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INVARIANT STRUCTURES IN GAUGE THEORIES AND CONFINEMENT

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1 Introduction

Though gauge theories are studied during more than sixty years there are still quite a number of little secrets and unanswered questions in this branch of theoretical physics. Especially this refers to the Yang-Mills theories which are much more complex than electrodynamics and are not so well understood – after all physicists spent much less time for their study. The difficulties root in the problem of going beyond the perturbation theory, mainly at large distances.

Why does the study of non-Abelian gauge theories meet those difficulties? What makes them so different from the ordinary (non-gauge) field theories? The answer is simple: it is the existence of constraints. This circumstance, though trivial, is the very new property that in a considerable degree invalidates efficiency of the old method (perturbation theory) and demands developing new approaches. Constraints [1] are conditions on canonical variables; they do not contain the time derivatives, only the space ones, so they presuppose some static nonlocal structures to exist. By themselves the nonlocal excitations are not unfamiliar objects in field theories. Introduction of a static source (e.g. an interaction term $j_{cl}\varphi$, $\partial_t j_{cl} = 0$,) leads to a nonlocal excitation of a scalar field φ in the neighbourhood of the source. The nonlocal structures due to constraints are of another type – they bear no dynamics and are a visible manifestation of gauge invariance.

Why is then electrodynamics, a typical theory with constraints, at the same time a pattern for other local (non-gauge) field theories? It is the perturbation theory which being in fact the only regular method in QED prevents from uncovering its nonlocal features. The typical nonlocal object in electrodynamics is the physical electron, i.e. a charged fermion with its Coulomb field. The constraint (the Gauss law $(\nabla, \mathbf{E}) = j_0$) identifies the electric charge with the static electric field (more precisely, with its flux through an enclosing surface). One attributes to the "physical electron" a gauge invariant nonlocal operator (see Sec. 2) $\hat{\Psi} = \exp(-ie\Delta^{-1}(\nabla, \mathbf{A}))\hat{\psi}$ [2] - [4] where the exponential describes the Coulomb field, and in the zero approximation of perturbation theory it coincides with the bare gauge non-invariant operator $\hat{\psi}$. It is just this local operator that is used in the standard QED.

The example of QED teaches us that physical objects are described by some nonlocal gauge-invariant field configurations, and real dynamics should appear as motion and interaction of these objects (including, of course, local fields too if they are, like $\mathbf{E}$ and $\mathbf{H}$, gauge invariant). It suggests that before turning to dynamics, one should first solve constraints, i.e. one has to find all gauge invariant objects.

Such a strategy is, evidently, not necessary in QED because of smallness of its running coupling constant $\alpha(Q^2)$ (at all accessible energies) and absence of an asymptotic freedom in it. But it seems inevitable in the Yang-Mills theories because there always exist physically important distances at which the coupling constant α_s is not small, and where the perturbation theory is inapplicable. It means that before approximating dynamics one should take into consideration constraints, which makes non-Abelian gauge theories so different from electrodynamics. The aim of the present article is to establish and to classify the gauge invariant structures. It is shown that a priori in electrodynamics besides the above-mentioned nonlocal operator $\hat{\Psi}$ and the local fields $\mathbf{E}$, $\mathbf{H}$ there may exist "charged" objects with electric fields on lines and on two-dimensional planes. (We stress the importance of knowledge of all such external fields because they are responsible for static interparticle potential, see Secs. 2,4.) In the non-Abelian theories gauge-invariant field configurations are connected with path-ordered exponents. The most important property, established and used in the paper, is that the P-exponents are *the only* fundamental "gauge covariant" objects there. All gauge invariant configurations of fields, though complex, are made of them.

This observation is tightly connected with the problem of confinement. By itself this problem is pretty complex and has many different aspects. But one its feature – the existence of a linearly rising with distance potential – can be established for some models in a relatively simple way. Moreover, one can show that in gluodynamics between static quarks there could be no other forces. This comes from the fact that the only proper gauge-invariant object consists of the P-exponent (i.e. of the "string") connecting two opposite charges.

The paper is organized as follows.

In Sec.2 we choose a naive (physical) approach to the problem of finding and classification of all gauge-invariant objects taking the Lagrangian as a starting point and a basis for the investigation. Both Abelian and non-Abelian theories with the gauge group $SU(n)$are considered. Gauge-invariant configurations in electrodynamics are listed; the theories with gauge groups $SU(3)$ and $SU(2)$ are considered separately.

In Sec.3 this problem is investigated from another, pure geometrical point of view. The Yang-Mills fields are treated in the framework of the principal fiber bundle approach. The only meaningful geometrical element of the theory is by definition the path-ordered exponent (in an infinitesimal form), so that every geometrical object is built of P- exponents. This conceptually quite different approach confirms conclusions of the previous section.

In Sec.4 we study interparticle forces arising due to external static fields accompanying charges, both in the Abelian and non-Abelian cases.

In Sec.5 we consider connection of these results with the problem of confinement. It is shown that in pure gluodynamics interaction of the static quarks is given by a linear potential, i.e. in this case one has "strong confinement". Different forms of confinement depending on the physical parameters of the system (such as masses, string tension etc.) are listed.

2 Invariant structures in gauge theories. A physical approach.

Our aim is to "solve" constraints in classical gauge theories, i.e. to find all gauge-invariant configurations of gauge and matter fields in the space-time. For simplicity we assume that the gauge group G is $SU(n)$. One can easily construct a lot of different gauge invariants. They are local or nonlocal composite fields, the latter being built with the help of ordered exponents. But knowledge of all these constructions does not still exclude the existence of invariants of another type. For example, in electrodynamics besides invariants with the P-exponents there is the above-mentioned nonlocal field $\hat{\Psi}$. The factors made of gauge fields, entering into the invariants, represent the physical static fields surrounding charges [2], so the knowledge of *all* the invariants is important for establishing static forces between colored charges (and, hence, for the problem of confinement). For instance, in electrodynamics the Coulomb field of a charged object given by $\hat{\Psi}$ is spread through all the space, and as a result, there are no confining forces, while squeezing of an electric field in a tube leads to the linearly rising potential. Proof of the absence of the other than string-like invariants would be a major step in proving the existence of confining forces. Thus, listing of all gauge invariants is the problem of highest priority both for understanding the most important features of a theory and for its successful description. In this and the following sections we shall study invariants of classical fields. In the present section we choose a rather straightforward strategy [5] which nevertheless allows us to elucidate the problem and makes the final answer almost trivial. In the next section we show that a mathematically more elaborate consideration confirms the obtained results.

2.1 The Lagrangian and polylocal tensors

We are going to find gauge-invariant combinations of fields. To do this one has first to find all the "gauge tensors" (g-*tensors*), i.e. objects transforming homogeneously under gauge transformations. The problem is not trivial due to the existence of inhomogeneously transforming objects (the gauge fields A_μ). It is rather easy to find all the local tensors constructed of fields and, may be, of their derivatives at a given space-time point. We shall mainly be interested in the nonlocal configurations of fields depending on 2, 3, ... points of space-time (*polylocal* g-tensors).

What is the information we have at our disposal? Suppose that the theory is given by the standard

Lagrangian

$$\mathcal{L} = -\frac{1}{4} Tr\ F_{\mu\nu}^2 + \bar{\psi}(i\hat{D} - m_q)\psi \quad , \tag{2.1}$$

where $F_{\mu\nu} = i(D_\mu D_\nu - D_\nu D_\mu)$, $D_\mu = \partial_\mu - iA_\mu^a \lambda_a \equiv \partial_\mu - iA_\mu$, $\hat{D} = \gamma_\mu D_\mu$; λ_a, γ_μ are the Gell-Mann and Dirac matrices, respectively, $Tr\lambda_a\lambda_b = \delta_{ab}$, $a, b = 1, 2, ..., N = \dim G = n^2 - 1$ ($G = SU(n)$) and m_q is a mass of the matter field ψ which may realize any non-trivial representation of G; we assume for certainty that ψ transforms according to an elementary one. The coupling constant g is put equal to unity in Eq.(2.1). If necessary it can be introduced by substitution $A_\mu \to gA_\mu$, $\psi \to g\psi$, $\mathcal{L} \to \mathcal{L}' = \mathcal{L}/g^2$. The Lagrangian (2.1) is invariant under gauge transformations

$$A'_\mu(x) = U(x)A_\mu U^+(x) + iU(x)\partial_\mu U^+(x), \quad U \in G, \quad UU^+ = 1, \tag{2.2}$$

$$\psi'(x) = U(x)\psi(x), \quad F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^+(x), \quad D'_\mu = U(x)D_\mu U^+(x). \tag{2.3}$$

We call homogeneously transforming objects *local* g-tensors if they change according to rules analogous to Eqs.(2.3). They may be composed of local fields taken at the same point of space-time (in this case they are local composed fields in the old sense like that $\bar{\psi}(x)\psi(x)$); but they may be built of fields taken at different points like elements of the holonomy group or the nonlocal field Ψ. Therefore, a local g-tensor may be an essentially nonlocal field. We are interested in all nonlocal homogeneously transforming fields. Tensors transforming like

$$T'(x_1, ..., x_{n+m}) = U(x_1) \otimes \cdots \otimes U(x_n) T(x_1, ..., x_{n+m}) U^+(x_{n+1}) \otimes \cdots \otimes U^+(x_{n+m}) \tag{2.4}$$

will be called *polylocal* tensors of rank $n + m$ (or polylocal g-tensors). It is sufficient to study irreducible polylocal tensors, i.e. those which are not direct products of tensors of lower ranks.

