IC/92/133

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

nth ROOTS OF NORMAL CONTRACTIONS

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UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

MIRAMARE-TRIESTE

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and
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ABSTRACT

Given a complex separable Hilbert space H and a contraction A on H such that A^n, $n \geq 2$ some integer, is normal it is shown that if the defect operator $D_A = (1 - A^*A)^{1/2}$ is of the Hilbert–Schmidt class, then A is similar to a normal contraction, either A or A^2 is normal, and if A^2 is normal (but A is not) then there is a normal contraction N and a positive definite contraction P of trace class such that $\|A - N\|_1 = \frac{1}{2}\|P \oplus P\|_1$ (where $\|\cdot\|_1$ denotes the trace norm). If T is a compact contraction such that its characteristic function admits a scalar factor, if $T = A^n$ for some integer $n \geq 2$ and contraction A with simple eigen–values, and if both T and A satisfy a "reductive property", then A is a compact normal contraction.

MIRAMARE – TRIESTE
July 1992

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1. INTRODUCTION

Let $B(H)$ denote the algebra of operators, i.e. bounded linear transformations, on a complex separable Hilbert space H into itself. Given normal $T \in B(H)$, the spectral theorem (and the functional calculus) for normal operators assures the existence of an $A \in B(H)$ such that $A^n = T$ for every integer $n \geq 2$. The operator A may not, however, be normal. The problem of finding sufficient conditions on A for A to be normal has been considered by a number of authors, amongst them Embry [4], Putnam [9], Radjavi and Rosenthal [11] and Stampfli [13]. Thus, for example, if zero is not in the numerical range of A and $n = 2$, then A is normal [9]; if zero does not belong to the spectrum of A, then A is similar to a normal operator [13]; and if the spectra of A and $\exp\left(\frac{2\pi i k}{n}\right) A$ have empty intersection for $k = 1, 2, \ldots, n-1$, then A is normal [4]. Restricting ourselves to those contractions A for which the defect operator $D_A = (1 - A^*A)^{1/2}$ is of Hilbert–Schmidt class, it was shown in [3] that if A satisfies a "reductive property", then A is a normal contraction of type $C_{11} \oplus C_0$. In this note we consider the case in which this "reductive property" is not satisfied. It is shown that A is then similar to a normal contraction (of type $C_{11} \oplus C_0$). Moreover, either A or A^2 is normal, and if A^2 is normal (but A is not), then there exists a normal contraction N and a positive definite contraction P of trace class such that $\|A - N\|_1 = \frac{1}{2}\|P \oplus P\|_1$, where $\|\cdot\|_1$ denotes the trace norm. As a consequence it is shown that if either there exists a line through the origin in the complex plane with a singleton set intersection with the spectrum of the pure part of A, or, the spectrum of the pure part of A lies in the sector $0 \leq arg\ \lambda \leq \frac{\pi}{2}$, then the pure part of A acts on the trivial space (and so A is normal). We also consider the case of a general contraction A, given that the 'characteristic function' of A satisfies a (particular) property, and the case in which T is compact, not necessarily normal, and the 'characteristic function' of T admits a 'scalar factor'.

The technique that we use in the proof of these results depends upon the results of Gilfeather [6] on the structure of nth roots of normal operators, and is an extension of the technique of [3].

2. SOME NOTATION

The kernel and the range of $A \in B(H)$ will be denoted by ker A and $ran\, A$, respectively; $\ker^{\perp} A$ and $\overline{ran\, A}$ will denote the orthogonal complement (in H) of ker A and the closure of $ran\, A$, respectively. The operator X is said to be a quasi–affinity if X is injective and has dense range. The operators A and B are quasi–similar if there exist quasi–affinities X and Y such that $AX = XB$ and $BY = YA$. We shall denote the spectrum (the point spectrum) of A by $\sigma(A)$ (respectively, $\sigma_p(A)$). The unit disc in $\mathbb{C}$ (= the complex plane) will be denoted by $\mathbf{D}$, C_1 shall denote the class of trace class operators and C_2 shall denote the class of Hilbert–Schmidt operators. Recall that an operator A is said to be pure if there exists no non–trivial reducing subspace M of A such that $A|M$, restriction of A to M, is normal.

We say that the contraction A is c.n.u. (= completely non–unitary) if there exists no non-trivial reducing subspace M of A such that $A|M$ is unitary. The contraction A belongs to the class $C_{.0}$ (class $C_{.1}$) of contractions if $A^{*^n} \to 0$ strongly as $n \to \infty$ (respectively, $\inf_n \|A^{*^n}x\| > 0$ for all non–zero $x \in H$). Classes $C_{0.}$ and $C_{1.}$ are defined by considering A^* (instead of A), and we define classes $C_{\alpha\beta}, \alpha, \beta = 0, 1$, by $C_{\alpha\beta} = C_{\alpha.} \cap C_{.\beta}$. The contraction A belongs to the class C_0 if there exists an inner function ϕ such that $\phi(A) = 0$. Amongst all inner functions ϕ such that $\phi(A) = 0$ there is a minimal one (that is, one which is a divisor in the Hardy space H^∞ of all others), called the minimal function of A [7]. Letting $D_A = (1 - A^*A)^{1/2}$ denote the defect operator of the contraction A, we define the defect spaces $\mathcal{D}_A$ and $\mathcal{D}_{A^*}$ by $\mathcal{D}_A = \overline{D_A H}$ and $\mathcal{D}_{A^*} = \overline{D_{A^*}H}$. The characteristic function Θ_A of A is a contractive operator–valued function in $H^\infty(\mathcal{D}_A, \mathcal{D}_{A^*})$. If $A \in C_{.0}$, Θ_A is inner (that is, $\Theta_A(e^{it})$ is isometric a.e.) and the (unitarily equivalent) functional model $S(\Theta_A)$ of A acts on the space $H(\Theta_A) = H^2(\mathcal{D}_{A^*}) \ominus \Theta_A H^2(\mathcal{D}_A)$ [7, p.248]. (Here H^2 denotes the Hardy space H^2.)

