Here we take recourse to the classical argumentation by Riemann [1]. The infinitesimal element of distance ds is a function of the coordinates x and their differentials dx

$$ds = f(x, dx) , \tag{2.3}$$

which is homogeneous of the first-order in dx's

$$f(x, \lambda\, dx) = \lambda\, f(x, dx) . \tag{2.4a}$$

and is positive defined

$$ds \geq 0 . \qquad (2.4b)$$

Of course the possibilities are infinitely many. Let us restrict our considerations to monomial functions. Then we will have

$$ds = \left[G_{\mu_1 \cdots \mu_r}(x)\, dx^{\mu_1} \cdots dx^{\mu_r}\right]^{1/r} . \qquad (2.5)$$

In order for this quantity to be positive r must be an even number.

The simplest choice is $r=2$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu , \qquad (2.6)$$

which corresponds to Riemannian geometry. The coefficients $g_{\mu\nu}$ are the components of the covariant metric tensor. The determinant of the metric is defined by

$$g = \frac{1}{d!}\, \epsilon^{\mu_1 \cdots \mu_d}\, \epsilon^{\nu_1 \cdots \nu_d}\, g_{\mu_1 \nu_1} \cdots g_{\mu_d \nu_d} . \qquad (2.7)$$

The inverse metric is defined by

$$g^{\mu\nu} = \frac{1}{(d-1)!}\frac{1}{g}\epsilon^{\mu\mu_1 \cdots \mu_{d-1}}\, \epsilon^{\nu\nu_1 \cdots \nu_{d-1}}\, g_{\mu_1 \nu_1} \cdots g_{\mu_{d-1} \nu_{d-1}} , \qquad (2.8)$$

and satisfies

$$g^{\mu\lambda}\, g_{\lambda\nu} = \delta^\mu_\nu . \qquad (2.9)$$

The next possibility is $r=4$. In this case the line element is given by

$$ds^4 = G_{\mu\nu\lambda\rho}\, dx^\mu\, dx^\nu dx^\lambda\, dx^\rho . \qquad (2.10)$$

It is clear that this possibility is observationally excluded at the scale of distances of our daily life. However, a Riemannian behaviour can be obtained for *separable spaces*. A space is said to be separable if $G_{\mu\nu\lambda\rho}$ is of the form

$$G_{\mu\nu\lambda\rho} = g_{(\mu\nu}\, g_{\lambda\rho)} = \frac{1}{3}\left(g_{\mu\nu}\, g_{\lambda\rho} + g_{\mu\lambda}\, g_{\nu\rho} + g_{\mu\rho}\, g_{\nu\lambda}\right) . \qquad (2.11)$$

In this case formula (2.10) reduces to (2.6) and therefore all the predictions and formulae obtained for a generic $G_{\mu\nu\lambda\rho}$ must reduce to those for Riemannian geometry when applied to a separable metric.

The determinant of the metric $G_{\mu\nu\lambda\rho}$ is defined as

$$G = \frac{1}{d!}\, \epsilon^{\mu_1 \cdots \mu_d} \cdots\cdots \epsilon^{\rho_1 \cdots \rho_d}\, G_{\mu_1 \nu_1 \lambda_1 \rho_1} \cdots G_{\mu_d \nu_d \lambda_d \rho_d} , \qquad (2.12)$$

where the ϵ's can be chosen as the usual completely antisymmetric Levi-Civita symbols. The inverse metric is defined by

$$G^{\mu\nu\lambda\rho} = \frac{1}{(d-1)!}\frac{1}{G}\, \epsilon^{\mu\mu_1 \cdots \mu_{d-1}} \cdots\cdots \epsilon^{\rho\rho_1 \cdots \rho_{d-1}}$$

$$\times\, G_{\mu_1 \nu_1 \lambda_1 \rho_1} \cdots G_{\mu_{d-1} \nu_{d-1} \lambda_{d-1} \rho_{d-1}} . \qquad (2.13)$$

The previous inverse metric satisfies the relations

$$G^{\mu\alpha\beta\gamma}\, G_{\nu\alpha\beta\gamma} = \delta^\mu_\nu . \qquad (2.14)$$

That the previous is true can be verified by hand in the two-dimensional case and with computer algebraic manipulation for three and four dimensions [11].

In the case of a separable metric the previous formulae reduce to

$$G = g^2 , \qquad (2.15a)$$

$$G^{\mu\nu\lambda\rho} = \frac{3}{d+2}\, g^{(\mu\nu}\, g^{\lambda\rho)} . \qquad (2.15b)$$

Up to now the connection Γ and the metric are unrelated. They can be related through a metricity condition. In Riemannian geometry this metricity condition reads

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu}\, g_{\rho\nu} - \Gamma^\rho_{\lambda\nu}\, g_{\mu\rho} = 0 . \qquad (2.16)$$

The number of unknowns for a symmetric Γ and the number of equations (2.16) are the same, *viz.* $d^2(d+1)/2$. Therefore, since this is an algebraic linear system, the solution is unique and is given by the familiar Christoffel symbols of the second kind

$$\Gamma^\lambda_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\}(g) = \frac{1}{2}\, g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) . \qquad (2.17)$$

In the case of a fourth-rank metric a condition analogous to (2.16) would read

$$\nabla_\mu G_{\alpha\beta\gamma\delta} = \partial_\mu G_{\alpha\beta\gamma\delta} - \Gamma^\nu_{\mu\alpha}\, G_{\nu\beta\gamma\delta} - \Gamma^\nu_{\mu\beta}\, G_{\alpha\nu\gamma\delta}$$

$$- \Gamma^\nu_{\mu\gamma}\, G_{\alpha\beta\nu\delta} - \Gamma^\nu_{\mu\delta}\, G_{\alpha\beta\gamma\nu} = 0 . \qquad (2.18)$$

However, in this case, the number of unknowns Γ is, as before, $d^2(d+1)/2$, while the number of equations is

$$\frac{1}{24}\, d^2\, (d + 1)\, (d + 2)\, (d + 3) > d^2\, (d + 1)/2 . \qquad (2.19)$$

Therefore the system is overdetermined and some differentio-algebraic conditions must be satisfied by the metric. Since, in general, such restrictions will not be satisfied by a generic metric one must deal with Γ and G as independent objects. A metricity condition can be imposed consistently only if the number of independent components of the metric is less than that naively implied by (2.18). The maximum acceptable number of independent components is d(d+1)/2. This can be achieved, for instance, if the metric is a separable one. One can furthermore verify that in this case the metricity condition (2.18) reduces to the usual metricity condition (2.16) and therefore Γ is precisely that for Riemannian geometry, i.e., the Christoffel symbol of the second kind.

3. General Relativity

In this section we recall some of the fundamentals of General Relativity. Even when they are standard results we include them here in order to see how the new approach departs from the standard one.

In General Relativity space-time is conceived as a Riemannian manifold and the metric $g_{\mu\nu}$ is identified with the gravitational field.