What are the irreducible polylocal g-tensors in the theory given by the Lagrangian (2.1)? First, let us answer a simpler question: what are the g-tensors in the vicinity of a point x?. We have no information about the system except that contained in the Lagrangian plus the transformation laws (2.2), (2.3). Thus, any complex polylocal field may be composed only of those elements which enter into the Lagrangian. They are the local g-tensors $\bar{\psi}$, ψ, $F_{\mu\nu}$ and D_μ ; the latter one, being a g-tensor, is not a genuine local operator because it contains a derivative operator ∂_μ. Note that the transformation law (2.3) for D_μ is determined by Eq.(2.2): we obtain it by multiplying both sides of Eq.(2.2) by $-i$ and adding to them an operator ∂_μ (∂_μ in Eq.(2.3) acts as an operator, i.e. $\partial_\mu U = [\partial_\mu, U] + U\partial_\mu \equiv (\partial_\mu U) + U\partial_\mu$). We can rewrite Eq.(2.2) in another form: multiplying it by $+idx^\mu$ and adding unity to both its sides we obtain

$$P'(x + dx,\ x) = U(x + dx)\ P(x + dx,\ x)\ U^+(x) \tag{2.5}$$

where

$$P(x + dx,\ x) \equiv 1\ +\ iA_\mu(x)dx^\mu \equiv P_{dx}. \tag{2.6}$$

Evidently, P_{dx} is an operator of the infinitesimal parallel translation along dx. Equation (2.2) tells us that in the neighbourhood of the point x there exists a bilocal g-tensor $P(x + dx, x)$. One cannot construct other g-tensors in the vicinity of x except those built of $P(x + dx, x)$. The full list of fundamental tensors at x reads: the local ψ, $\bar{\psi}$ and the bilocal one $P(x + dx, x)$. Tensors of any rank can be constructed of them and only of them, i.e. they are building blocks of the theory. From ψ, $\bar{\psi}$ one can construct local g-invariants, from P_{dx} and ψ, $\bar{\psi}$ both local and polylocal invariants. It is the latter ones, we are mainly interested in.

Of two bilocal g-tensors P in the neighbourhood of x one can obtain the only non-trivial irreducible bilocal tensor

$$P(x + dx_1 + dx_2,\ x) = P(x + dx_1 + dx_2,\ x + dx_1)P(x + dx_1,\ x). \tag{2.7}$$

Therefore, only operations of this type give new interesting objects. Repeating the operation (2.7) we obtain the well-known path-ordered exponent (P-exponent)

$$P[c(x, x')] = \lim_{N\to\infty} \prod_{i=1}^{N} P(x_{i+1}, x_i) = P\exp\left\{i\int_{x'}^{x} A_\mu(x)dx^\mu\right\}, \tag{2.8}$$

where $x_{i+1} = x_i + \Delta x_i$, $c(x,x)$ symbolizes a contour of integration in space-time. Thus, the P-exponent $P[c(x,x')]$ ($\equiv P_{xx'}$ in the standard notation) is the only bilocal building block in a gauge theory. Note that this is true for an Abelian theory too. Any other polylocal tensors should contain as substructures P-exponents, invariants, invariant tensors like the antisymmetric ones $\epsilon_{\alpha\beta\ldots}$, and nothing else. All the local g- tensors made of D_μ (like those $[D_\mu, D_\nu]$) are contained in P-exponents.

2.2 Polylocal g-invariants

There are two types of nonlocal gauge-invariant objects: those made of D_μ and of $P_{xx'}$. The nonlocal tensors made of D_μ have the form $f(D_\mu^2)$ where f is a non-polynomial function. But they are not genuine polylocal tensors and they are not specific for gauge theories. For example, consider the nonlocal operator D_μ^{-2}. Its action on tensors assumes integration over the whole space-time: $(D_\mu^{-2})\psi(x) = \int d^4x'(D_\mu^{-2})_{xx'}\psi(x')$. We state that the kernel $(D_\mu^{-2})_{xx'}$ is not a bilocal g-tensor because a typical g-invariant constructed with its help involves fields at all space-time points: $Inv = \int d^4xd^4x'\bar{\psi}(x)(D_\mu^{-2})_{xx'}\psi(x')$, and it has nothing to do with instantaneous field cofigurations studied in this paper. The nonlocality of D_μ^{-2} is not specific for gauge theory – it is nonlocal even in the absence of the gauge field ($A_\mu = 0$). We conclude that invariant structures with D_μ^{-2} cannot appear as a result of "solving" constraints. In the following we shall not be interested in such objects [1].

Thus, we have to list all the invariants which could be constructed of the exponents $P_{xx'}$, fields ψ, $\bar{\psi}$ and invariant tensors $\epsilon_{\alpha\beta\ldots}$, $\epsilon^{\alpha\beta\ldots}$, where $\epsilon_{\alpha\beta\ldots}$ is the unit fully antisymmetric tensor. The irreducible invariants are: $Tr\ P_{xx}$ – a local invariant (P_{xx} is an element of the holonomy group), $\bar{\psi}_x\psi_x$, $\bar{\psi}_x P_{xx'}\psi_{x'}$, and all the invariants of this type made of ψ, $\bar{\psi}$ and polylocal tensors constructed of strings $P_{xx'}$ with the use of the invariant tensors $\epsilon_{\alpha\beta\ldots}$. For example, in chromodynamics the simplest of them is the "nucleonic configuration" $\epsilon P_{xx_1}P_{xx_2}P_{xx_3}\psi(x_1)\psi(x_2)\psi(x_3)$.

Besides these skeleton objects obtained by multiplying P-exponents, one may construct new tensors by "implantation" of local invariants or tensors into strings. Examples:

$$P_{xy}\ F_{\mu\nu}(y)\ P_{yx'}, \tag{2.9}$$

$$P_{xy}\ F_{\mu\nu}(y)\ P_{yz}\ F_{\mu\nu}(z)\ P_{zx'}, \tag{2.10}$$

$$P_{xy}\ F^2_{\mu\nu}(y)\ P_{yx'}, \tag{2.11}$$

$$P_{xy}\ Tr(F^2_{\mu\nu})P_{yx'}\ . \tag{2.12}$$

We note that (2.9) is a tensor of a new type (it is the Lorentz tensor too); it may be obtained from the string $P[c(x,x')]$ by forming an infinitesimal loop at y on its contour $c(x,x')$ (remember: $P\exp(i\oint A_\mu dx^\mu) \approx 1 + i\int F_{\mu\nu}d\sigma_{\mu\nu}$). The same can be said about (2.10); besides, this object is not a true scalar in general relativity – the summation runs over the indices of tensors taken at different space-time points, so the small gravity destroys the illusion of its general invariance. The next configuration (2.11) suits for the construction of invariants. Note, however, that it is composed of two objects of type (2.9), with $x' = y$ and $x = y$. The last configuration (2.12) is reducible. We conclude that all such objects are either unacceptable or contained in P-exponents.

In what follows we give a comparative analysis of simplest invariant structures in electrodynamics and in the Yang-Mills theories with gauge groups $SU(2)$ and $SU(3)$ (chromodynamics).

2.3 Invariant structures in electrodynamics

Let us study the simplest gauge theory – electrodynamics in more detail. The analysis of the previous subsections is applicable to the case of the Abelian gauge group too. But we prefer to give

[1] Note, however, that the kernels D_μ^{-2} and $(\gamma_\mu D_\mu)^{-1}$ as well as D_k^{-2} and $(\gamma_k D_k)^{-1}$, $k = 1,2,3$, may be expressed via the exponential line integral considering D_μ^2, $\gamma_\mu D_\mu$ and D_k^2, $\gamma_k D_k$ as some Hamiltonians for quantum mechanical systems [6].

an independent consideration of the subject. It is easily seen that in electrodynamics there exist the following linear in the electromagnetic field A_μ objects B transforming as $B(x) \to B(x) + \Lambda(x)$ if A_μ transforms as $A_\mu \to A_\mu + \partial_\mu \Lambda$ (assuming [2] $\Lambda(x) \to 0$ when $x \to \infty$):

$$B_1 = \int_{-\infty}^{x} A_\mu \, dx^\mu \,, \qquad (2.13)$$

$$B_2 = \Delta_2^{-1} (\nabla_\perp \,, \mathbf{A}_\perp), \quad \Delta_2 = (\nabla_\perp \,, \nabla_\perp) = \partial_1^2 + \partial_2^2, \qquad (2.14)$$

$$B_3 = \Delta^{-1} (\nabla \,, \mathbf{A}), \qquad (2.15)$$

$$B_4 = -\Box^{-1} \partial_\mu A_\mu, \qquad \Box \equiv -\partial_\mu^2 = -\partial_0^2 + \Delta. \qquad (2.16)$$

Of them and of the matter field ψ we form local g-invariants (composite fields)

$$\Psi_k(x) = e^{-iB_k(x)} \psi(x) \equiv P_k(x)\psi(x), \qquad k = 1, ..., 4, \qquad (2.17)$$

assuming that ψ transforms as $\psi(x) \to \exp(i\Lambda(x))\psi(x)$. Evidently, $\Psi_k(x)$ are local g-invariants. Their quanta correspond to charged particles with electric fields surrounding them.