3. RESULTS

We start by proving some lemmas which help obtain an insight into the structure of contractions A for which A^n is normal.

Lemma 1 If A is a contraction such that A^n is normal for some integer $n \geq 2$, then there exist direct sum decompositions $H = H_n \oplus H_p$ and $A = A_n \oplus A_p$ such that $A_n = A|H_n$ is a normal $C_{11} \oplus C_{00}$ type contraction and $A_p = A|H_p$ is a pure C_{00} contraction.

Proof Since every operator has a direct sum decomposition of type normal $\oplus$ pure, the existence of the decompositions $H = H_n \oplus H_p$ and $A = A_n \oplus A_p$ is guaranteed. Decomposing A_n into its unitary and c.n.u. parts we see that A_n is of type $C_{11} \oplus C_{00}$. We are thus left with having to prove the C_{00} type of A_p.

Recall that the c.n.u. contraction A_p has a triangulation

$$\begin{bmatrix} A_{01} & * & * \\ 0 & A_{00} & * \\ 0 & 0 & A_{1.} \end{bmatrix} \quad \text{of type} \quad \begin{bmatrix} C_{01} & * & * \\ 0 & C_{00} & * \\ 0 & 0 & C_{1.} \end{bmatrix} \tag{1}$$

(see [7, p.75]). Suppose A_{01} is non–trivial. Letting $B = \begin{bmatrix} A_{00} & * \\ 0 & A_{1.} \end{bmatrix}$, the normality of A_p^n implies that A_{01}^n is subnormal (with normal extension A_p^n). Since $A_{01}^* \in C_{10}$, there exists a quasi–affinity X and an isometry V such that $VX = XA_{01}^*$ [7, p.71]. Hence $V^nX = XA_{01}^{*^n}$. The contractions A_{01}^n and V^n being subnormal, this implies (by the Putnam–Fuglede theorem [10]) that V^n and $A_{01}^{*^n}$ are unitarily equivalent unitary operators. Since the nth root of a unitary operator is unitary, A_{01}^* (and so also A_{01}) is unitary – a contradiction. Hence A_{01} acts on the trivial space, and $A_p = B$. We show now that $A_p = A_{00}$.

Since $A_{1.} \in C_{1.}$, there exists an isometry V_1 and a quasi–affinity X_1 such that $V_1X_1 = X_1A_{1.}$, or, $V_1^nX_1 = X_1A_{1.}^n$. Considering B^* in the form $\begin{bmatrix} A_{1.}^* & * \\ 0 & A_{00}^* \end{bmatrix}$, the normality of B^n implies $A_{1.}^{*^n}$ is a subnormal operator. Hence, once again, applying the Putnam–Fuglede theorem, $A_{1.}^*$ is unitary. But then B^* has a unitary direct summand – a contradiction since B is c.n.u. Hence $A_{1.}^*$ also acts on the trivial space, and $A_p = A_{00} \in C_{00}$.

□

Given the contraction A_p of Lemma 1, a structure theorem of Gilfeather [6, Section 3] implies the existence of a direct sum decomposition $H_p = \bigoplus_{j=0}^{\infty} H_{pj}$ such that each H_{pj} reduces A_p, $A_{p0} = A_p|H_{p0}$ is nilpotent, and each $A_{pj} = A_p|H_{pj}$, $j = 1, 2, \ldots$, is similar to a normal contraction N_j. Letting $N = \bigoplus_{j=1}^{\infty} N_j$, it follows that $A_p' = \bigoplus_{j=1}^{\infty} A_{pj}$ is quasi–similar to the normal contraction N. Thus, if $0 \notin \sigma_p(A)$, then the contraction A of Lemma 1 is the direct sum of a unitary operator and a C_{00} contraction quasi–similar to a normal contraction.

Assume that the defect operator D_{A_p} of the contraction A_p (of Lemma 1) is in C_2. Then $D_{A_{pj}} \in C_2$ for all $j = 0, 1, 2, \ldots$ Consequently, A_{p0} acts on the trivial space (see the proof of [3, Theorem 1]), and $A_p = A_p' = \bigoplus_{j=1}^{\infty} A_{pj}$. Since a C_{00} contraction T such that $D_T \in C_2$ is of the class C_0 [14], A_p' (and each A_{pj} for $j = 1, 2, \ldots$) is of the class C_0. The quasi–similarity of A_p' and N (similarity of A_{pj} and $N_j, j = 1, 2, \ldots$) implies that N (respectively, N_j) is also of the class C_0. Since quasi–similar C_0 contractions have the same spectrum and the spectrum of a normal C_0 contraction consists of simple eigen–values (that is, characteristic values of index one), $\sigma(A_p') = \sigma_p(A_p')$ (respectively, $\sigma(A_{pj} = \sigma_p(A_{pj}))$ and the minimal function of A_p' (respectively, A_{pj}) is a Blaschke product with simple zeros [7, pp.126 and 135].