In order to describe the dynamics of the gravitational field we need to construct an invariant which might be used as Lagrangian. In Riemannian geometry the simplest invariant which can be constructed is

$$R(g, \Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma) , \tag{3.1}$$

which in the case of a metric space is rewritten as

$$R(g) = g^{\mu\nu} R_{\mu\nu}(g) . \tag{3.2}$$

The analytical formulation of General Relativity takes as its starting point the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{EH}(g) = k_E R(g) (-g)^{1/2} , \tag{3.3}$$

where $k_E = c^4/8\pi G_N$ is the Einstein gravitational constant; G_N being the Newton constant. The full Lagrangian must consider also the contributions of matter

$$\mathcal{L}_1 = \mathcal{L}_{EH} + \mathcal{L}_{MATTER} . \tag{3.4}$$

Variation of the Lagrangian with respect to the metric

$$\frac{\delta \mathcal{L}_1}{\delta g^{\mu\nu}} = 0 , \tag{3.5}$$

gives the Einstein field equations

$$k_E [R_{\mu\nu}(g) - \frac{1}{2} R(g) g_{\mu\nu}] = T_{2,\mu\nu} , \tag{3.6}$$

where

$$T_{2,\mu\nu} = - (-g)^{-1/2} \frac{\delta \mathcal{L}_{MATTER}}{\delta g^{\mu\nu}} , \tag{3.7}$$

is the energy-momentum tensor of matter; the 2 stands for the fact that the energy-momentum tensor is related to Riemannian, second-rank, geometry.

As a starting point for General Relativity one can also consider the "Palatini" Lagrangian

$$\mathcal{L}_P(g, \Gamma) = k_E g^{\mu\nu} R_{\mu\nu}(\Gamma) (-g)^{1/2} . \tag{3.8}$$

In this case one must also consider the contributions of matter

$$\mathcal{L}_2 = \mathcal{L}_P + \mathcal{L}_{MATTER} . \tag{3.9}$$

Now the connection and the metric are varied independently in a procedure known as the Palatini variational principle. Variation of the Lagrangian with respect to Γ gives

$$\frac{\delta \mathcal{L}_2}{\delta \Gamma^{\lambda}_{\mu\nu}} = \frac{\delta \mathcal{L}_P}{\delta \Gamma^{\lambda}_{\mu\nu}} + \frac{\delta \mathcal{L}_{MATTER}}{\delta \Gamma^{\lambda}_{\mu\nu}} = 0 . \tag{3.10}$$

In all known cases of physical interest one has [12]

$$\frac{\delta \mathcal{L}_{MATTER}}{\delta \Gamma^{\lambda}_{\mu\nu}} = 0 . \tag{3.11}$$

In this case eq. (3.10) reduces to a metricity condition equivalent to (2.16), therefore the connection is given by the Christoffel symbol of the second kind for the metric $g_{\mu\nu}$. Variation with respect to the metric

$$\frac{\delta \mathcal{L}_2}{\delta g^{\mu\nu}} = 0 , \tag{3.12}$$

gives

$$k_E [R_{\mu\nu}(\Gamma) - \frac{1}{2} g^{\lambda\rho} R_{\lambda\rho}(\Gamma) g_{\mu\nu}] = T_{2,\mu\nu} . \tag{3.13}$$

If we now use the previously obtained metricity condition these equations reduce to the original Einstein field equations (3.6). Therefore, the procedures of imposing the metricity condition and of applying the variational principle commute.

Einstein field equations can be rewritten in the Landau form

$$k_E R_{\mu\nu} = S_{2,\mu\nu} , \tag{3.14}$$

where

$$S_{2,\mu\nu} = T_{2,\mu\nu} - \frac{1}{2} T_2 g_{\mu\nu} , \tag{3.15}$$

with

$$T_2 = g^{\mu\nu} T_{2,\mu\nu} , \tag{3.16}$$

is the reduced energy-momentum tensor.

Einstein field equations have been applied to many physical situations. The first classical test of any theory of gravitation is in the solar system. In this case one needs to solve Einstein field equations in vacuum for a spherically symmetric field. The solution is the exterior Schwarzschild metric. Using this metric one can give account for the anomalous shift of the perihelion of inner planets and for the bending of light rays near the solar surface, to an accuracy of 1 per cent or better.

The next test concerns the coupling of gravity to matter. This is achieved, for instance, when considering the large scale structure of the universe where gravity becomes coupled to a perfect fluid.

Here we undertake a more radical approach consisting in considering the possibility that General Relativity is an incomplete theory. However, it is in good agreement with observation in the vacuum case, the Schwarzschild solution. Therefore, what is failing in General Relativity is the way in which the gravitational field couples to matter.

Some hints, on how this problem can be approached, come from high-energy physics. When one tries to quantise General Relativity one discovers that there are irremovable ultraviolet divergences. This is taken as indicative of the fact that at small distances the geometry of space-time may be different from the Riemannian one. The current view is that General Relativity, with its Riemannian structure, is only the low-energy, large distance, manifestation of a more general theory at small distances. Many possibilites have been explored mainly in the direction of modifying the affine structure of space-time. Up to our knowledge, modifications of the metric structure of space-time have not yet been attempted. As stated in the introduction the purpose of this work is to explore this possibility and it is to this problem that we turn our attention now.

4. Fourth-Rank Gravity

In this section we develop a theory for the gravitational field based on fourth-rank geometry. General Relativity is recovered only in the vacuum case. Before entering in formal developments there is a natural question to be answered: How a field theory for the fourth-rank metric would connect with the usual theories of gravitation based on the second-rank metric? In order to answer this question one must take recourse to the concept of separable spaces. In fact, in this case the line element factorises and we are back to

the Riemannian case. Thus, the metricity condition (2.18) has a solution: the connection is the Christoffel symbol of the second kind for the metric $g_{\mu\nu}$, and therefore the space is Riemannian. This would explain why the universe, if described by a fourth-rank metric, at large scales it looks Riemannian. The problem is now to obtain this Riemannian behaviour as the low-energy regime of some field theory.

4.1. Fourth-Rank Gravitational Equations

As in General Relativity, in order to describe the dynamics of the gravitational field we need to construct a geometrical invariant. At our disposal we have the metric, which is a fourth-rank tensor, and the Ricci tensor, which is second-rank one. From the metric alone it is impossible to construct any invariant, therefore we must combine the metric and the Ricci tensor. The simplest invariant is

$$\langle R^2\rangle(\Gamma, G) = G^{\mu\nu\lambda\rho} R_{\mu\nu}(\Gamma)\, R_{\lambda\rho}(\Gamma) \, . \tag{4.1}$$

The Lagrangian must have dimensions of an energy density. In this case the coupling constant will be just the Einstein gravitational constant. Based on this simple dimensional analysis we choose

$$\mathcal{L}_{GRAV}(\Gamma, G) = \kappa_E \langle R^2\rangle^{1/2}(\Gamma, G)\; G^{1/4} \, . \tag{4.2}$$

Other choices of the Lagrangian will need the use of $\hbar$ in the coupling constant. The previous is the analogous of the Palatini Lagrangian for General Relativity. But now, since there is no metricity condition, a Lagrangian anologous to the Einstein-Hilbert one simply does not exist. Therefore we can drop the dependence on Γ and G without risk of confusion.