In Eq.(2.13) the integration is done over a straight space-like line, so Ψ_1 describes charged particles with a static electric field on a line (a charge with the only line of force (see Sec.4)). A curved line is attributed to the excited electromagnetic field. As is well-known, factors $\exp(-iB_1)$ with time-like lines of integration describe soft photons responsible for infrared divergences [8].

The composite field Ψ_2 describes charges with static electric fields on the plane $x_3 = const$, while Ψ_3 corresponds to the familiar case of charged particles surrounded by the Coulomb fields [2] - [4]. The case of B_4 differs from the previous ones because it assumes integration over time: $\Box^{-1}\partial_\mu A_\mu(x) = \int d^4x' (\Box^{-1})_{xx'} \partial_\mu A_\mu(x') = (2\pi)^{-4} \int d^4k \; k^{-2} \exp(ikx) ik_\mu a_\mu(k)$; for a static field we have $a_\mu(k) \equiv \delta(k_0) a_\mu(\mathbf{k})$, and Ψ_4 becomes equivalent to Ψ_3.

We see that contrary to our expectations in the Abelian case there are much more possibilities than we could expect from the general approach sketched in Sec.2.1. The resolution of the paradox lies in the demonstration of composite nature of structures Ψ_k , $k = 2, 3$. Let us show that their electric fields are composed of "elementary" strings made of B_1 given by Eq.(2.13). Take, for example, $P_2(x) = \exp(-iB_2(x))$; consider

$$P_2(x, N) = \prod_{j=1}^{N} \exp\left(-ig \int_{-\infty}^{x} A_\mu(y_j) \, dy_j^\mu \right); \qquad (2.18)$$

here we introduce the coupling constant g explicitly. The integration contours in Eq.(2.18) are straight lines in the plane (x_1, x_2) going from x to infinity, and angles between neighboring lines $\Delta\varphi_j = 2\pi/N$ tend to zero when $N \to \infty$. Introducing a new constant $e = Ng$ and taking e fixed when $N \to \infty$ we have

$$\lim_{N\to\infty} P_2(x, N) = \lim_{N\to\infty} \exp\left[\frac{ie}{2\pi} \sum_j \Delta\varphi_j \int_0^\infty dr \, A_r(r, \varphi_j, \frac{\pi}{2}) \right]$$

$$= \exp\left[\frac{ie}{2\pi} \int_0^{2\pi} d\varphi \int_0^\infty dr \; A_r \right] \equiv \exp(ieI_2). \qquad (2.19)$$

Here $r = |\mathbf{x} - \mathbf{y}|$, $A_\mu dy_j^\mu = A_\mu (dy_j^\mu / dr) dr \equiv A_r(r, \varphi_j, \theta) dr$, i.e. A_r is the radial component of the 2-dimensional vector $\mathbf{A} = (A_1, A_2)$; the plane $x_3 = 0$ corresponds to $\theta = \pi/2$. The integral in the

[2] Incorporation of transformations with $\Lambda(x) \to const$, $x \to \infty$ implies in fact inclusion into the gauge group of global transformations (with $\partial_\mu \Lambda = 0$). Such an extension is possible and allows one of us to prove the superselection rule for the electric charge [7]; moreover, it leads to the conclusion that in the infinite Universe with all the matter confined into a finite volume the total electric charge is zero [7].

exponent (2.19) can be rewritten in the form

$$I_2 = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\infty} dr\ A_r = \frac{1}{2\pi}\int d^2y(\partial_r \ln r)A_r = -\frac{1}{2\pi}\int d^2y \ln r\ \left[r^{-1}\partial_r(rA_r)\right]. \qquad (2.20)$$

The final expression is nothing else than $-B_2$ because $(2\pi)^{-1}\ln|\mathbf{x}-\mathbf{y}| = \Delta_2^{-1}(\mathbf{x},\mathbf{y})$ is the kernel of the operator Δ_2^{-1}, and $r^{-1}(\partial_r rA_r) = (\nabla_\perp, \mathbf{A}_\perp)$ (in polar coordinates $(\nabla, \mathbf{A}) = r^{-1}(\partial_r rA_r) + r^{-1}\partial_\varphi A_\varphi$; here by construction $A_\varphi = 0$, see Eq.(2.18)). We conclude that $\lim P_2(x,N) = P_2(x)$ when $N \to \infty$.

Analogous consideration can be performed for $P_3(x) = \exp(-iB_3(x))$. Now there is a "2-dimensional" set of straight lines in the product substituting (2.18). Parametrizing it by two spherical angles φ_i, θ_j with $dy_i^\mu \to dy_{ij}^\mu$, so that to a line (i,j) there corresponds the solid angle $4\pi/N = \sin\theta_j\ \Delta\theta_j\Delta\varphi_i$ and taking $e = Ng$ being fixed when $N \to \infty$, we obtain an analog of the limit (2.19)

$$\begin{aligned}\lim_{N\to\infty} P_3(x,N) &= \lim_{N\to\infty} \exp\left[\frac{ie}{4\pi}\sum_{i,j}\sin\theta_j\ \Delta\theta_j\Delta\varphi_i \int_0^{\infty} dr\ A_r(r,\varphi_i,\theta_j)\right] \\ &= \exp\left[\frac{ie}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\ \sin\theta \int_0^{\infty} dr\ A_r\right] \equiv \exp(ieI_3), \end{aligned} \qquad (2.21)$$

where $A_\mu dy_{ij}^\mu \equiv A_r(r,\varphi_i,\theta_j)dr$. The integral in the exponent can be rewritten in the form

$$I_3 = \frac{1}{4\pi}\int d^3y\ r^{-2}A_r = -\frac{1}{4\pi}\int d^3y(\partial_r r^{-1})\ A_r = \frac{1}{4\pi}\int d^3y\ r^{-1}\left[r^{-2}\partial_r(r^2A_r)\right] = -B_3. \qquad (2.22)$$

The last equality in Eq.(2.22) is due to the identity $(-1/4\pi)|\mathbf{x}-\mathbf{y}|^{-1} \equiv \Delta^{-1}(\mathbf{x},\mathbf{y})$ and the formula for the divergence in the spherical coordinates $(\nabla, \mathbf{A}) = r^{-2}\partial_r(r^2A_r) + (r\sin\theta)^{-1}[\partial_\theta(\sin\theta\ A_\theta) + \partial_\varphi A_\varphi]$, assuming that $A_\theta = A_\varphi = 0$. It completes an anatomy of the Coulomb field.

We conclude that according to the first principles a particle surrounded by the Coulomb field is not the simplest charged object. Rather, it is a very complex object, the simplest one being a particle with one line (see also Sec.4). The experimental consequences of the hypothesis that N in Eq.(2.21) is large but finite has been studied in [9],[10].

2.4 Invariant structures in theories with gauge groups $SU(3)$ and $SU(2)$

For an obvious reason it is important to know invariant structures in the theory with the gauge group $SU(3)$ (chromodynamics). As usual we are interested only in the field configurations of the lowest rank and energy. The invariant tensors (besides the trivial one δ_α^β) are

$$\epsilon_{\alpha\beta\gamma}\ , \qquad \epsilon^{\alpha\beta\gamma}\ . \qquad (2.23)$$

Of ψ, $\bar\psi$, $P_{xx'}$ and these invariant totally antisymmetric unite tensors one can construct the following non-trivial invariants

$$\begin{aligned} Tr\ P[c(x,x)], \quad \Theta(x,x') &= \epsilon^{\alpha\beta\gamma}(P_{xx'})_\alpha^{\alpha'}(P'_{xx'})_\beta^{\beta'}(P''_{xx'})_\gamma^{\gamma'}\epsilon_{\alpha'\beta'\gamma'}, & (2.24)\\ \mathcal{M}_{xx'} &= \bar\psi(x)\ P_{xx'}\ \gamma_5\ \psi(x')\ , & (2.25)\\ \mathcal{B}_{x_1x_2x_3} &= \epsilon^{\alpha\beta\gamma}(P_{xx_1})_\alpha^{\alpha'}(P_{xx_2})_\beta^{\beta'}(P_{xx_3})_\gamma^{\gamma'}\psi_{\alpha'}(x_1)\psi_{\beta'}(x_2)\psi_{\gamma'}(x_3)\ , & (2.26)\end{aligned}$$

etc.; P, P', P'' in Eq.(2.24) differ by contours. The fields (2.24) represent the simplest local and bilocal physical configurations of a pure gluonic field. The configuration (2.25) is usually referred to as a mesonic, while (2.26) – as a barionic fields. Existence of the antisymmetric invariant tensors (2.23) implies that there are infinitely many topologically nonequivalent bilinear in ψ, $\bar\psi$ invariants because strings in this case may branch. Such string configurations may be represented by graphs with directed lines. With the string $P_\alpha^\beta(x,y) \equiv P_{\alpha x}^{\beta y}$ we associate a line connecting points x, y, an arrow being directed from the upper index to the lower one (i.e. the ends of the line are associated with indices too). Vertices with three outgoing or incoming lines correspond to the invariant tensors $\epsilon_{\alpha\beta\gamma}$ or $\epsilon^{\alpha\beta\gamma}$, respectively. Some simplest zero- , two- and three-point configurations are shown in Fig.1

The first two graphs represent the invariants (2.24). The third one corresponds to a bilocal g-invariant composed of the string P_{xy} with a quark (the field ψ_α, an open circle in the figure) and an antiquark (the field $\bar{\psi}^\alpha$, a black circle) at its ends. So, lines go from quark to antiquark fields. Another graphs in the figure give examples of bilocal and trilocal structures. Note of caution: it is graphs of a new type – they represent an instantaneous topology of strings connecting different space points. The graph j , for instance, is senseless.