The following lemma, though not necessary to the proof of Theorem 1, will be required in the proof of Theorem 1′. Also it leads to a stronger (than required in the proof of Theorem 1) conclusion in Lemma 3.

Lemma 2 Suppose the contraction A_p of Lemma 1 has a triangulation

$$A_p = \begin{bmatrix} E & X \\ 0 & F \end{bmatrix}.$$

If $D_{A_p} \in C_2$, then $X \in C_1$.

Proof Clearly, $D_E \in C_2$. Since for a weak contraction $E \in C_0$, $\dim \ker(E) = \dim \ker(E^*) < \infty$ [7, p.323], where $\dim \ker(E)$ denotes the dimension of the null space of E, E is a Fredholm operator. The hypothesis $1 - E^*E \in C_1$ implies $X^*E \in C_1$. By the Fredholm property of E, there exist operators E' and P such that $E'E^* = 1 - P$, where $ran\, P = \ker(E^*)$. Consequently, $(1 - P)X = E'E^*X \in C_1$, and $X = (1 - P)X + PX \in C_1$.

Let $C(D, E) : B(H) \to B(H)$ denote the derivation $C(D, E)(Y) = DY - EY$, and let $C^n(D, E)$ denote n–times application of $C(D, E)$ (i.e., $C^n(D, E)(Y) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} D^{n-r} Y E^r$).

□

Lemma 3 Suppose the contraction A_p of Lemma 1 satisfies $D_{A_p} \in C_2$. Then there exists a trace class operator Z such that $C^n(A_p, -A_p)(Z) = 0$.

Proof Let $A_p = \overset{\infty}{\underset{j=1}{\oplus}} A_{pj}$, where $A_{pj} \in C_0$ and A_{pj} is similar to a normal contraction for all $j = 1, 2, \ldots$ (see above). The minimal function $m_{A_{pj}}$ of A_{pj} is a Blaschke product with simple zeros, and the eigen–spaces corresponding to distinct eigen–values of A_{pj} describe a 'basic system' (in the sense of [1, p.104]) of invariant subspaces of A_{pj} (and N_{pj}) such that the restriction of A_{pj} (respectively, N_j) to each of these subspaces is normal ([7, p.135] and [1]). Let $M_{j\lambda}$ be the eigen–space corresponding to the eigen–value λ of A_{pj}. Then A_{pj} and N_j have upper triangulations

$$A_{pj} = \begin{bmatrix} D_{j1} & X \\ 0 & -D_{j2} \end{bmatrix} \begin{bmatrix} M_{j\lambda} \\ H_j \ominus M_{j\lambda} \end{bmatrix} \quad \text{and} \quad N_j = \begin{bmatrix} N_{j1} & 0 \\ 0 & N_{j2} \end{bmatrix} \begin{bmatrix} M'_{j\lambda} \\ H_j \ominus M'_{j\lambda} \end{bmatrix}. \tag{2}$$

Let S_j be the invertible operator such that $S_j\, A_{pj} = N_j S_j$. Letting S_j have the corresponding matrix representation

$$S_j = \begin{bmatrix} S_{j1} & S_{j3} \\ S_{j4} & S_{j2} \end{bmatrix} \; (: M'_{j\lambda} \oplus (H_j \ominus M'_{j\lambda}) \to M_{j\lambda} \oplus (H_j \ominus M_{j\lambda})) \;,$$

we have

$$S_{j4}\, D_{j1} = N_{j2}\, S_{j4} \;.$$

The minimal functions m_1 and m_2 of D_{j1} and $-D_{j2}$ (respectively) are co–prime (with $m_1\, m_2 = m_{A_{jp}}$). Since N_{j2} and $-D_{j2}$ have the same minimal function, $\sigma(D_{j1}) \cap \sigma(N_{j2}) = \emptyset$ and it follows from Rosenblum's Corollary [11] that $S_{j4} = 0$. The invertibility of S_j now implies the invertibility of S_{j1} and S_{j2}. Since

$$S_{j1}\, D_{j1} = N_{j1}\, S_{j1} \quad \text{and} \quad -S_{j2}\, D_{j2} = N_{j2}\, S_{j2} \;,$$

$$\begin{bmatrix} S_{j1}^2 & S_{j1} S_{j3} \\ 0 & S_{j2}^2 \end{bmatrix} \begin{bmatrix} D_{j1} & X \\ 0 & -D_{j2} \end{bmatrix} = \begin{bmatrix} D_{j1} & 0 \\ 0 & -D_{j2} \end{bmatrix} \begin{bmatrix} S_{j1}^2 & S_{j1} S_{j3} \\ 0 & S_{j2}^2 \end{bmatrix} ,$$

i.e., the contractions A_{pj} and $D_{j1} \oplus -D_{j2}$ are similar. By Lemma 2, $X \in C_1$. This since $\sigma(D_{j1}) \cap \sigma(-D_{j2}) = \emptyset$, implies the existence of a (unique) C_1 operator Y_j satisfying

$$D_{j1}\, Y_j + Y_j\, D_{j2} = X \;,$$

or,

$$C(D_{j1}, -D_{j2})(Y_j) = X$$

(see [5]).