The total Lagrangian must consider also the contributions of matter and is given by

$$\mathcal{L} = \mathcal{L}_{GRAV} + \mathcal{L}_{MATTER} \, . \tag{4.3}$$

Variation with respect to the connection gives

$$\frac{\delta \mathcal{L}}{\delta \Gamma^{\lambda}_{\mu\nu}} = \frac{\delta \mathcal{L}_{GRAV}}{\delta \Gamma^{\lambda}_{\mu\nu}} + \frac{\delta \mathcal{L}_{MATTER}}{\delta \Gamma^{\lambda}_{\mu\nu}} = 0 \, , \tag{4.4}$$

where

$$\frac{\delta \mathcal{L}_{GRAV}}{\delta \Gamma^{\lambda}_{\mu\nu}} = \frac{\partial \mathcal{L}_{GRAV}}{\partial \Gamma^{\lambda}_{\mu\nu}} - d_\rho \frac{\partial \mathcal{L}_{GRAV}}{\partial(\partial_\rho \Gamma^{\lambda}_{\mu\nu})}$$

$$= \gamma^{\alpha\beta}[\tfrac{1}{2}(\delta^\nu_\lambda \Gamma^\mu_{\alpha\beta} + \delta_\lambda{}^\mu \Gamma^\nu_{\alpha\beta}) + \delta_\alpha{}^\mu \delta_\beta{}^\nu \Gamma^\sigma_{\lambda\sigma} - \delta_\beta{}^\nu \Gamma^\mu_{\lambda\alpha} - \delta_\alpha{}^\mu \Gamma^\nu_{\lambda\beta}]G^{1/4}$$

$$- d_\rho[\gamma^{\alpha\beta}(\delta^\rho_\lambda \delta^\mu_\beta \delta^\nu_\alpha - \frac{1}{2}\delta^\rho_\beta(\delta^\mu_\lambda \delta^\nu_\alpha + \delta^\nu_\lambda \delta^\mu_\alpha))G^{1/4}], \tag{4.5}$$

with

$$\gamma^{\alpha\beta} = \langle R^2\rangle^{-1/2}\ G^{\alpha\beta\gamma\delta}\ R_{\gamma\delta}\ . \tag{4.6}$$

(for simplicity, we have omitted κ_E). As mentioned previously, in all known cases of physical interest the matter Lagrangian does not depend on the connection; this fact is independent on the metric properties of space-time. Therefore the second term in (4.4) vanishes and one remains with a metricity condition which has the solution

$$\Gamma^\lambda_{\mu\nu} = \{^\lambda_{\mu\nu}\}(\gamma) = \frac{1}{2}\gamma^{\lambda\rho}(\partial_\mu\gamma_{\nu\rho} + \partial_\nu\gamma_{\mu\rho} - \partial_\rho\gamma_{\mu\nu}), \tag{4.7}$$

i.e., the connection is the Christoffel symbol of the second kind for the tensor γ, which we have assumed to be regular. We can therefore write

$$R_{\mu\nu}(\Gamma) = R_{\mu\nu}(\gamma). \tag{4.8}$$

Furthermore

$$\langle R^2\rangle^{1/2} = \langle R^2\rangle^{-1/2}\ G^{\mu\nu\lambda\rho}R_{\mu\nu}R_{\lambda\rho} = \gamma^{\mu\nu}R_{\mu\nu}(\gamma) = R(\gamma)\cdot \tag{4.9}$$

Variation with respect to $G_{\mu\nu\lambda\rho}$

$$\frac{\delta\mathcal{L}}{\delta G^{\mu\nu\lambda\rho}} = \frac{\partial\mathcal{L}}{G^{\mu\nu\lambda\rho}} - d_\sigma\left[\frac{\partial\mathcal{L}}{\partial(\partial_\sigma G^{\mu\nu\lambda\rho})}\right] = 0\ , \tag{4.10}$$

gives

$$\frac{1}{2}\kappa_E\frac{1}{\langle R^2\rangle^{1/2}}[R_{(\mu\nu}R_{\lambda\rho)} - \frac{1}{2}\langle R^2\rangle G_{\mu\nu\lambda\rho}] = T_{\mu\nu\lambda\rho}\ , \tag{4.11}$$

where

$$T_{\mu\nu\lambda\rho} = -G^{-1/4}\ \frac{\delta\mathcal{L}_{MATTER}}{\delta G^{\mu\nu\lambda\rho}}\ , \tag{4.12}$$

is the fourth-rank analogous of the energy-momentum tensor. With the use of eq. (4.9) we can rewrite eq. (4.11) as

$$\frac{1}{2}\kappa_E\frac{1}{R}[R_{(\mu\nu}R_{\lambda\rho)} - \frac{1}{2}R\ G_{\mu\nu\lambda\rho}] = T_{\mu\nu\lambda\rho}\ . \tag{4.13}$$

More information can be obtained from eq. (4.13) by observing that the energy-momentum tensor must decompose in one part proportional to the metric and one part which is a separable tensor. In order to accommodate all the symmetries it is necessary to write

$$T_{\mu\nu\lambda\rho} = \frac{1}{2}\langle S_4^2\rangle^{-1/2}\ [S_{4,(\mu\nu}S_{4,\lambda\rho)} - \frac{1}{2}\langle S_4^2\rangle G_{\mu\nu\lambda\rho}]\ , \tag{4.14}$$

where

$$\langle S_4^2\rangle = G^{\mu\nu\lambda\rho}S_{4,\mu\nu}S_{4,\lambda\rho}\ . \tag{4.15}$$

In this case the field equations reduce to the simple form

$$\kappa_E\ R_{\mu\nu}(\gamma) = \pm\ S_{4,\mu\nu}\ . \tag{4.16}$$

and, as a further consequence we have

$$\kappa_E^2\ R^2(\gamma) = \kappa_E^2\ \langle R^2\rangle = \langle S_4^2\rangle = S_4^2(\gamma)\ , \tag{4.17}$$

where $S_4(\gamma) = \gamma^{\mu\nu}S_{4,\mu\nu}(\gamma)$. One would be tempted to replace $S_{4,\mu\nu}$ by the reduced energy-momentum tensor appearing in (3.14). However, that tensor was derived from a Lagrangian containing a metric $g_{\mu\nu}$, an object which is, in principle, absent in fourth-rank geometry. The ± sign is of main relevance for the applications.

4.2. The Different Energy Regimes

The field equations (4.13) exhibit three energy regimes: low, medium and high. In the low-energy regime there is no matter and therefore the geometry is Riemannian, $G_{\mu\nu\lambda\rho} = g_{(\mu\nu}g_{\lambda\rho)}$, as can be read from (4.13). In this case furthermore the field equations reduce to the Einstein field equations in vacuum. In the medium-energy regime the geometry is still Riemannian, $G_{\mu\nu\lambda\rho} = g_{(\mu\nu}g_{\lambda\rho)}$, but now there is matter in the game. This possibility is not excluded as a closer analysis of eqs. (4.13) reveals. Finally, we have the true high-energy regime in which there is matter and the geometry is truly fourth-rank.

4.2.1. The Low-Energy Regime

In the low-energy regime $\mathcal{L}_{MATTER}=0$ and then the field equations reduce to

$$\frac{1}{\langle R^2\rangle^{1/2}}[R_{(\mu\nu}R_{\lambda\rho)} - \frac{1}{2}\langle R^2\rangle G_{\mu\nu\lambda\rho}] = 0\ . \tag{4.18}$$

In order to further analyse this equation it is better to start from the Lagrangian

$$\mathcal{L}_{GRAV,\Lambda} = \kappa_E\ [\langle R^2\rangle^{1/2} + \Lambda]\ G^{1/4}, \tag{4.19}$$

containing a cosmological constant Λ. The original gravitational Lagrangian is obtained as

$$\mathcal{L}_{GRAV} = \mathcal{L}_{GRAV,0} = \lim_{\Lambda\to 0} \mathcal{L}_{GRAV,\Lambda} . \tag{4.20}$$