Question: can a string change its topology with time? The string topology changes if and only if strings can break or branch. It is easily seen that in pure gluodynamics (no matter term in (2.1)) strings in the fundamental representation preserve their topology. Indeed, strings cannot break because in this theory there are no quarks, and open strings are forbidden by the gauge invariance [7]. Strings cannot branch either because, for example, transition from the loop a in Fig.1 to the configuration b assumes the existence of a vertex trilinear in glue fields (to ensure transition $A \to A\,A$) made with the use of the product of tensors (2.23) $\epsilon^{\alpha\beta\gamma}\epsilon_{\alpha'\beta'\gamma'}$ (see Eq.(2.24) for $\Theta(x,x')$). Such a combination is absent in the gluonic Lagrangian and may appear only in the effective action as a trilinear vertex containing the invariant symmetrical tensor $d^{abc} \sim Tr(\lambda^a\{\lambda^b,\lambda^c\})$ (the sign $\sim$ means: equal up to a constant). This is easily seen from the equalities

$$\epsilon^{\alpha\beta\gamma}A_\alpha^{\alpha'}A_\beta^{\beta'}A_\gamma^{\gamma'}\epsilon_{\alpha'\beta'\gamma'} \sim Tr(A\{A,A\}) \sim d^{abc}A^aA^bA^c \tag{2.27}$$

(A is a certain matrix, $A = A^a\lambda^a$, $TrA = 0$; in the first equality we used the identity $\epsilon^{\alpha\beta\gamma}\epsilon_{\alpha'\beta'\gamma'} = \delta_{\alpha'}^\alpha\delta_{\beta'}^\beta\delta_{\gamma'}^\gamma$ + *permutations with proper signs*). But effective vertices of this type never appear in pure gluodynamics. The latter from the very beginning contains only the structure constants f^{abc}, and from these constants one cannot construct the invariant symmetrical tensor d^{abc} [11] because $Tr(F^a\{F^b,F^c\}) = 0$, where $(F^a)_{bc} = f^{abc}$, i.e. in gluodynamics $\langle d^{abc}A_\mu^a(x)A_\nu^b(y)A_\rho^c(z)\rangle_0 = 0$. Hence, in pure gluodynamics strings in the fundamental representation cannot branch. We may expect the appearance of effective d-vertices only after the introduction of quarks (QCD, Lagrangian (2.1)). It turns out, however, that even in QCD they are absent, at least in lowest orders of perturbation theory. Indeed, the simplest triangle Feynman graph a in Fig.2 does not give them because it is accompanied by a graph with opposite directions of arrows, so that their sum is proportional to $Tr(\lambda^a[\lambda^b,\lambda^c]) \sim f^{abc}$. This is the case for more complex graphs too (e.g. for graph of order g^7 with six γ-vertices). In local limit such vertices vanish identically in all order of perturbation theory simply because $d^{abc}F_{\mu\nu}^aF_{\nu\rho}^bF_{\rho\mu}^c \equiv 0$. They appear only in the presence of some other vector fields interacting with quarks. For example, effective d-vertices arise from graph b in Fig.2, as it has an even number of γ- vertices in the fermion loop; the extra line there represents a photon. Due to this circumstance the string branching is accompanied by emission of a photon, and each vertex in Fig.1 enters with the factor $g^{3/2}e^{1/2} \sim \alpha_s(\alpha/\alpha_s)^{1/4}$, where α is the QED fine structure constant. Note that a certain function of momenta also enters there (square root of the corresponding vertex function).

The theory with the gauge group $SU(2)$ differs drastically from chromodynamics, because the complex conjugate representation of the 2×2 U-matrices ($UU^+ = 1, \det U = 1$) is unitary equivalent to the original one. The invariant antisymmetric tensors $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ transform spinors with the lower indices into those with the upper ones and vice versa. As a result, strings are not directed here. Furthermore, the tensors $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ have rank two , so the strings cannot branch. We conclude that a non-Abelian $SU(2)$ gauge theory differs strongly from any other theory with group $SU(n)$, $n > 2$ Rather, it is closer to electrodynamics, because strings there also do not branch.

A few words about more complex objects composed of strings. Till now we considered only an elementary representation which is sufficient for modern hadron physics - all the observed strongly interacting particles are assumed to be made of quarks realizing the fundamental representation of the gauge group $SU(3)$. However, strings can be taken in any representation. If, for instance, there is a matter field φ in an adjoint representation, one can construct invariants either bilinear in φ with the aid of a string in the adjoint representation, or the linear ones but with two strings in the fundamental representation and a quark and an antiquark at their ends. Generalization to higher representations is trivial.

In Sec.2.3 we observed that besides objects with the "1-dimensional external field" there are also those with 2- and 3-dimensional ones, the latter corresponding to the ordinary charged particles surrounded by the Coulomb fields. They can be considered as complex objects composed of infinitely many strings. It is important to prove or disprove the existence of such objects in chromodynamics (see Sec.4). Fortunately, no objects of this type are admissible in non-Abelian gauge theories: the corresponding source would realize in this case a certain very high representation of the gauge group, indefinite in the limit of an infinite number of strings. But we know from experiment that in QCD colored objects realize the lowest representations of the $SU(3)$ group. Thus, quarks and antiquarks are connected by "strings" with two ends and more or less rich topology (see Fig.1, c, d, e).

3 Invariant structures in gauge theories. A geometrical approach

As is well known [12], gauge fields are nothing but connections in the principal fiber bundle theory, i.e. they admit a natural geometrical interpretation. In the present section we analyze the problem of invariants (or polylocal g-tensors) from this quite different point of view. Though all the statements of this section are rigorous, we avoid the style admitted in mathematical literature [13]. Rather we simply introduce the needed objects and describe their properties to elucidate the geometrical nature of gauge field theories. All the consideration can be made rigorous, though much more lengthy.

3.1 Gauge fields as geometrical objects

Let $P(M,G)$ be the principal fiber bundle over base M with a semisimple compact group G. The space P may be considered as the base space M (it may be the Minkowski space) with a group manifold G attached to each its point x (the fibre over x) so that $M = P/G$, P being a differentiable manifold. Locally, i.e. in an open set $U \subset M$, a point $u \in P$ may be considered as a pair $u = (x,g)$ where $x \in U$, $g \in G$. In other words, we may always introduce locally a coordinate system so that x are coordinates in the open neighbourhood U and g are coordinates in corresponding fibers (the group manifolds). If a covering of M by a system of open neighbourhoods is defined, then one may determine local coordinates for all the manifold P. The principal fiber bundle P with thus defined coordinates on it is called a principal coordinate bundle in the sense of Steenrod [12]. The existence of this coordinate bundle is part of the definition of the principal fiber bundle. The group G acts on P as a group of right translations: $ug' = (x, gg')$, $g' \in G$, i.e. G translates points along the fiber which is isomorphic to G. The existence of a group action on the bundle space P gives rise to other possible coordinate bundles [12].

Further, we need the notion of a local cross section σ which may be considered as a surface in P isomorphic to some open set $U \subset M$, which crosses each fibre only once, i.e. $\sigma = \sigma(x) = (x, g(x))$. The existence of a global covering of base M by a system of open sets $\{U\}$ allows to define a system of local cross sections on the whole space P.

G acts on P as a translation along the fiber so at any point $u \in P$ one can define a tangent space G_u to each fiber. It is linearly isomorphic to the Lee algebra $\mathcal{A}(G)$ of the group G (a tangent space to a group manifold is a corresponding Lie algebra); G_u is called a *vertical linear space*. We denote this isomorphism by $\Sigma : \mathcal{A}(G) \rightarrow G_u$; the operator $\Sigma(A) \in G_u$ ($A \in \mathcal{A}(G)$) is called a fundamental vector field.

Introduction of the connection in P is equivalent to introduction at each point $u \in P$ of a space Q_u which is an orthogonal complement to G_u in the total tangent space T_u at $u \in P$, i.e. $T_u = G_u \oplus Q_u$; G_u and Q_u are called the vertical and horizontal subspaces, respectively.