Let $\hat{Y}_j = \begin{bmatrix} 0 & Y_j \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_{j\lambda} \\ H_j \ominus M_{j\lambda} \end{bmatrix}$; then $\hat{Y}_j^{*} \hat{Y}_j = 0 \oplus |Y_j|^2$, and so $\hat{Y}_j \in C_1$. Let Z denote the (infinite) matrix with $\hat{Y}_j$ as its (j,j)–th entry and zeros elsewhere. Then, with $A_p = \overset{j-1}{\underset{r=1}{\oplus}} A_{pr} \oplus A_{pj} \overset{\infty}{\underset{r=j+1}{\oplus}} A_{pr}$, a simple calculation shows that

$$\begin{aligned} C(A_p, -A_p)(Z) &= 0_1 \oplus C(A_{pj}, -A_{pj})(\hat{Y}_j) \oplus 0_2 \\ &= 0_1 \oplus \hat{X} \oplus 0_2 \;, \end{aligned}$$

where 0_1 is the zero operator on $\overset{j-1}{\underset{r=1}{\oplus}} H_r$, 0_2 is the zero operator on $\overset{\infty}{\underset{r=j+1}{\oplus}} H_r$ and $\hat{X} : H_j \to H_j$ is the operator $\hat{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$. Consequently,

$$C^n(A_p, -A_p)(Z) = 0_1 \oplus \begin{bmatrix} 0 & C^{n-1}(D_{j1}, D_{j2})(X) \\ 0 & 0 \end{bmatrix} \oplus 0_2 \;.$$

Consider now A_p^n. Since

$$A_p^n = \begin{bmatrix} \overset{j-1}{\underset{r=1}{\oplus}} A_{jr}^n & 0 & 0 \\ 0 & \begin{bmatrix} D_{j1}^n & C^{n-1}(D_{j1}, D_{j2})(X) \\ 0 & (-D_{j2})^n \end{bmatrix} & 0 \\ 0 & 0 & \overset{\infty}{\underset{r=j+1}{\oplus}} A_{pr} \end{bmatrix} \begin{bmatrix} \overset{j-1}{\underset{r=1}{\oplus}} H_r \\ H_j \\ \overset{\infty}{\underset{r=j+1}{\oplus}} H_r \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \overset{j-1}{\underset{r=1}{\oplus}} A_{pr}^n & 0 \\ 0 & D_{j1}^n \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ C^{n-1}(D_{j1}, D_{j2})(X) & 0 \end{bmatrix} \\ 0 & \begin{bmatrix} (-D_{j2})^n & 0 \\ 0 & \overset{\infty}{\underset{r=j+1}{\oplus}} A_{pr}^n \end{bmatrix} \end{bmatrix} \begin{bmatrix} \overset{j-1}{\underset{r=1}{\oplus}} H_r \oplus M_{j\lambda} \\ (H_j \ominus M_{j\lambda}) \overset{\infty}{\underset{r=j+1}{\oplus}} H_r \end{bmatrix} ,$$

and both A_p^n and $\overset{j-1}{\underset{r=1}{\oplus}} A_{pr}^n \oplus D_{j1}^n$ are normal, we must have

$$C^{n-1}(D_{j1}, D_{j2})(X) = 0 \;. \tag{3}$$

Hence

$$C^n(A_p, -A_p)(Z) = 0 \,, \;\; Z \in C_1 \;.$$

□

Lemma 3 proves more than what shall be required in the proof of our main result; all that we shall require is that the operator X satisfies Eq.(3).

Theorem 1 If A is a contraction such that $D_A \in C_2$ and A^n is normal for some integer $n \geq 2$, then A is similar to a normal contraction. Furthermore, either A is normal or A^2 is normal.

Proof Following the notation of Lemmas 1 and 3, let $A_p = \overset{\infty}{\underset{j=1}{\oplus}} A_{pj}$. (Recall that if $D_A \in C_2$, then A_{p0} acts on the trivial space.) Suppose the pure contraction A_p is non–trivial. Since $\sigma(A_p) = \sigma_p(A_p)$ is simple, either 0 is a simple eigen–value of A_p or $0 \notin \sigma(A_p)$. We consider these cases separately, and show that whereas $0 \in \sigma(A_p)$ implies A_p acts on the trivial space, the hypothesis $0 \notin \sigma(A_p)$ implies A_p^2 is normal.

Suppose $0 \in \sigma(A_p)$. Then there is a j, $j = 1, 2, \ldots$, such that $0 \in \sigma(A_{pj}) = \sigma_p(A_{pj})$. For $\lambda = 0$, define $M_{j0}, D_{j1}, D_{j2}, N_{j1}$ and N_{j2} as in the proof of Lemma 3. Then $\sigma(D_{j1}) \cap$

$\sigma(-D_{j2}) = \{0\} \cap \sigma(-D_{j2}) = \emptyset$. Since $D_{j2} = S_{j2}^{-1}(-N_{j2})S_{j2}$, $S_{ji}(i = 1, 2)$ as in the proof of Lemma 3, we have $C^{n-1}(D_{j1}, -N_{j2})(X\, S_{j2}^{-1}) = 0$. Hence

$$\lim_{m\to\infty} \|C^m(D_{j1}, -N_{j2})(X S_{j2}^{-1})\|^{1/m} = 0$$

The contractions D_{j1} and N_{j2} being normal, this implies

$$C(D_{j1}, -N_{j2})(X S_{j2}^{-1}) = 0 = C(D_{j1}^*, -N_{j2}^*)(X S_{j2}^{-1})$$