The field equations

$$\frac{\delta \mathcal{L}_{GRAV,\Lambda}}{\delta G^{\mu\nu\lambda\rho}} = 0 , \tag{4.21}$$

are now

$$\frac{1}{R} [R_{(\mu\nu} R_{\lambda\rho)} - \frac{1}{2} R (R + \Lambda) G_{\mu\nu\lambda\rho}] = 0 . \tag{4.22}$$

The only sensible solution is when the square bracket is zero. For this to be the case one should have

$$G_{\mu\nu\lambda\rho} = g_{(\mu\nu} g_{\lambda\rho)} , \tag{4.23a}$$

$$R_{\mu\nu} = \frac{1}{\sqrt{2}} [R (R + \Lambda)]^{1/2} g_{\mu\nu} . \tag{4.23b}$$

The contravariant fourth-rank metric is

$$G^{\mu\nu\lambda\rho} = \frac{1}{2} g^{(\mu\nu} g^{\lambda\rho)} . \tag{4.24}$$

The tensor $\gamma^{\mu\nu}$ is given by

$$\gamma^{\mu\nu} = [\frac{1}{2} \frac{1}{R} (R + \Lambda)]^{1/2} g^{\mu\nu} . \tag{4.25}$$

On the other hand

$$R^2 = \langle R^2 \rangle = 2 R (R + \Lambda) . \tag{4.26}$$

The previous is equivalent to

$$R (R + 2 \Lambda) = 0 . \tag{4.27}$$

We choose the solution

$$R = - 2 \Lambda . \tag{4.28}$$

Then

$$R_{\mu\nu} = - \Lambda g_{\mu\nu} . \tag{4.29}$$

Finally, the expression for $\gamma^{\mu\nu}$ reduces to

$$\gamma^{\mu\nu} = \frac{1}{2} g^{\mu\nu} . \tag{4.30}$$

The field equations reduce, in the limit $\Lambda\to 0$, to

$$R_{\mu\nu}(g) = 0 . \tag{4.31}$$

Other limit procedures can produce a different numerical factor in (4.30). However, the Ricci tensor is independent of this factor such that (4.31) is unchanged.

One can conclude that this theory coincides with General Relativity only in the vacuum case and that departures will be obtained only in the presence of matter.

4.2.2. The Medium-Energy Regime

In the medium-energy regime the metrics $G_{\mu\nu\lambda\rho}$ and $g_{\mu\nu}$ are related through (2.11). In this case it is therefore reasonable to replace $S_{4,\mu\nu}$ with that appearing in (3.14)

$$K_E R_{\mu\nu}(\gamma) = \pm S_{2,\mu\nu}(g) . \tag{4.32}$$

However, the field equations (4.32) are not equivalent to Einstein field equations since the Ricci tensor appearing there is for the tensor $\gamma_{\mu\nu}$ and not for the metric $g_{\mu\nu}$. The previous choice is a delicate point since other mechanisms of coupling the fourth-rank geometry with "second-rank" matter can be conceived. For example one can consider

$$K_E G_{\mu\nu}(\gamma) = \pm T_{2,\mu\nu}(g) , \tag{4.33}$$

which is not equivalent to (4.32). We have explored this and other possibilities and we have concluded that (4.32) is the correct choice. Which sign is to be chosen in (4.32) must be decided by considering some application.

The large scale geometrical structure of our universe seems to be well described by Riemannian geometry, and since matter is involved, its description belongs to the previously mentioned medium-energy regime. We explore this possibility in section 7.

5. Cosmography

In this section we collect the observational results concerning the structure of the universe and its evolution. Further details can be found in [7, 8, 9, 13, 14].

The observed isotropy and homogeneity of the universe gives as the only possible Riemannian geometry for the universe a Friedman-Robertson- Walker (FRW) geometry. FRW spaces are characterised by the cosmic radius a(t) and by the constant k=1,0,-1, corresponding to a closed, spatially flat, and open universe, respectively. The curvature properties of a FRW geometry can be rewritten in terms of the cosmological parameters H, the Hubble constant, and q, the deacceleration parameter, which can, in principle, be determined from the observed distance versus redshift Hubble diagram [7, 13, 14].

The observed redshift of galaxies shows that the universe was very dense in the past and that it has evolved to the present diluted state. Since matter cannot be compressed beyond the Planck density one must consider the universe as evolving from an initial hot ball. One can furthermore estimate the radius of the initial hot ball to be $a_0=10^{60} L_{PLANCK}$.

In order to have compatibility with the observed homogeneity and isotropy of the universe, cosmic matter must be described as a perfect fluid. The perfect fluid is characterised by ρ, the energy density, and p, the pressure, and they are related by the state equation of matter, $p/\rho=y$. For $y=1/3$ one has a radiation dominated, or ultrarelativistic, perfect fluid; for $y\approx 0$ one has instead a non-relativistic, or almost pressureless, perfect fluid. Based on the observation one can therefore conclude that the universe has evolved from a relativistic to a non-relativistic regime.

In the framework of the FRW geometry one can introduce a critical density parameter ρ_c which sets the scale of energy densities. One can then introduce the cosmic density parameter $\Omega=\rho/\rho_c$.

The parameters H, q and Ω are observable and any proposed cosmological theory must be confronted with these observed values.

Observation shows isotropy at large scales. In order to exhibit this behaviour the different regions of the universe have to be in causal contact from the beginning. Observation shows furthermore that our universe is almost spatially flat, k=0, i.e., its geometry is almost Euclidean.

Any proposed cosmological model must agree with the previously described observational results.

5.1. Isotropy, the Cosmological Principle and FRW Spaces

The observation shows that the universe is isotropic and homogeneous. These facts give as the only possible Riemannian geometry a FRW metric. In this case the line element is

$$ds^2 = dt^2 - a^2(t)\, d\ell^2 , \qquad (5.1)$$

where

$$d\ell^2 = (1 - k r^2)^{-1} dr^2 + r^2 d\Omega^2 , \qquad (5.2a)$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 . \qquad (5.2b)$$

In the previous a(t) is the cosmic scale factor and $d\ell^2$ is the line element of a maximally symmetric three-dimensional space-like section. The radial coordinate r is written in units such that the constant k takes the values 1, 0 or -1. The parameter k characterises the geometry of the space-like sections of the universe. For k=1 the universe is closed; for k=0 it is flat; for k=-1 it is open.

The Ricci tensor is given by

$$R_{\mu\nu} = -\frac{2}{a^2}[a\ddot{a} - (k + \dot{a}^2)]\, \delta^0_\mu \delta^0_\nu - \frac{1}{a^2}[a\ddot{a} + 2(k + \dot{a}^2)]\, g_{\mu\nu} . \qquad (5.3)$$

Hence, the scalar curvature is

$$R = -\frac{6}{a^2}[a\ddot{a} + (k + \dot{a}^2)] . \qquad (5.4)$$

These quantities can be parametrised in terms of the cosmological parameters

$$H = \frac{\dot{a}}{a} , \qquad (5.5a)$$

which is the Hubble "constant", and which is a true constant only for a de Sitter space; and

$$q = -\frac{a\ddot{a}}{\dot{a}} = -1 - \frac{\dot{H}}{H^2} , \qquad (5.5b)$$

which is the deacceleration parameter. The importance of this parametrisation resides in the fact that H and q can be determined from the observed Hubble diagram.

5.2. Observed Values of the Cosmological Parameters

The situation with the observed values of the cosmological parameters is not a happy one. In fact, due to several practical difficulties, they are quite inaccurate. In some cases this inaccuracy does not allow to discriminate the validity of different cosmological models.