A gauge field A_μ in this approach is a component of the connection form projected on a certain cross section σ in P. The connection form ω (for a given connection) has the following properties: $\omega(X) = 0$ if and only if $X \in Q_u$ – the horizontal space (by definition $\omega(X) = \omega_i X^i$ for any $X \in T_u$, where index i numerates components (coordinates) of the 1-form ω and the field X in a certain basis [3]), and $\omega(\Sigma(A)) = A$ for any $A \in \mathcal{A}(G)$. The 1-form ω is defined on the whole space P. Its projection to a cross section σ is [4].

$$\sigma^*(\omega) \equiv \omega_\sigma = A_\mu dx^\mu \; . \tag{3.1}$$

This projection may be determined in a local coordinate system as a formal replacement of du by $d\sigma(x)$ in ω and u by σ in its coefficients.

Gauge transformations in P are locally (i.e. in a neighbourhood $U \subset M$) defined with the help of functions $g\ : U \to G$ so that a local cross section changes according to the law $\sigma \to \sigma g$, $g = g(x)$. The gauge transformations in P induce the transformations of the 1-form ω_σ components, i.e. of the fields A_μ. Indeed, the transformation $\sigma \to \sigma g \equiv (x, g'(x)g(x))$ induces the transformation $\omega_\sigma \to \omega_{\sigma g} = A^g_\mu dx^\mu$, A^g_μ are [12]

$$A_\mu \to A^g_\mu = g^{-1} A_\mu g \; + \; g^{-1}\partial_\mu g \; . \tag{3.2}$$

For this reason the coefficients A_μ are identified with the Yang-Mills fields and are called gauge potentials.

A local tensor (matter field) ψ in the bundle P may be defined as a function on P realizing a linear representation T of the group G, which should satisfy the following condition [14]

$$\psi(ug) = T_{g^{-1}}\psi(u), \quad g \in G \tag{3.3}$$

where T_g is an element $g \in G$ in the representation T. The conjugated field ψ^* is analogously defined as an element of the conjugated representation T^*

$$\psi^*(ug) = \psi^*(u)T_g \; , \quad g \in G \; . \tag{3.4}$$

Functions $\psi(\sigma(x)) \equiv \psi_\sigma(x)$ and $\psi^*(\sigma(x)) \equiv \psi^*_\sigma(x)$ are tensors on a cross section σ. The transformation law for these tensors induced by the transformation $\sigma \to \sigma g$ follows from the relations (3.3) and (3.4)

$$\psi_\sigma \;\to\; \psi(\sigma g) = T_{g^{-1}}\psi_\sigma \; , \qquad \psi^*_\sigma \;\to\; \psi^*(\sigma g) = \psi^*_\sigma T_g \; . \tag{3.5}$$

The scalar product

$$(\psi_a, \psi_a) = \psi^*_a(\sigma)\psi^a(\sigma) \; , \tag{3.6}$$

where a enumerates components of the field ψ in the representation T, is the simplest gauge-invariant.

Thus, we see that, as in Sec.2, the description of invariants depending on A_μ is reduced to the description of polylocal tensors in P which realize the tensor representation $(\prod_1^n \otimes T) \otimes (\prod_1^m \otimes T^*)$ of G, i.e. to finding tensors T which transform under gauge transformations (3.2) as

$$T(\sigma_1 g, ..., \sigma_n g; \sigma'_1 g, ..., \sigma'_m g) = \left(\prod_{i=1}^n \otimes T_{g^{-1}}(x_i)\right) T(\sigma_1, ..., \sigma_n; \sigma'_1, ..., \sigma'_m) \left(\prod_{j=1}^m \otimes T_g(x'_j)\right) \tag{3.7}$$

where $\sigma_i = \sigma(x_i)$ and $\sigma'_j = \sigma(x'_j)$. The law (3.7) is the general transformation law for polylocal tensors depending on A_μ and ψ_a, ψ^*_a.

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[3]Let M be a manifold, T_y be a tangent space to M at a point $y \in M$. Then in a local coordinate system, in any open neighbourhood of y one may define $X = X^i\partial/\partial y^i \in T_y$ and $\omega = \omega_i dy^i$; by definition $\omega(X) = \omega_i X^i$.

[4]In this section, following the notation accepted in mathematical literature we substitute $A_\mu \to iA_\mu$

3.2 Polylocal tensors

The connection form ω is the fundamental geometrical object in the bundle P containing gauge potentials. All other structures depending on its components A_μ have to be expressed via ω_σ because there are no other objects in the bundle P. Therefore, tensors depending on A_μ must be also built of ω_σ. However, the 1-form ω_σ transforms non-homogeneously under gauge transformations. Consequently, one has to find homogeneously transforming functions of ω_σ. Using (3.2) we find that

$$1 - \omega_\sigma \equiv P(\sigma(x+dx), \sigma(x)) = P_\sigma(x+dx, x) \tag{3.8}$$

is a bilocal tensor

$$P_{\sigma g}(x+dx, x) = g^{-1}(x+dx) P_\sigma(x+dx, x) g(x) \,. \tag{3.9}$$

The infinitesimal bilocal tensor P_σ is the only (up to a factor) linear function of ω_σ transforming homogeneously. Therefore, any finite tensor (functional of A_μ) should be constructed of P_σ, and we may simply repeat the reasoning of Sec.2, beginning from Eq. (2.6).

Remark. In building tensors of bilocal tensor (3.8), one should take into consideration the representation under which the matter fields transform. It can be done by the substitution

$$\omega_\sigma \to \omega_\sigma^\Sigma = \Sigma(A_\mu) dx^\mu \tag{3.10}$$

in (3.8), i.e. the fundamental field $\Sigma(A_\mu)$, a linear operator in the representation space $\mathcal{T}$, substitutes for the element A_μ of Lie algebra in A_σ. To determine the action of the field $\Sigma(A_\mu)$ on a local tensor, we use the definition of $\Sigma(A_\mu)$ according to which $\Sigma(A_\mu) \in G_*$ so the field $\Sigma(A_\mu)$ is a generator of the parallel transport along a curve belonging to G_*. We may take such a curve passing through a point $\sigma(x)$ in the form $u_\tau = \sigma(x)\exp(\tau A_\mu n^\mu) \equiv \sigma(x) g_\tau(A_\mu n^\mu)$ where $n_\mu n^\mu = 1$, $g_\tau \in G$ and τ is a parameter of the curve. Apparently, $n_\mu \Sigma(A_\mu)$ is a tangent vector field for the curve and, hence, its action on a local tensor may be defined as its variation under the infinitesimal parallel transport along the curve g_τ. Let ψ be a function on P satisfying the condition (3.2), then (see also [12])

$$\Sigma(A_\mu)\psi_\sigma = \lim_{\tau \to 0} \tau^{-1} \left(T_{g_\tau^{-1}} \psi - \psi \right) |_\sigma = -\mathcal{T}(A_\mu)\psi|_\sigma = -\mathcal{T}(A_\mu)\psi_\sigma \tag{3.11}$$

where $T_{g_\tau^{-1}}\psi_\sigma$ gives a tensor ψ_σ transported along the curve g_τ and $\mathcal{T}(A_\mu)$ is an element of the Lie algebra $\mathcal{A}(G)$ in the representation $\mathcal{T}$. Using the relation (3.11) one may easily show that under gauge transformations the field $\Sigma(A_\mu)$ changes according to the law (3.2) where $A_\mu \to \Sigma(A_\mu)$ and $g \to T_g$. Therefore, any tensor built of $P_\sigma^\Sigma = 1 - \omega_\sigma^\Sigma$ has the gauge transformation law coinciding with (3.7).

Construction of invariants is based on the gauge transformation law (3.2) induced by the projection of ω on σ. Question: does there exist another projection of ω, $\tilde{\omega}_\sigma$ with coefficients $\tilde{A}_\mu$ which are nonlinear functionals of A_μ and undergo the same gauge transformation law (3.2)? According to the reconstruction theorem [12],[13] the connection form in P may be reproduced from local forms $\tilde{\omega}_\sigma$. However, the *should correspond to the same (given) connection in P*. Hence, the forms ω_σ and $\tilde{\omega}_\sigma$ may differ only by a coordinate system in fibers, i.e. by a cross section choice. Therefore, $\tilde{\omega}_\sigma = \omega_{\sigma\phi}$ where $\phi \in G$ depends on A_μ. It follows from the obvious equalities $\tilde{\omega}_{\sigma g} = \omega_{\sigma\phi g} = \omega_{\sigma g \phi'}$ where $\phi'(A) = \phi(A^g)$ that ϕ transforms homogeneously, $\phi(A) \to \phi(A^g) = g^{-1}\phi(A) g$. Thus, $\phi(A)$ is a local tensor and may be represented as a series over the covariant derivative $D_\mu = \partial_\mu - \Sigma(A_\mu)$ since it is the only linear in A_μ local tensor. D_μ are generators of a horizontal transport orthogonal to fibers [12]. However, $\phi \in G$ and consequently, acting on any local tensor, ϕ translates it along a fiber. Hence every term of the ϕ series should be composed of such combinations of D_μ which generate the vertical transport, i.e. they must belong to the Lie algebra $\mathcal{A}(G)$ [13]. Thus

$$\phi^\Sigma(A) = \sum_{n=0}^{\infty} C^{\mu_1\mu_2\cdots\mu_n\mu\nu}(x) D_{\mu_1} D_{\mu_2} \cdots D_{\mu_n} \Sigma(F_{\mu\nu}) + const \tag{3.12}$$

where $\Sigma(F_{\mu\nu}) = [D_\mu, D_\nu]$ is the vertical field

According to the Ambrous-Singer theorem [13] (see also [15]), $D_{\mu_1} D_{\mu_2} \cdots D_{\mu_n} \Sigma(F_{\mu\nu}), n = 0, 1, \ldots,$ so $\phi^\Sigma(A)$ should be an element of the holonomy group and there should exist a closed contour $C(x)$ such that

$$\phi(A) = P \exp \oint_{C(x)} A_\mu dx^\mu \; . \tag{3.13}$$

We see that ϕ is built of P_σ (or P_{dx}).