(see [10, Lemma 2]). Thus $\overline{ran(X S_{j2}^{-1})}$ reduces D_{j1}, $\ker^\perp(X S_{j2}^{-1})$ reduces $-N_{j2}$, and $D_{j1}|\overline{ran(X S_{j2}^{-1})}$ and $-N_{j2}|\ker^\perp(X S_{j2}^{-1})$ are unitarily equivalent normal contractions. Since $0 \notin \sigma(D_{j2}) = \sigma(-N_{j2})$, we must have $X S_{j2}^{-1} = 0$, or, $X = 0$. Hence $A_{pj} = D_{j1} \oplus -D_{j2}$, and so A_p has a normal direct summand (when decomposed with respect to $H_p = M_{j0} \oplus (H_{pj} \ominus M_{j0}) \overset{j-1}{\underset{r=1}{\oplus}} H_{pr} \overset{\infty}{\underset{r=j+1}{\oplus}} H_{pr}$). Since A_p is pure, A_p must act on the trivial space (so that $A = A_n$ is normal).

Consider now the case $0 \notin \sigma(A_p)$. Clearly, $A_p^{2^n}$ is normal, $A_p^2 \in C_0$ and $\sigma(A_p^2) = \{\mu^2 \in \mathbb{C} : \mu \in \sigma_p(A_p)\}$. A consideration of the proof for the case $0 \in \sigma(A_p)$ shows that A_p^2 will not have a non-trivial pure part, B_p say, as long as there exists a $\lambda \in \sigma(B_p)$ such that $-\lambda \notin (B_p)$. Since such a $\lambda \in \sigma(B_p)$ always exists if $\sigma(B_p) \subset \{\mu^2 \in \mathbb{C} : \mu \in \sigma(A_p)\}$, A_p^2 must be normal. Hence $A^2 = A_n^2 \oplus A_p^2$ is normal. The normality of A_p^2 along with the hypothesis $0 \notin \sigma(A_p)$ also implies that A_p (is invertible and) similar to a normal contraction [13]. Hence A is similar to a normal contraction, and the proof is complete.

□

The conclusion A^2 is normal (but A is not) in Theorem 1 implies that A has a representation

$$A = B \oplus \begin{bmatrix} C & P \\ 0 & -C \end{bmatrix},$$

where B is normal (necessarily of type $C_{11} \oplus C_0$), and $C(\in C_0)$ and P are commuting normal operators with P positive definite [11, 6]. By Lemma 2, P is trace class. Let V be a unitary operator such that $C = V|C|$ (where $|C|$ is the positive square root of C^*C); let X_0 and N be defined by

$$X_0 = \begin{bmatrix} C & \frac{1}{2}P \\ \frac{1}{2}V^2P & -C \end{bmatrix} \quad \text{and} \quad N = B \oplus X_0 .$$

Then N is normal and

$$\|A - N\|_1 = \left\| 0 \oplus \begin{bmatrix} 0 & \frac{1}{2}P \\ -\frac{1}{2}V^2P & 0 \end{bmatrix} \right\|_1 = \frac{1}{2}\|P \oplus P\|_1 ,$$

where $\|\cdot\|_1$, denotes the C_1-norm. Theorem 1 thus implies the following.

Theorem 1' If A is the contraction of Theorem 1, then either A is normal or there exists a normal contraction N and a positive definite $P \in C_1$ such that $\|A - N\|_1 = \frac{1}{2}\|P \oplus P\|_1$.

If, in Theorem 1, $\sigma(A)$ is a subset of the real line, then A^2, B abd C (above) are self-adjoint. Choosing X_0 to be the self-adjoint operator $\begin{bmatrix} C & \frac{1}{2}P \\ \frac{1}{2}P & -C \end{bmatrix}$ and letting N denote the self-adjoint operator $B \oplus X_0$, we then have:

Corollary 1 If the spectrum of the contraction A of Theorem 1 is a subset of the real line, then A is similar to a self-adjoint contraction. Furthermore, either A is self-adjoint or there exists a self-adjoint contraction N and a positive definite $P \in C_1$ such that $\|A - N\|_1 = \frac{1}{2}\|P \oplus P\|_1$.

Corollary 2 The contraction A of Theorem 1 is normal if either of the following conditions hold.

(i) A_p has empty point spectrum.

(ii) A satisfies the property that if the restriction of A to an invariant subspace is normal, then the subspace reduces A.

(iii) There exists a line L through the origin (of $\mathbb{C}$) such that $L \cap \sigma(A_p)$ is a singleton set.

(iv) $\sigma(A_p)$ lies in the sector $\{\lambda \in \mathbb{C} : 0 \le arg\lambda \le \frac{\pi}{2}\}$.

(v) A_p and A_p^n have the same commutant.