It must be emphasized that for our purposes we need values of the cosmological parameters determined in a model independent way. The Hubble diagram must at the same time provide the Hubble constant, H, and the deacceleration parameter, q. However, the Hubble diagram shows a big dispersion for big values of H such that no relaible value for q exists today. The energy density is determined from the cosmic virial theorem and the infall to the Virgo cluster [7, 13].

There exists a wide range of reported values for the cosmological parameters [7, 8, 14] depending on both the nature of the performed observation and the interpretation of the observed data. We use the values reported in [7]

$$h_{OBS} = 0.5 - 1.0 \,, \qquad (5.6)$$

with preferred values closer to 0.5 than to 1.0. The determination of q from deviations from the linear Hubble law is almost imposible with the present day accuracy of the existing observations.

Since H is positive one can conclude that the universe has been evolving from a very dense state to a very diluted present universe.

Since matter cannot be compressed beyond the Planck density one can estimate the radius of the initial hot ball universe.

5.3. The Matter Content of the Universe. The Perfect Fluid

In order to be compatible with the observed homogeneity and isotropy of the universe, cosmic matter must be described as a perfect fluid.

A perfect fluid is characterised by the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p\, g_{\mu\nu} \,, \qquad (5.7)$$

where ρ and p are the energy density and pressure of cosmic matter, and

$$u_\mu = (g_{oo})^{1/2}\, \delta^o_\mu \,, \qquad (5.8a)$$

such that

$$g^{\mu\nu} u_\mu u_\nu = 1 \,. \qquad (5.8b)$$

The reduced energy-momentum tensor, defined in (3.15), is

$$S_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - \frac{1}{2} (\rho - p)\, g_{\mu\nu} \,. \qquad (5.9)$$

In order to relate the energy density ρ and the pressure p one needs a state equation. Two well understood regimes are the ultra-relativistic regime, radiation dominated, in which $y=p/\rho=1/3$, and the non-relativistic regime, "dust" matter dominated, in which $y\approx0$.

The coupling of gravity to matter needs the Einstein gravitational constant k_E. Combining this with the FRW geometry we obtain the critical density

$$\rho_c = 3\, k_E H^2 = 1.96 \times 10^{-28}\, h^2\, g/cm^3 \,, \qquad (5.10)$$

where

$$h = \frac{H}{100km/sec/Mpc} \,. \qquad (5.11)$$

This leads to the introduction of the cosmic energy density parameter

$$\Omega = \rho/\rho_c \,, \qquad (5.12)$$

in such a way that ρ_c sets the scale of energy densities.

The observed value for the energy density ratio is

$$\Omega_{OBS} \approx 0.1 - 0.3 \,. \qquad (5.13)$$

There is furthermore a safety upper bound

$$\Omega_{SAFE} \approx 0.18 \,. \qquad (5.14)$$

From the microwave background radiation one observes that the present value of the total entropy in the universe is so large as to be of order 10^{87} [8, 9, 10]. On the other hand one would expect entropy to be governed by the third thermodynamical principle, the statement that the entropy is always a non-decreasing function of time. One is therefore faced to the problem of determining whether the entropy of the universe has always been as large as it is today, dS=0, or if it has evolved from a smaller value. From an intuitive point of view it is quite unprobable that dS=0.

Now, with the observed values of Ω and H we can estimate a_o as follows. The mass of the universe is assumed to be a conserved quantity. At t=0 we assume that mass was compressed at Planck density. This allows to determine a_o to be

$$a_o = \left[\frac{M_{UNIV}}{M_{PLANCK}}\right] L_{PLANCK} \,. \qquad (5.15)$$

We have estimated the mass of the universe to be

$$M_{UNIV} \approx \rho_c c^3 H^{-3} \approx 10^{60} M_{PLANCK} \,. \qquad (5.16)$$

Therefore

$$a_o \approx 10^{20} L_{PLANCK} \,. \qquad (5.17)$$

We can furthermore introduce a time scale

$$t_{CLAS} = a_o/c \approx 10^{20}\, t_{PLANCK} \approx 10^{-23}\, s \,. \qquad (5.18)$$

In all our estimations this is the time at which quantum effects are no more important and one enters into a classical regime.

Our estimate contains an error of a few orders of magnitude which is not relevant to our analysis.

5.4. Flatness and Causality

Further observations show that our present day universe is almost flat, i.e., its geometry is almost Euclidean; this means that

$$k_{OBS} = 0 \ . \qquad (5.19)$$

The other observation concerns the isotropy of the universe over large regions of space; this means that all regions were causally connected in the past. For this to be the case one should have

$$r_H > a \ , \qquad (5.20)$$

where r_H is the horizon radius which sets the size of the region in which causal contact can be achieved.

For a FRW space the horizon radius is

$$r_H(t) = a(t) \int_0^t \frac{du}{a(u)} , \qquad (5.21)$$

which is the maximum distance that light signals can travel during the age t of the universe.

6. The Standard Model of Cosmology

The standard model of cosmology is based on the Einstein field equations. They provide the coupling of gravity, or geometry, given by a FRW metric, to matter, described by a perfect fluid.

The Einstein-Friedman equations are equivalent to

$$\rho = 3\, k_E \frac{1}{a^2} (k + \dot{a}^2) > 0 \ , \qquad (6.1a)$$

$$p = -\, k_E \frac{1}{a^2} [2\, a\, \ddot{a} + (k + \dot{a}^2)] \ . \qquad (6.1b)$$

From them one can easily deduce that

$$T\, dS = dU + p\, dV = 0 \ . \qquad (6.2)$$

Therefore, the standard model of cosmology predicts that the expansion of the universe is an adiabatic isentropic process such that there is no entropy production.

In terms of the cosmological parameters eqs. (6.1) can be rewritten as

$$\Omega = 1 + \frac{k}{\dot{a}^2} , \qquad (6.3a)$$

$$q = \frac{1}{2} (1 + 3\, y)\, \Omega \ . \qquad (6.3b)$$

In order to verify the validity of the previous equations we will consider them for the present and the early universe.

6.1. The Present Universe

The observed values of Ω and k, (5.13), (5.14) and (5.19), do not fit into eq. (6.3a). There exist two possibilities in front to this *impasse*. The first one consists into assuming k_{OBS}=0, Ω_{PRED}=1. This is preferred by some authors for "aesthetic or philosophical reasons" [7]. This takes us to the "missing mass" problem.

The second possibility is to accept Ω_{OBS}<1, then one should have k_{PRED}=-1, as deduced from (6.3a), which corresponds to an open universe. If the non-relativistic regime has lasted for quite a long time then we can integrate the field equations (6.1b) to

$$a\, (k + \dot{a}^2) = \alpha > 0 \ , \qquad (6.4)$$

where α is an integration constant. Since we assume k=-1, it is not possible to have $\dot{a}$=0, i.e., there is no maximum and the universe will expand forever. A further integration gives

$$t = \lambda + (a^2 + \alpha\, a)^{1/2} - \frac{\alpha}{2} \ln[2\, (a^2 + \alpha\, a)^{1/2} + 2\, a + \alpha] \ , \qquad (6.5)$$

where λ is a further integration constant. For very large a

$$t \approx a \ , \qquad (6.6)$$

such that

$$\dot{a} \approx 1 \ . \qquad (6.7)$$

Then Ω goes to the value

$$\Omega_\infty \approx 0 \ . \qquad (6.8)$$

Hence, it is clear that the observed cosmological data do not fit into the field equations and that even under strong assumptions on the observed values of the cosmological parameters the situation cannot be much improved.