We come to the following conclusion. All invariants can be built of the path-ordered exponents

$$P^\Sigma_{xx'} = P \exp \left(- \int_C \Sigma(A_\mu) dx^\mu \right) \; , \tag{3.14}$$

the matter fields ψ and ψ^* and of invariant tensors (like $\epsilon_{ab\ldots}$) independent of A_μ, ψ and ψ^* in the representations $\mathcal{T}$ and $\mathcal{T}^*$.

4 Static interparticle forces. Calculations

We have already mentioned in the previous sections that external fields surrounding charged objects appear as a manifestation of the first-class constraints and are responsible for the static interparticle forces. In the present section we study this aspect of the problem in detail both in classical and quantum theories. In the Abelian theory besides the familiar Coulomb forces there may also exist logarithmically and linearly rising static potentials. In pure gluodynamics we find confinement for external classical sources, i.e. their static interaction is given by a linear potential. We start with electrodynamics which serves as a pattern for study non-Abelian models.

4.1 Electrodynamics. A classical theory

As is well known, in electrodynamics there are two first-class constraints

$$\pi_0 = 0, \tag{4.1}$$

$$(\nabla, \mathbf{E}) - j_0 = 0, \tag{4.2}$$

where $\pi^\mu = \partial\mathcal{L}/\partial\dot{A}_\mu$ is canonically conjugated to A_μ momentum,$\pi^k = E^k$ and j_0 is the zero component of the electric current of charged fields ($j_\mu = -\partial\mathcal{L}/\partial A_\mu$). The constraint (4.1) is trivial (it says that the component A_0 is an unphysical variable), while (4.2) contains important physical information. It involves a number of physical variables and states that one of them is unnecessary for description of the dynamical system. In the integral form Eq.(4.2) identifies the flux of lines of electric forces through some closed surface with an electric charge inside it, stating that these two notions are physically equivalent (i.e. indistinguishable). Analogous statement about fields can be made addressing the local Eq (4.2) - one of the fields can be eliminated. Usually it is called "unphysical". But which of the fields entering into Eq.(4.2) is unphysical? For instance, rewrite Eq.(4.2) in the forms

$$\mathbf{E}_\parallel = \Delta^{-1}\nabla j_0, \quad \mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_\perp, \quad (\nabla, \mathbf{E}_\perp) = 0, \tag{4.3}$$

$$\mathbf{E}_{pl} = \Delta_2^{-1}\nabla_\perp(j_0 - \partial_3 E_3), \quad \mathbf{E} = \mathbf{E}_{pl} + \mathbf{n}_3 E_3, \tag{4.4}$$

$$E_3 = \partial_3^{-1}(j_0 - (\nabla_\perp, \mathbf{E}_{pl})); \quad (\nabla_\perp, \nabla_\perp) = \partial_1^2 + \partial_2^2 \equiv \Delta_2; \tag{4.5}$$

in the last two formulas $\mathbf{n}_3^2 = 1$, $(\mathbf{n}_3, \mathbf{E}_{pl}) = 0$, i.e. $\mathbf{E}_{pl}$ is a "planar" vector. According to Eqs.(4.3)-(4.5) one can eliminate as an unphysical variable either $\mathbf{E}_\parallel$ or $\mathbf{E}_{pl}$, or else E_3. What is the criterion for the choice?

[5] As is above-mentioned, the linear isomorphism Σ preserves the structure of the Lie algebra $\mathcal{A}(G)$ in G_u therefore $[\Sigma(A_\mu), \Sigma(A_\nu)] = \Sigma([A_\mu, A_\nu])$, hence, the strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \in \mathcal{A}(G)$ and $\Sigma(F_{\mu\nu}) \in G_u$

We know from Sec.2 that in electrodynamics there are three types of static gauge invariant objects given by Ψ_k (Eq.(2.17)), i.e. charged particles in principle may possess different exterior static fields. In the case of Ψ_3 the electric field component $\mathbf{E}_{\parallel}$ (Eq.(4.3)) is attached to a "charge", so it can be eliminated from dynamics, because its evolution is determined by the motion of charged matter. Only the other two components $\mathbf{E}_{\perp}$ can propagate independently. Hence, we can describe the dynamics in terms of matter fields and the field $\mathbf{A}_{\perp}$, $(\nabla, \mathbf{A}_{\perp}) = 0$ ("radiation gauge", i.e. the radiated field is $\mathbf{A}_{\perp}$). On the contrary, in the cases Ψ_1 and Ψ_2 the field components $\mathbf{E}_{pl}(E_1, E_2)$ and E_3 respectively (Eqs.(4.4), (4.5)) are attached to charges. They also can be eliminated because their future is predestinated by equations of motion of charged matter. Of course, the elimination gives rise to some additional terms in the Hamiltonian $H_0 = \int d^3x[\mathbf{E}^2 + \mathbf{H}^2]/2$. For example, substituting $\mathbf{E}_{\parallel}$ given by Eq.(4.3) into H_0 we obtain the familiar expression

$$H_0 = \frac{1}{2}\int d^3x \left[\mathbf{E}_{\perp}^2 + \mathbf{H}^2 - j_0\Delta^{-1}j_0\right], \tag{4.6}$$

where the last term describes the Coulomb interaction. Analogously, we obtain the following static interactions for the cases (4.4), (4.5) when the components $\mathbf{E}_{pl}$, E_3 are eliminated (for simplicity we take "plane" $j_0 \to \delta(x_3)\tilde{j}_0(\mathbf{x}_{\perp})$ and "linear" $j_0 \to \delta_{\perp}(\mathbf{x})\tilde{j}_0(x_3)$ sources respectively).

$$\frac{1}{2}\int d^3x \left(\nabla_{\perp}\Delta_2^{-1}j_0, \nabla_{\perp}\Delta_2^{-1}j_0\right) = -\frac{1}{2}\delta(0)\int d^2x \tilde{j}_0\Delta_2^{-1}\tilde{j}_0, \tag{4.7}$$

$$\frac{1}{2}\int d^3x \partial_3^{-1}j_0\partial_3^{-1}j_0 = -\frac{1}{2}\delta_{\perp}(0)\int dx_3\tilde{j}_0\partial_3^{-2}\tilde{j}_0, \tag{4.8}$$

where $\delta_{\perp}(\mathbf{x}) = \delta(x_1)\delta(x_2)$, and $(\partial_3^{-1})_{x_3x_3'} = \theta(x_3 - x_3')$. For two point sources [10] in the latter case $\tilde{j}_0 = \delta(x_3 - x_3') - \delta(x_3 - x_3")$, $\partial_3^{-1}j_0 = \delta_{\perp}(\mathbf{x})[\theta(x_3 - x_3') - \theta(x_3 - x_3")]$, one rewrites Eq.(4.8) as $(1/2)\delta_{\perp}(0)|x_3' - x_3"|$, thus obtaining a linearly rising potential.

We conclude that static interparticle forces depend on the configurations of static external fields, which appear as a consequence of the secondary first-class constraints.