Proof In view of the proof of Theorem 1, the proof in cases (i)–(iv) is obvious. For case (v), let A_{pj} in (2) have the representation

$$A_{pj} = \begin{bmatrix} D_{j1} & X \\ 0 & D_{j2} \end{bmatrix}, \quad D_{j1} \text{ normal and } \quad \sigma(D_{j1}) \cap \sigma(D_{j2}) = \emptyset .$$

The normality of A_{pj}^n then implies

$$Q(X) = \sum_{r=0}^{n-1} D_{j1}^{n-r-1}\, X\, D_{j2}^r = 0 .$$

Hence

$$D_{j1}\, Q(X) - Q(X) D_{j2} = D_{j1}^n X - X D_{j2}^n = 0 ,$$

or,

$$\begin{bmatrix} D_{j1}^n & 0 \\ 0 & D_{j2}^n \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_{j1}^n & 0 \\ 0 & D_{j2}^n \end{bmatrix}$$

Defining the operator Z by the (infinite) matrix with $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ as its (j, j)-th entry and zeros elsewhere we have $A_p^n Z = Z A_p^n$. Hence, from the hypothesis, $A_p Z = Z A_p$, i.e., $D_{j1}X = XD_{j2}$. By Rosenblum's Corollary, X must be the zero operator, implying thereby that A_p has a normal direct summand. Consequently, A_p acts on the trivial space.

□

Call the property satisfied by A in (ii) of the corollary above property $(P2)$. Suppose that T is a contraction satisfying property $(P2)$ with $D_T \in C_2$. Then $T = T_n \oplus T_p$, where T_n

is normal (of type $C_{11} \oplus C_0$) and $T_p \in C_0$ is pure. Suppose now that $\sigma(T_p)$ consists of simple eigen-values. Then the restriction of T_p to the eigen-space corresponding to an eigen-value is normal, and so because of property ($P2$) this eigen-space reduces T_p. Thus T_p acts on the trivial space, and $T = T_n$ is normal. If now $T = A^n$, for some contraction A and integer $n \geq 2$, then the pure part A_p of A is a C_0-contraction with $\sigma(A_p)$ consisting of simple eigen-values. (A_p cannot have a C_{11} part for the reason that if it were to have such a part, then the normality of A_p^n would imply A_p has a unitary part.) Theorem 1 thus implies the following:

Corollary 3 Let T be a contraction satisfying property ($P2$). Suppose $D_T \in C_2$ and $\sigma_p(T)$ consists of simple eigen-values. If $T = A^n$ for some contraction A and integer $n \geq 2$, then either A or A^2 is normal.

Corollary 4 If the contraction of A of Theorem 1 also satisfies the hypothesis $AY - YA \in C_1$ form some $Y \in C_2$, then trace $(AY - YA) = 0$.

Proof Since A is similar to a normal contraction, $A = S^{-1}NS$ for some normal contraction N and invertible operator S. Then $S(AY - YA)S^{-1} = NSYS^{-1} - SYS^{-1}N \in C_1$, where $SYS^{-1} \in C_2$. By [16], trace $(NSYS^{-1} - SYS^{-1}N) = 0$; hence trace $(AY - YA) = 0$.

□

We consider now a general contraction A such that A^n is normal. Then, as already seen, there exist decompositions

$$H = H_0 \oplus H_n \oplus H_p \quad \text{and} \quad A = A_0 \oplus A_n \oplus A_p \qquad (4)$$

such that $A_0 = A|H_0$ is nilpotent, $A_n = A|H_n$ is normal and $A_p = A|H_p \in C_{00}$ is pure. Furthermore, there exist decompositions $H_p = \bigoplus_{j=1}^{\infty} H_{pj}$ and $A_p = \bigoplus_{j=1}^{\infty} A_{pj}$ such that $A_{pj} = A_p|H_{pj}$ ($\in C_{00}$) is similar to a normal contraction $N_j (j = 1, 2, \ldots)$. Recall that if Θ denotes the characteristic function (associated with the functional model) of the c.n.u. contraction A_p, then Θ is inner (from both sides), $\sigma(A_p) = \sigma(\Theta)$ and $\sigma_p(A_p) = \{\lambda \in \mathbf{D} : \ker\ \Theta(\lambda) \not\equiv 0\}$ (see [7] and [8, Lecture 3]). Corresponding to an upper triangulation

$$A_p = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix},$$

there corresponds a regular factorization $\Theta = \Theta_2\Theta_1$, where the characteristic functions of T_1 and T_2 coincide with (the purely contractive parts) of Θ_1 and Θ_2, respectively. (Since Θ is inner from both sides, all regular divisors of Θ are strong regular divisors [7, p.308].) Clearly, $\sigma(A_p) = \sigma(T_1) \cup \sigma(T_2) = \sigma(\Theta_1) \cup \sigma(\Theta_2)$, and if $\sigma(T_1) = \sigma_p(T_1)$, then Θ_1 is a Blaschke product.

Let b_λ^k denote the Blaschke factor with λ as its (only) zero, $k = k(\lambda)$ being the multiplicity of the zero at λ. Let Θ denote the characteristic function of A_p. The hypotheses of Theorem 1 imply that if ϑ_λ are the divisors of the inner function Θ representing the A_p-invariant subspaces $K_\lambda = \{\ker(A_p - \lambda) : \lambda \in \sigma_p(A_p)\}$, then $A_p|K_\lambda \in C_0$, the minimal function of $A_p|K_\lambda$ is b_λ, $A_p|K_\lambda$ is normal, and $\vartheta_\lambda \wedge \vartheta_{\lambda'} \equiv 1$ (i.e., ϑ_λ and $\vartheta_{\lambda'}$ are relatively prime) for $\lambda \not\equiv \lambda'$.