6.2. The Early Universe

At early times matter is described by the state equation y=1/3. In this case the Einstein field equations reduce to

$$a \ddot{a} + (k + \dot{a}^2) = 0 . \qquad (6.9)$$

The solution is

$$a = (\alpha t + a_0^2)^{1/2} , \qquad (6.10a)$$

$$a = (c^2 t^2 + a_0^2) , \qquad (6.10b)$$

for k=0,-1, respectively. The horizon radii are given by

$$r_H = 2 \frac{c}{\alpha} (\alpha t + a_0^2)^{1/2} [(\alpha t + a_0^2)^{1/2} - a_0] , \qquad (6.11a)$$

$$r_H = a_0 (1 + x^2)^{1/2} \ln[x + (1 + x^2)^{1/2}] , \qquad (6.11b)$$

where $x=ct/a_0$. In both cases causality is not violated for $t>t_{CLAS}$. Our result differs from the standard one since we have considered a universe evolving from an initial hot ball with a finite radius rather than from an initial singularity.

6.3. Comments

Since $\Omega_{OBS}<1$ one should expect an open universe with $k_{PRED}=-1$. However, the large scale structure of the universe tells us that k is rather close to zero so that Ω must be close to unity; but this is more or less excluded by the cosmological data [7]. This ambiguous situation is one of the most serious puzzles of the standard model of cosmology known as the flatness problem.

One can easily conceive the local isotropy of the universe however the global isotropy remains inconceivable since, in the standard model of cosmology, regions separated by distances exceeding the light horizon, i.e. causally disconnected regions in the universe, cannot be correlated with each other. Researchers dealing with the microwave background radiation show that 10^5 years after its birth the universe was isotropic at a scale by far larger than the size of causal contact r_H [7, 8, 14]. This violates for sure the causality principle and this is another ambiguous puzzle of the standard model, the so called horizon or causality problem.

The previous inconsistencies can be removed if either the true value of the Hubble constant h_{OBS} is less than that implied by the present

observations or if the true value of the energy density parameter Ω_{OBS} is greater than the observed one. The former possibility is related to the incompleteness of the hot big bang theory while the latter asks for more accurate measurements. The inflationary universe scenario deals with the first scheme; cf. [8, 9, 10] for a detailed account on inflationary cosmology and further references.

Another question the standard model is unable to answer concerns the problem of the large total entropy in the universe, a problem in fact related to the flatness problem [8, 9, 10]. One of the predictions of standard cosmology is that the expansion of the universe is an adiabatic isentropic process. Therefore, from the point of view of standard cosmology the entropy of the universe has always been as large as it is today. If the inflationary scenario solves, among others, the flatness and causality problems, it is because it predicts a brief reheating period that follows the supercooling, inflation, and during which the entropy is created.

7. Fourth-Rank Cosmology

The large scale geometry of the universe seems to be Riemannian. Furthermore there is matter present. Therefore, in the context of fourth-rank gravity, the description of the cosmic properties of the universe belongs to the previously mentioned medium-energy regime of our model. Therefore the metric $G_{\mu\nu\lambda\rho}$ will be separable in terms of a metric $g_{\mu\nu}$ which we assume to be the FRW metric. Matter is described by a perfect fluid; therefore we use the same energy-momentum tensor appearing in (5.7).

When fourth-rank gravity is applied to cosmology one should deal with the equations analogous to the Einstein-Friedman equations of standard cosmology. In fourth-rank gravity however matter enters the field equations in a non-linear way. An essential difference with respect to General Relativity is the fact that the equations determining the evolution of the universe involve not only the energy density and the pressure but also their time derivatives. Therefore in order to correctly deal with these equations one should provide a time dependent state equation. However, the two extreme situations we are interested in, *viz.* the ultrarelativistic and the non-relativistic regimes, had lasted for such a long time that we can confidently work under the assumption of a time independent state equation.

In order to be consequent with $k_{OBS}=0$ we must assume q<0. In this case the flatness and horizon problems do find a solution. Our approach is however restricted to $y<0.236$. Therefore, a relativistic regime $y=1/3$ is excluded. We consider an early universe in which matter is described by the state equation $y=0.236$.

In General Relativity $q>0$ and $a(t)$ is a convexe function of t. From this fact one deduces the existence of a singularity at some time in the past. This qualitative prediction of General Relativity is considered a success of the theory. However, an evolution from an initial singularity may also be conceived with a concave function, $q<0$, as one can see from the figure.

If we insist in keeping $q>0$, and in order to agree with $k_{OBS}=0$, we must assume $p<0$. The generalised Friedman equations are compatible with the observed values of Ω and k, so that the flatness problem can be overcome. For early times the radius of the universe grows linearly while the horizon radius grows faster for a time bigger than t_{CLAS}. At earlier times quantum gravity effects become important and our classical approach breaks down such that there is no violation of causality.

Since our model can find solution for both the flatness and horizon problems it appears that this treatment contains in a natural way the basic ingredients of both the standard model of cosmology and inflation. In the language of inflationary cosmology we can therefore reinterpret our assumption of negative pressure as related to the presence at early times of a cosmological constant.

7.1. Preliminaries

The field equations are

$$\kappa_E R_{\mu\nu}(\gamma) = \pm [(\rho + p)\, u_\mu u_\nu - \frac{1}{2}(\rho - p)\, g_{\mu\nu}] \, . \tag{7.1}$$

From here one directly obtains

$$\kappa_E R = \kappa_E \langle R^2\rangle^{1/2} = \kappa_E (G^{\mu\nu\lambda\rho} R_{\mu\nu} R_{\lambda\rho})^{1/2} = \kappa_E (\tfrac{1}{2} g^{\mu\nu} g^{\lambda\rho} R_{\mu\nu} R_{\lambda\rho})^{1/2}$$

$$= \frac{1}{\sqrt{2}} (\rho^2 - 2\, p\, \rho + 5\, p^2)^{1/2} \, . \tag{7.2}$$

On the other hand we have

$$\gamma^{\mu\nu} = \frac{1}{R} G^{\mu\nu\lambda\rho} R_{\lambda\rho} = \frac{1}{3\kappa_E R} [(\rho + p)\, u^\mu u^\nu - (\rho - 2p)\, g^{\mu\nu}] \, . \tag{7.3}$$

The inverse of (7.3) is given by

$$\gamma_{\mu\nu} = D\, u_\mu u_\nu - C\, g_{\mu\nu} \, , \tag{7.4}$$

where

$$C = \frac{3}{\sqrt{2}} \frac{(1-2y+5y^2)^{1/2}}{(1-2y)} \, , \tag{7.5a}$$

$$D = \frac{1}{\sqrt{2}} \frac{(1+y)(1-2y+5y^2)^{1/2}}{y(1-2y)} \, . \tag{7.5b}$$

Further useful quantities are

$$J = D - C = \frac{1}{\sqrt{2}} \frac{1}{y} (1 - 2\, y + 5\, y^2)^{1/2} \, , \tag{7.6}$$

$$(1 + 3\, y)\, C - 3\, (1 - y)\, J = -\, 3\, \frac{(1-4y-y^2)}{(1-2y)} J \, . \tag{7.7}$$

There is a global $\pm$ sign in (7.2)-(7.4) which we have fixed at will since the final result is independent of this choice.

The generalised Friedman equations will agree with $k_{OBS}=0$ if $q<0$, $p>0$, or $p<0$, $q>0$.