4 2 Electrodynamics. A quantum theory

In this subsection we calculate the mean value of electric field $\hat{\mathbf{E}}$ for the static configurations given by Eq.(2.17), $k = 1, 2, 3$ and the corresponding interaction energies of a couple of point-like sources. Using the canonical commutation relations for $\hat{A}_j, \hat{E}^k$: $[\hat{A}_j(\mathbf{x}), \hat{E}^k(\mathbf{y})] = i\delta_j^k\delta(\mathbf{x} - \mathbf{y})$ we find the following commutation relations for operators $\hat{\mathbf{E}}$ and $\hat{P}_k$ (Eq.(2.17)), writing however for future use $\hat{P}_{xx'}$ instead of $\hat{P}_1$

$$\hat{E}^j(\mathbf{y})\hat{P}_{xx'} = \hat{P}_{xx'}\left[\hat{E}^j(\mathbf{y}) + \int_{x'}^{x} dz^j\delta(\mathbf{z} - \mathbf{y})\right], \tag{4.9}$$

$$\hat{E}^j(\mathbf{y})\hat{P}_{2x} = \hat{P}_{2x}\left[\hat{E}^j(\mathbf{y}) + \int d^2z(\nabla_{\perp}^j\Delta_2^{-1})_{\mathbf{xz}}\delta(\mathbf{z} - \mathbf{y})\right], \tag{4.10}$$

$$\hat{E}^j(\mathbf{y})\hat{P}_{3x} = \hat{P}_{3x}\left[\hat{E}^j(\mathbf{y}) + \int d^3z(\nabla^j\Delta^{-1})_{\mathbf{xz}}\delta(\mathbf{z} - \mathbf{y})\right] \tag{4.11}$$

(all the times are equal $y_0 = x_0 = x_0'$). Then one gets for the mean values of $\hat{E}^j$ in the states Ψ_k assuming $\langle\hat{E}^j\rangle_0 = 0$

$$\langle\hat{P}_{1x}^+\hat{E}^j(\mathbf{y})\hat{P}_{1x}\rangle_0 = \int_{-\infty}^{x} dz^j\delta(\mathbf{z} - \mathbf{y}), \tag{4.12}$$

$$\langle\hat{P}_{2x}^+\hat{E}^j(\mathbf{y})\hat{P}_{2x}\rangle_0 = \int d^2z(\partial_{\perp}^j\Delta_2^{-1})_{\mathbf{xz}}\delta(\mathbf{z} - \mathbf{y}), \tag{4.13}$$

$$\langle \hat{P}^+_{3x}\hat{E}^j(\mathbf{y})\hat{P}_{3x}\rangle_0 = (\partial^j\Delta^{-1})_{xy} = \frac{y-x}{4\pi|\mathbf{y}-\mathbf{x}|^3} \ . \tag{4.14}$$

We observe that the average (4.12) is zero everywhere except the integration line, where it is (in our units) a constant equal to $\delta^{(2)}(0)$ (the factor $\delta^{(2)}(0)$ appears because the radius of the line is zero). Thus, we have a single line of the electric force; it is identical with the electric field of a point-like charge in 1-dimensional electrodynamics. Further, one easily recognizes in (4.13) and (4.14) the electric fields of point-like charges respectively in 2- and 3-dimensional theories. The former behaves as the electric field of a charged straight line, while the latter is the standard Coulomb field. These formulae establish correspondence between structures (2.17) and the average electric fields.

The forces between any two point-like charges can be easily found from expressions (4.12)-(4.14) (by the formula $\mathbf{F} = q\mathbf{E}$), but we prefer to obtain the interaction energy directly, calculating the energy of the external electric field. Consider configurations $\bar{\psi}(x)P_{xx'}\psi(x')$, $\Psi^+_k(x)\Psi_k(x')$, $k = 2, 3$. We are interested in the energy of excited (i.e. "induced" by charge) electromagnetic field, so we omit the matter fields ψ in these expressions to avoid unnecessary complications. Using the formulae $\hat{P}^+_{xx'}\hat{P}_{xx'} = 1$, $\hat{P}^+_{kx}\hat{P}_{kx} = 1$ and Eqs.(4.9)-(4.11), one has

$$\langle \hat{P}^+_{xx'}\hat{H}_0\hat{P}_{xx'}\rangle_0 = C_0 + \frac{1}{2}\int_{x'}^{x} dz_1^j \int_{x''}^{x} dz_2^j \delta(z_1 - z_2) = C_0 + \frac{1}{2}\delta^{(2)}(0)|\mathbf{x}-\mathbf{x}'|, \tag{4.15}$$

$$\begin{aligned}\langle \hat{P}^+_{2x'}\hat{P}^+_{2x}\hat{H}_0\hat{P}_{2x}\hat{P}_{2x'}\rangle_0 &= C_0 + \frac{1}{2}\int d^2z_1 d^2z_2 \left[(\partial^j_\perp\Delta_2^{-1})_{xz_1} + (\partial^j_\perp\Delta_2^{-1})_{x'z_1}\right]\delta(\mathbf{z}_1-\mathbf{z}_2) \times \\ &\times \left[(\partial^j_\perp\Delta_2^{-1})_{xz_2} + (\partial^j_\perp\Delta_2^{-1})_{x'z_2}\right], \end{aligned}\tag{4.16}$$

$$\begin{aligned}\langle \hat{P}^+_{3x'}\hat{P}^+_{3x}\hat{H}_0\hat{P}_{3x}\hat{P}_{3x'}\rangle_0 &= C_0 + \frac{1}{2}\int d^3z_1 d^3z_2 \left[(\nabla\Delta^{-1})_{xz_1} + (\nabla\Delta^{-1})_{x'z_1}\right]\delta(\mathbf{z}_1-\mathbf{z}_2) \times \\ &\times \left[(\nabla\Delta^{-1})_{xz_2} + (\nabla\Delta^{-1})_{x'z_2}\right] \ , \end{aligned}\tag{4.17}$$

where $\hat{H}_0$ is the electromagnetic field Hamiltonian, $C_0 \equiv \langle\hat{H}_0\rangle_0$; the effect comes from the term $\langle\hat{P}^+\hat{\mathbf{E}}^2\hat{P}\rangle$. For the linear structure $\hat{P}_{xx'}$ one obtains the potential (4.15) linearly rising with distance (opposite charges). It justifies identification of the P-exponent with a string. Subtracting from the r.h.s.'s of Eqs.(4.16) and (4.17) self-energies of sources (infinite constants) one gets the interaction energies $(\Delta_2^{-1})_{xx'} = (1/2\pi)\ln|\mathbf{x}-\mathbf{x}'|$ and $(\Delta^{-1})_{xx'} = (-1/4\pi)|\mathbf{x}-\mathbf{x}'|^{-1}$, the latter being the Coulomb potential.

Thus, as it should be, the static interaction of charged objects depends on their exterior static fields, and in the case of P-exponent $P_{xx'}$ it is given by the linearly rising potential (4.15). If the integration contour in Eq.(4.9) is not the straight line, then the length of the contour substitutes for the distance $|\mathbf{x}-\mathbf{x}'|$ in Eq.(4.15).

Remark. The above analysis elucidates connection between the configuration of external fields and the corresponding gauge conditions.The Coulomb (radiation) gauge $(\nabla, \mathbf{A}) = 0$ is natural for studying electrodynamics of charged objects of type Ψ_3, Eq.(2.17). The Coulomb field $\mathbf{A}_\parallel$ $(\mathbf{A} = \mathbf{A}_\parallel + \mathbf{A}_\perp,\ (\nabla, \mathbf{A}_\perp) = 0)$ is not an independent variable and is attached to ψ. The field component $\mathbf{A}_\perp$ propagates independently and describes a radiation, i.e. by writing in this case $(\nabla, \mathbf{A}) = 0$ one identifies in fact $\mathbf{A} = \mathbf{A}_\perp$. On the contrary, for objects of type Ψ_1 with the integration contour along the axis "3" the component $A_3 = \mathbf{n}_3\mathbf{A}$ is attached to the charged field $(\mathbf{A} = \mathbf{n}_3 A_3 + \mathbf{A}_\perp,\ \mathbf{n}_3\mathbf{A}_\perp = 0)$, so A_3 is eliminated from dynamics as an independent variable, and the physical field $\mathbf{A}$ $(\equiv \mathbf{A}_\perp)$ should satisfy the condition $\mathbf{n}_3\mathbf{A}$. It is the axial gauge. Note that in the latter case the choice of the axis depends on the problem, i.e. on the direction of the straight line connecting charges. Of course, in both the cases one can choose any other gauge but it will not be natural – it complicates the description.

4.3 Non-Abelian gauge theories

As it was shown in Secs.2,3, in non-Abelian gauge theories for any fixed representation of a charged matter field there exist structures only with a fixed number of "incoming" or "outgoing" strings attached to the "charge". Let us find static forces between charges in the simplest case of excitation born by the operator (2.25) (ψ realizes an elementary representation of $SU(n)$, $n \geq 2$). The answer is almost evident, nevertheless, we give a short derivation of the potential for the sake of completeness. *Only quantum theory is considered.*

The equal-time canonical commutators in this case are

$$\left[\hat{A}^a_j(\mathbf{x}) \, , \, \hat{E}^k_b(\mathbf{y}) \right] = i\delta^k_j \delta^a_b \delta(\mathbf{x} - \mathbf{y}) \, . \tag{4.18}$$

The commutation relation (4.9) for a path-ordered exponent changes its form. Introducing parameter-dependent λ-matrices one rewrites the P-exponent in the form

$$\hat{P}_{xx'} = P \exp \left[i \int_0^1 \hat{A}^b_\mu(z(\sigma))\lambda_b(\sigma)\dot{z}^\mu(\sigma)d\sigma \right] \tag{4.19}$$

where $\dot{z}^\mu(\sigma) = dz^\mu/d\sigma$, and $z(0) = x'$, $z(1) = x$. Now one can ignore noncommutativity of the matrices [16], and instead of (4.9) we have

$$\begin{aligned} \hat{E}^j_a(\mathbf{y})\hat{P}_{xx'} &= \hat{P}_{xx'}\hat{E}^j_a(\mathbf{y}) + \\ &+ P \left[\exp \left(i \int_0^1 \hat{A}^c_\mu(z(\sigma))\lambda_c(\sigma)\dot{z}^\mu(\sigma)d\sigma \right) \int_0^1 \lambda_a(\tau)\dot{z}^j(\tau)\delta\left(\mathbf{z}(\tau) - \mathbf{y}\right) d\tau \right] . \end{aligned} \tag{4.20}$$