Theorem 2 Let A be the contraction of Lemma 1, and let Θ denote the characteristic function of A_p (A_p as in (4)). If there exists a regular factorization $\Theta = \Theta_0\ b_\lambda^k$ such that $\Theta_0 \wedge b_\lambda^k \equiv 1$, $\Theta_0 \wedge b_{-\lambda}^k \equiv 1$ and $A_p|\Theta\ b_\lambda^{-k}(H^2 \ominus b_\lambda^k H^2)$ is normal, then A_p' acts on the trivial space and $A = A_0 \oplus A_n$.

Proof Since $A_p = \bigoplus_{j=1}^{\infty} A_{pj}$, the hypothesis imply the existence of a $j (j = 1, 2, \ldots)$ such that the characteristic function Θ_j of A_{pj} has a regular factorization $\Theta_j(\mu) = \Theta_{j0}(\mu)\ b_\lambda^k(\mu)$, where $\Theta_{j0}(\mu) \wedge b_\lambda^k(\mu) \equiv 1$, $\Theta_{j0}(\mu) \wedge b_{-\lambda}^k(\mu) \equiv 1$ and $D_{j1} = A_{pj}|\Theta_{j0}\ b_\lambda^{-k}(H^2 - b_\lambda^k H^2) \in C_0$ is normal. The similarity of A_{pj} to normal N_j implies $k = 1$ (so that $b_\lambda^k = b_\lambda$ is a simple Blaschke factor), and

$$\Theta_{j0}\ b_\lambda^{-1}(H^2 - b_\lambda H^2) = \ker(A_{pj} - \lambda)$$

[8, p.82]. Letting A_{pj} have triangulation (2), if follows (as in the proof of Theorem 1) that $C(D_{j1}, D_{j2}) = 0$. Since

$$\sigma(-D_{j1}) \cap \sigma(D_{j2}) = \{-\lambda\} \cap \sigma(D_{j2}) = \sigma(b_{-\lambda}) \cap \sigma(\Theta_{j0}) = \emptyset,$$

we must have $X = 0$. But then A_{pj}, and A_p, has a normal direct summand – a contradiction. Hence A_p acts on the trivial space, and $A = A_0 \oplus A_n$.

□

The hypotheses of Theorem 2 ensure a triangulation of the type (2) for A_{pj}. More generally, Theorem 2 remains valid for contractions A for which A_p has a triangulation

$$A_p = \begin{bmatrix} B & X_1 \\ 0 & -D \end{bmatrix} \qquad (5)$$

for some normal contractions B and D satisfying $\sigma(B) \cap \sigma(D) = \emptyset$. A concrete example of this occurs in the case in which 0 is not a point of the numerical range of A and A^2 is normal (see [9] where such a proposition is considered, and [11] where such a representation for A_p is given). If the contraction A, A^n is normal, is such that $\sigma(A)$ lies in two sectors of the plane each of width less than $\frac{2\pi}{n}$, then A_p has triangulation (5) (upto unitary equivalence), where B, D and X_1 are commuting normal operators with X_1 positive definite [6, Corollary 4.3]. Clearly, $C(B, D)(X_1) = 0 = C(B^*, D^*)(X_1)$, so that B and D are unitarily equivalent. We may therefore represent A_p, upto unitary equivalence, in such a case by

$$A_p = \begin{bmatrix} B & X_1 \\ 0 & -B \end{bmatrix}.$$

Corollary 5 If the contraction A of Lemma 1 has its spectrum in two sectors of the plane each of width less than $\frac{2\pi}{n}$ and with both sectors lying to one side of the origin in the same half-plane, then A is an invertible normal contraction.

Proof Since $0 \notin \sigma(A)$, A_0 acts on the trivial space and $A = A_n \oplus A_p$ is invertible. Since $\sigma(B) \cap \sigma(-B) = \emptyset$, A_p acts on the trivial space.

We consider now compact contractions T satisfying property $(P2)$ (i.e., the property stated in (ii) of Corollary 2), and for which $T = A^n$ for some contraction A. Recall that the characteristic function $\Theta_T(\lambda)$ of T is said to admit a scalar factor $\delta(\lambda)$ ($\not\equiv 0$) if there exists a contractive analytic function $\{\mathcal{D}_{T^*}, \mathcal{D}_T, \Omega(\lambda)\}$ such that

$$\Omega(\lambda)\,\Theta_T(\lambda) = \delta(\lambda)\, I_{\mathcal{D}_T} \quad \text{and} \quad \Theta_T(\lambda)\,\Omega(\lambda) = \delta(\lambda)\, I_{\mathcal{D}_{T^*}}$$

[7, p.264].

Theorem 3 Let T be a compact contraction satisfying property $(P2)$ such that its characteristic function admits a scalar factor. Suppose $T = A^n$ for some contraction A satisfying property $(P2)$ and integer $n \geq 2$. If the eigen–values of A are all simple (or, the pure part of A has empty point spectrum), then A is a compact normal contraction.