7.2. Positive Pressure

The first step is to calculate the Ricci tensor for the metric $\gamma_{\mu\nu}$. Let us start by writing the corresponding line element

$$ds^2 = J\, dt^2 + C\, a^2\, d\ell^2 \, . \tag{7.8}$$

Now we introduce the new time coordinate

$$d\tau = J^{1/2}\, dt \, . \tag{7.9}$$

Then the line element is written as

$$ds^2 = d\tau^2 + A^2\, d\ell^2 \, , \tag{7.10}$$

with

$$A = [C^{1/2}\, a](\tau) \, . \tag{7.11}$$

The previous is nothing else but a FRW line element with Euclidean signature: we can therefore use eqs. (5.3) with $a^2 \to -A^2$. The Ricci tensor is then given by

$$R_{\mu\nu}(\gamma) = -\frac{2}{A^2} [A\, A'' - (-\, k + A'^2)]\, u_\mu u_\nu$$

$$-\frac{1}{A^2} [A\, A'' + 2\, (-\, k + A'^2)]\, \gamma_{\mu\nu} \, , \tag{7.12}$$

where primes denote derivatives with respect to τ. In the system of coordinates involving t the previous expression is given by

$$R_{\mu\nu}(\gamma) = -\frac{2J}{A^2}[A\,A'' - (-k + A'^2)] \qquad \delta^0_\mu \delta^0_\nu$$

$$-\frac{1}{A^2}[A\,A'' + 2\,(-k + A'^2)]\,\gamma_{\mu\nu}\,, \tag{7.13}$$

Comparison with the Ricci tensor obtained from the field equations, eq. (7.1), gives

$$-3\,k_E\,J\,\frac{A''}{A} = \pm\frac{1}{2}(1 + 3\,y)\,\rho\,, \tag{7.14a}$$

$$-k_E\,\frac{C}{A}[A\,A'' + 2\,(-k + A'^2)] = \pm\frac{1}{2}(1 - y)\,\rho\,. \tag{7.14b}$$

These field equations can be rewritten as

$$6\,k_E\,C\,\frac{1}{A^2}(-k + A'\ ) = \mp\frac{3}{2}\frac{(1-4y-y^2)}{(1-2y)}\rho, \tag{7.15a}$$

$$-3\,\frac{(1-4y-y^2)}{(1-2y)}\,J\,A\,A'' + 2\,(1 + 3\,y)\,C\,(-k + A'^2) = 0\,, \tag{7.15b}$$

where we have used (7.7). The field equations written in this form are of practical use since the first one allows to determine the value of k when evaluated at the present time. The second one allows to determine the evolution of the early universe.

The entropy variation is governed by

$$T\,dS = \frac{(1+3y)(1-y^2)}{(1-4y-y^2)}\,\rho\,a^2\,da + \rho\,a^3\,\frac{2(1+8y^2+16y^3-5y^4)}{(1-2y)(1-2y+5y^2)(1-4y-y^2)}\,dy\,. \tag{7.16}$$

Since the radius of the universe grows at a rate much larger that that by which y decreases, the previous quantity is positive. Hence the model predicts, in a natural way, an increasing total entropy of the universe. Thus, fourth-rank cosmology predicts an adiabatic non-isentropic, and therefore irreversible, expansion of the universe.

The previous field equations can be easily solved when y behaves like a constant.

7.2.1. Constant y

In the two physical regimes in which we are interested y behaves like a constant. In this case the relevant equations are obtained with the simple replacements

$$A \to C^{1/2}\,a\,, \tag{7.17a}$$

$$(\,)' \to J^{-1/2}\,(\,)^{\cdot}\,. \tag{7.17b}$$

In this case eq. (7.14a) is rewritten like

$$-3\,k_E\,\frac{\ddot{a}}{a} = \pm\frac{1}{2}(1 + 3\,y)\,\rho\,. \tag{7.18}$$

Equations (7.15) reduce to

$$6\,k_E\,\frac{1}{a^2}(-k + \frac{C}{J}\dot{a}^2) = \mp\frac{3}{2}\frac{(1-4y-y^2)}{(1-2y)}\,\rho\,, \tag{7.19a}$$

$$-3\,\frac{(1-4y-y^2)}{(1-2y)}\,a\,\ddot{a} + 2\,(1 + 3\,y)\,(-k + \frac{C}{J}\dot{a}^2) = 0\,. \tag{7.19b}$$

Equations (7.18) and (7.19a) can be rewritten in terms of the cosmological parameters as

$$q = \pm\frac{1}{2}(1 + 3\,y)\,\Omega\,, \tag{7.20}$$

$$\frac{4}{3}(-\frac{k}{\dot{a}^2} + \frac{C}{J}) = \mp\frac{(1-4y-y^2)}{(1-2y)}\,\Omega\,, \tag{7.21}$$

respectively.

Let us observe that eq. (7.20) with the upper sign is the same we obtain in General Relativity, *viz.*, (6.3b). That equation has lead to the common belief that q>0. From this fact one deduces that the universe has evolved from an initial singularity. Since matter cannot be compressed beyond the Planck density it is more reasonable to consider an initial hot ball with finite radius. The existence of such hot ball is not excluded if q<0, as can be seen from the figure. Furthermore, the present observational status of the Hubble diagram does not allow to determine the sign of q.

7.2.2. The Present Universe

In order to obtain compatibility with k_{OBS}=0 we must choose the lower sign. In this case (7.21) reduces to

$$\Omega = \frac{4y}{1-4y-y^2}\,. \tag{7.22}$$

This quantity is positive for $0 \leq y < \sqrt{5}-2 \simeq 0.236$, and negative for y>0.236. Therefore, this model is applicable only for y<0.236. The previous equation can be inverted to

$$y = \frac{1}{\Omega}\left[\sqrt{4\,(1+\Omega)^2 + \Omega^2} - 2\,(1+\Omega)\right]. \tag{7.23}$$

The other equation reduces to

$$(1 - 4\,y - y^2)\, a\,\ddot{a} - 2\,y\,(1 + 3\,y)\,\dot{a}^2 = 0\,, \tag{7.24}$$

and can be applied to the early universe too.

For the observed values of Ω we obtain

$$y_{0.1} = 0.023\,, \tag{7.25a}$$

$$y_{0.3} = 0.058\,, \tag{7.25b}$$

$$y_{0.18} = 0.038\,. \tag{7.25c}$$

These are good bounds for almost pressureless perfect fluids.

For small y eq. (7.24) reduces to

$$\ddot{a} \approx 0\,, \tag{7.26}$$

therefore

$$a = \alpha\, t + \beta\,, \tag{7.27}$$

where α and β are integration constants.

7.2.3. The Early Universe

The early universe is more delicate to deal with. Since our approach cannot be applied to y=1/3 we must consider the early universe as described by $y \approx 0.236$. In this case we obtain $\dot{a}=0$ and it is clear that this corresponds to the minimum radius of the universe. The solution is

$$a = a_0\,. \tag{7.28}$$

The horizon radius is

$$r_H = c\,t\,. \tag{7.29}$$

Causality is not violated for $t > t_{CLAS}$.