Proof We show that (the c.n.u. contraction) T is normal. Letting T have representation (1) (of the proof of Lemma 1), it is seen that $VX = XA_{01}^*$ for some isometry V and quasi–affinity X. Since A_{01} is compact (because T is, and A_{01} is the restriction of T to an invariant subspace), $X^*V^*V^*X = X^*X$ is compact. Applying [2, Corollary 6.5] to the equation $A_{01}\,X^*X\,A_{01}^* = X^*X$ we have A_{01} is unitary. Consequently, A_{01} acts on the trivial space, and $T = B = \begin{bmatrix} A_{00} & * \\ 0 & A_{1\cdot} \end{bmatrix}$. Consider the contraction $\begin{bmatrix} A_{1\cdot}^* & * \\ 0 & A_{00}^* \end{bmatrix}$. Since $A_{1\cdot} \in C_{1\cdot}$, there exists an isometry V_1 and a quasi–affinity X_1 such that $V_1X_1 = X_1A_{1\cdot}$, or, $A_{1\cdot}^*X_1^* = X_1^*V_1^*$. Once again we conclude from the equation $A_{1\cdot}^*X_1^*X_1A_{1\cdot} = X_1^*X_1$ that $A_{1\cdot}^*$ acts on the trivial space, and hence that $T = A_{00} \in C_{00}$. Recall that a C_{00}–contraction is of the class C_0 if and only if its characteristic function admits a scalar factor [7, Theorem VI.5.1]. Hence $T \in C_0$. Decompose T into its normal and pure parts by $T = T_n \oplus T_p$. Since $\sigma(T_p) = \sigma_p(T_p) \cup \{0\}$ for the compact contradiction T_p, $\sigma(T_p) = \sigma(T_p) \cap \mathbf{D} = \sigma_p(T_p)$, and so if T_p has empty point spectrum, then $T = T_n$ is normal. If, instead, the eigen–values of T are simple, then the eigen–values of T_p are simple. Let $T_{p\lambda}$ denote the restriction of T_p to the subspace M_λ generated by the eigen–vectors corresponding to the eigen–value λ of T_p. Then $T_{p\lambda}$ is normal, and so, since T satisfies property $(P2)$ implies T_p satisfies property $(P2)$, M_λ reduces T_p. Hence, once again, we conclude that T_p acts on the trivial space, and $T = T_n$ is normal. Since A satisfies property $(P2)$, it now follows that A is the direct sum of a nilpotent and normal contraction. Since 0 is a simple eigen–value of A, the nilpotent part of A must be the trivial operator. This completes the proof.

□

Remark At this point, I must point our that Theorems 2 and 2′ of [3] are vacuous, for the reason that the hypotheses $D_T \in C_2$ and T is compact are incompatible. To my regret, it was only recently, long after the galley proof for [3] had been returned, that I realised this. For [3, Theorems 2 and 2′] to be meaningful the hypotheses must be changed, as in Theorem 3 above.

4. SUBNORMAL CONTRACTIONS

An operator T is said to be subnormal if it has a normal extension. A subnormal operator need not have a root (the unilateral shift provides such an example), and even where the roots exist none of the roots need be subnormal (see [17] for examples). In the following it will be shown that if $T = A^n$ is a subnormal contraction with $D_A \in C_2$, then A is the direct sum of contraction similar to a normal contraction and a contraction which is the quasi-affine transform of a unilateral shift.

Let A be a c.n.u. contraction for which $D_A \in C_2$ and A^n, n some integer ≥ 2, is subnormal. Let A have representation (1) (of the proof of Lemma 1). Then A_{01}^n is subnormal. Arguing as in the proof of Lemma 1 it follows that A_{01} acts as the trivial space, and so $A = \begin{bmatrix} A_{00} & * \\ 0 & A_{1\cdot} \end{bmatrix}$ is of type $\begin{bmatrix} C_{00} & * \\ 0 & C_{1\cdot} \end{bmatrix}$. Since $D_A \in C_2$, A_{00} is a C_0 contraction (by [14]). Consequently, the subnormal contraction A_{00}^n is also of class C_0. Recall that the pure part of a subnormal operator has empty point spectrum. Hence, if A_{00}^n has a pure part, then the spectrum of the pure part lies on the unit circle (in $\mathbb{C}$). Since a subnormal contraction with spectrum on the unit circle must be unitary (and A_{00} is c.n.u.), A_{00}^n must be a normal contraction (of class C_0). The operator A^n being subnormal satisfies property $(P2)$; hence $A^n = A_{00}^n \oplus A_{1\cdot}^n$. By Theorem 1, A_{00} is similar to a normal contraction. Since $A_{1\cdot} \in C_{1\cdot}$ is c.n.u. and $D_{A_{1\cdot}} \in C_2$, $A_{1\cdot}$ is the quasi–affine transform of a unilateral shift, i.e., there exists a quasi–affinity X and a unilaterial shift U such that $XA_{1\cdot} = UX$ [15]. Keeping in mind the decomposition of a contraction into the direct sum of its unitary and c.n.u. parts, we have proved:

Theorem 4 If A^n, $n \geq 2$ some integer, is a subnormal contraction and $D_A \in C_2$, then A is the direct sum of a contraction similar to a normal contraction and a contraction which is the quasi–affine transform of a unilateral shift.

Acknowledgments

The author is thankful to Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, where this work was done. He is grateful to Professor James Eells, Director of the Mathematical Section, for making this visit to ICTP possible and would like to thank Profesor G. Prasad for some inspiring correspondence.

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