7.3. Negative Pressure

The calculation for p<0 is almost identical to that for p>0. Therefore, we just show the results and conclusions. Since one is used to think of the evolution of the universe in terms of q>0 this was the case we favoured in

our previous works [15, 16]. In this case the field equations are compatible with flatness only if p<0. The entropy is again an increasing function of time. The analogous of (7.22) is

$$\Omega = -\frac{4y}{1-4y-y^2}\,. \tag{7.30}$$

The previous equation provides a state equation for matter. In fact this equation can be inverted to

$$y = -\frac{1}{\Omega}\left[\sqrt{4\,(1-\Omega)^2 + \Omega^2} - 2\,(1-\Omega)\right]. \tag{7.31}$$

For the reported values of Ω we obtain

$$y_{0.1} = -\,0.028\,, \tag{7.32a}$$

$$y_{0.3} = -\,0.10\,, \tag{7.32b}$$

$$y_{0.18} = -\,0.055\,. \tag{7.32c}$$

The lower bound corresponds approximately to non-relativistic cosmic matter, which is in agreement with present observations. The upper bound is to high to describe non-relativistic matter. In this case one has also a linear growing of the radius of the universe

$$a = \alpha\, t + \beta\,, \tag{7.33}$$

where α and β are integration constants.

For the early universe one can put y=-1/3. One then obtains

$$\ddot{a} = 0\,. \tag{7.34}$$

The solution is therefore

$$a = a_0\,(1 + v\,t/a_0)\,, \tag{7.35}$$

where v can be thought of as the initial velocity of expansion of the universe.

The horizon radius is given by

$$r_H = a_0\,\frac{c}{v}\,(1 + v\,t/a_0)\,\ln(1 + v\,t/a_0)\,; \tag{7.36}$$

causality is then not violated for $t > t_{CLAS}$.

8. Fourth-Rank Cosmology at the Ligth of General Relativity

We have seen that this model is able to solve the flatness and horizon problems. The two previous problems are usually approached through General Relativity plus some inflationary mechanism. In order to emphasize more this point of view we will try to reinterpret our field equations at the light of General Relativity.

The Ricci tensor for $\gamma_{\mu\nu}$ can be decomposed as

$$R_{\mu\nu}(\gamma) = R_{\mu\nu}(g) + Q_{\mu\nu} . \tag{8.1}$$

The extra term $Q_{\mu\nu}$ can be brought to the right-hand side of the field equations and absorbed in the energy-momentum tensor.

This allows to interpret the extra term as the effect of a field driving inflation. This is due to the fact that fourth-rank gravity behaves, at least in what concerns the horizon and flatness problems, like General Relativity plus inflation.

For the Ricci tensor we have

$$\begin{aligned} R_{\mu\nu}(\gamma) &= -\frac{2J}{A^2}[A A'' - (-k + A'^2)]\,\delta^0_\mu \delta^0_\nu - \frac{1}{A^2}[A A'' + 2(-k + A'^2)]\,\gamma_{\mu\nu} \\ &= [-(2J + D)\frac{A''}{A} + \frac{2C}{A^2}(-k + A'^2)]\,\delta^0_\mu \delta^0_\nu \\ &\quad - \frac{C}{A^2}[A A'' + 2(-k + A'^2)]\,g_{\mu\nu} . \end{aligned} \tag{8.2}$$

In the almost constant y regime the Ricci tensor reduces to

$$\begin{aligned} R_{\mu\nu}(\gamma) &= [-(2 + \frac{D}{J})\frac{\ddot{a}}{a} + \frac{2}{a^2}(-k + \frac{C}{J}\dot{a}^2)]\,\delta^0_\mu \delta^0_\nu \\ &\quad - \frac{1}{a^2}[\frac{C}{J} a \ddot{a} + 2(-k + \frac{C}{J}a^2)]\,g_{\mu\nu} . \end{aligned} \tag{8.3}$$

We can now easily evaluate $Q_{\mu\nu}$ which is given by the simple expression

$$Q_{\mu\nu} = \frac{D}{J}(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2})(g_{\mu\nu} - \delta^0_\mu \delta^0_\nu) = \frac{1+y}{1-2y}(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2})(g_{\mu\nu} - \delta^0_\mu \delta^0_\nu) \tag{.4}$$

The trace is given by

$$Q = 3\frac{1+y}{1-2y}(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}) . \tag{8.5}$$

Then we can define

$$\tilde{Q}_{\mu\nu} = Q_{\mu\nu} - \frac{1}{2} Q g_{\mu\nu} = -\frac{1+y}{1-2y}(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2})[\delta^0_\mu \delta^0_\nu + \frac{1}{2} g_{\mu\nu}] . \tag{8.6}$$

The tensors $R_{\mu\nu}(g)$, $S_{\mu\nu}$ and $\tilde{Q}_{\mu\nu}$ are all of the same order at any epoch of the evolution of the universe. Therefore, even when they decrease with increasing time, the term $\tilde{Q}_{\mu\nu}$ will persist and no recover of the standard

model is possible.

The previous result can be interpreted at the light of inflationary cosmology. The contribution of $Q_{\mu\nu}$ to the energy-momentum tensor can be seen as a vacuum potential energy in the early universe. This, in fact, is one of the most relevant features of inflation.

9. Concluding Remarks

Some of the results we have obtained are quite unexpected and surprising. Among them we have the fact that the gravitational theory developed in section 3 coincides with General Relativity only in the vacuum case and differs from it when matter is present; this is the regime in which General Relativity fails in its predictions. When applied to cosmology we obtain no contradiction with observation.

In this paper we focused our attention on some of the cosmological aspects of fourth-rank gravity. Our main result is the fact that in the medium-energy regime fourth-rank gravity combined with the cosmological principles can provide us with a good description of both, the early and the present day universe, a description at least free from the horizon and flatnes problems. These two among other problems of the standard hot big-bang model have puzzled physicists for many years. A satisfactory solution to these puzzles is contained in the clever idea of inflation. We expect the model proposed here to be an alternative to the solution of the previous difficulties with the standard model of cosmology.

We have shown that fourth-rank gravity is compatible with cosmological observations in the presence of cosmic matter with positive pressure. This posibility was excluded in our previous work[16] since we were induced to conceive the presence of a singularity in the past only in the case of a positive deacceleration parameter, $q>0$. A negative deacceleration parameter, $q<0$, does not exclude the existence of such a singularity in the past. The model predicts in a natural way an increasing total entropy of the universe. The only restriction on the model is its applicability only for $p/\rho<0.236$, therefore we consider an early universe in which the state equation of matter is $p/\rho=0.236$. The model is furthermore free from the flatness and horizon problems. For the present time, $\Omega_{SAFE}=0.18$, our model predicts a state equation $p/\rho=0.038$, which corresponds to an almost pressureless perfect fluid.

Other problems of the standard model are partially, if not totally, overcome in the framework of inflationary cosmology and had not been considered in this paper. The reason for this lack is that most of these problems rised when applying particle physics to cosmology; this is the case of the primordial monopoles, domain walls, gravitinos and strings. Therefore, in order to consider these problems in the context of fourth-rank cosmology one has first to look for a self-consistent field theory in such "curved space-times" used in fourth-rank gravity.

Another interesting problem which we have not considered here is the cosmological constant problem. However, in order to be considered consistently one should study fourth-rank gravity in the high-energy regime, i.e., one should construct and "extended quantum cosmology".

Finally, we should emphasize the question concerning the analogous in fourth-rank gravity of the equivalence principle. Since in the low and medium-energy regimes the geometry of this model reduces somehow to the Riemannian geometry, one can claim that there is an answer.

We hope further investigation will follow and give satisfactory improvements on the presently existing approaches.

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Figure caption

How a universe with q<0 can evolve from an initial hot ball with radius a_0 to a universe with constant expansion rate, $a \propto t$. The usual case, q>0, is included for comparison reasons.