Path ordering in the last term refers both to operators $\hat{A}^c_\mu$ and all the matrices λ. For calculation of the average of the gluonic field energy $\hat{H}^g_0 = \int d^3x [\mathbf{E}^2_a + \mathbf{H}^2_a]/2$ in the state $\mathcal{M}_{\mathbf{xx}'}|0\rangle$ (see Eq.(2.25): $|0\rangle$ is the physical vacuum) one needs the equal-time commutation relations ($x_0 = y_0$) for ψ:

$$\left[\hat{\psi}^+_\alpha(\mathbf{x}) \, , \, \hat{\psi}_\beta(\mathbf{y}) \right]_+ = \delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{y}) \tag{4.21}$$

where α, β stand both for spinor and color indices. To simplify formulae, we again neglect irrelevant to the problem factors $2n[\delta^{(3)}(0)]^2$ appearing when $m_q \to \infty$. Appearance in Eq.(4.21) of the Kronecker symbol $\delta_{\alpha\beta}$ allows us to represent the final expression for the gluonic average energy as a trace. We have in the limit $m_q \to \infty$:

$$\begin{aligned} \langle 0| \ \mathcal{M}^+_{\mathbf{xx}'}\hat{H}^g_0\mathcal{M}_{\mathbf{xx}'}|0\rangle &\doteq \langle \hat{H}^g_0 \rangle_0 + \\ + \ \frac{1}{2n}\langle Tr & \left\{ \tilde{P} \left[\int d^3\mathbf{y} \exp \left(-i \int_0^1 \hat{A}^c_\mu(z(\sigma'))\lambda_c(\sigma')\dot{z}^\mu(\sigma')d\sigma' \right) \int_0^1 \lambda_a(\tau')\dot{z}^j(\tau')\delta\left(\mathbf{z}(\tau') - \mathbf{y}\right) d\tau' \right] \times \right. \\ \times \ & \left. P \left[\exp \left(i \int_0^1 \hat{A}^c_\mu(z(\sigma))\lambda_c(\sigma)\dot{z}^\mu(\sigma)d\sigma \right) \int_0^1 \lambda_a(\tau)\dot{z}^j(\tau)\delta(\mathbf{z}(\tau) - \mathbf{y})d\tau \right] \right\} \rangle_0 \ . \end{aligned} \tag{4.22}$$

Here the equality $\doteq$ means: equal up to the normalization factor $2n[\delta^{(3)}(0)]^2$. In Eq.(4.22) we used the limits $\hat{\psi}^{(+)}|0\rangle \to 0$, $\bar{\hat{\psi}}^{(+)}|0\rangle \to 0$, $m_q \to \infty$, where $\hat{\psi}^{(+)}$ ($\bar{\hat{\psi}}^{(+)}$) contains the quark (antiquark) annihilation operators, and omitted the self-energy term. The symbol $\tilde{P}$ there indicates an antiordering (relative to P). Due to this circumstance the last term in Eq.(4.22) (i.e. the interaction energy of charges) takes the extremely simple form

$$
\begin{aligned}
V_{xx'} &= \frac{1}{2n} Tr\lambda_a^2 \int_0^1 d\tau \int_0^1 d\tau' \dot{z}^j(\tau)\dot{z}^j(\tau')\delta(\mathbf{z}(\tau)-\mathbf{z}(\tau')) = \\
&= \frac{1}{2n} Tr\lambda_a^2 \delta_\perp(0)|\mathbf{x}-\mathbf{x}'|,
\end{aligned} \tag{4.23}
$$

the last equality being valid for the straight line contour. We conclude that in the non-Abelian gauge theories the string-like external gluonic field given by the P-exponent (4.19) also leads to the linearly rising potential. As is clear from the previous sections, for *elementary* representations a single (possibly branched) string is ***the only allowed*** external field configuration; structures analogous to the Coulomb field are forbidden.

5 The problem of confinement

In this section we study consequences of the previous analysis. It is instructive to consider "pure gluodynamics" and the corresponding theory with matter separately. It is also useful to compare theories with different groups, as we did it in Sec.2.

5.1 Static interaction in gauge theories. Discussion

A. *Pure gluodynamics*

Under pure gluodynamics we understand any gauge theory without matter, including the Abelian case (e.g. electrodynamics). We discuss static interactions of classical sources (static quarks, i.e. massive particles, $m_q \to \infty$, realizing the elementary representations of the gauge group).

1. *Abelian theory (electrodynamics).* According to Sec.2, a priori there are three possible configurations of static electric fields of charged objects, when the field is non-zero (i) on a straight line, (ii) on a plane, and (iii) everywhere (the Coulomb field).

In the case (i) two external sources with opposite charges are connected by a string, and according to Eq.(4.15) their static interaction is given by a linearly rising potential. Note that the string cannot break here, however long it be, because there are no charged particles of finite masses, and a string without charge at the end is nonsense (it is not a gauge-invariant object). Further, the strings in the Abelian theory have no structure (they do not branch). We call it an "absolute" or strong confinement. The problem of the existence of objects of that sort in Nature remains open. Dirac [17] seriously considered this possibility.

The case (ii) is presumably of an academic interest because one should expect that such a field configuration is unstable. Ignoring this circumstance, one obtains for (ii) a potential logarithmically rising with distance; the situation is intermediate between the case (i), corresponding to the string, and the case (iii) when one has the Coulomb potential. The latter potential is not confining though one can still show that the total electric charge of the Universe should be zero [7]. Evidently, this statement should be true for charges of the other two types too.

2. *Gluodynamics. a) The gauge group $SU(2)$.* This theory is analogous to electrodynamics with strings (case (i)). Here strings also neither break nor branch; the existence of a linearly rising potential completes similarity of the two cases, i.e. we have strong confinement.

b) *The gauge group $SU(3)$.* This theory does not differ in fact from the previous one. It is evident that strings do not break but they do not branch either because in chromodynamics without matter (gluodynamics) strings cannot change their topology (Sec.2; in building invariants we can use only those entities which enter into the Lagrangian, but the gluonic Lagrangian does not contain invariant tensors (2.23)). So, the sources (heavy quarks) are connected by a simple string, and the situation is identical with that in the case a), i.e. we have static potential (4.23).

c) *The gauge group* $SU(n)$. It is evident that the situation here is identical with those in the cases a), b).

B. *Gauge theories with matter*

Introduction of matter changes dynamics radically; now (i) strings can break, (ii) they can branch (in theories with $G = SU(n)$, $n > 2$).

1. *Electrodynamics (case (i)) and theories with the gauge group* $SU(2)$. The only effect of matter in this case is the possibility of string breaking. A sufficiently long string can break, the critical length depending on its tension and on the masses and spins of neutral particles. Passing from sources to quarks at the ends of the string does not change the principal features of the picture. It is just what one usually expects to occur in reality (in chromodynamics) - a long enough string breaks, and at its newly born ends opposite charges are set, so that these objects are also gauge-invariant. We conclude that in this case one has a standard situation ("normal" confinement). In electrodynamics, cases (ii),(iii), the effect of matter depends on the particle masses and the "string" tension (see Subsec.5.2).

2. *Chromodynamics.* In contrast to these theories in chromodynamics, strings can branch. This complicates the description of static interaction. Now one has to calculate the interaction energy for every bilocal multistring configuration, some of which are represented in Fig.1(d,e). The sum of these functions of distance with proper weights gives the needed potential V. The general form of V in the case is

$$V(r) = \sum_{v} w_v E_v(r), \tag{5.1}$$

where

$$E_v(r) = \sum_{i} c_{vi} l_i \tag{5.2}$$

is the energy of a multistring configuration with v vertices, and l_i are the lengths of the strings connecting vertices. The weights w_v are positive and normalized, $\sum_v w_v = 1$, while some coefficients c_{vi} in (5.2) may be negative. This complicates the problem of evaluation or estimation of the quark potential.

5.2 The variety of confinements

Evidently, the term "confinement", in view of variety of physical situations connected with this phenomenon, is too general. Indeed, there are at least three different types of field theories exhibiting confinement: (i) an Abelian theory, (ii) a non-Abelian theory, (iii) a gauge theory without matter. One should also distinguish qualitatively different pictures, arising as a result of interplay of physical parameters, such as the string tension, the lowest particle masses. We observe the following possibilities:

1. *Weak confinement.* Confinement in its weakest form states that only colorless field excitations are physical. The statement refers both to Abelian and non-Abelian theories. As for the latter, it seems trivial because a colored object is always accompanied by a string (Secs.2.3), and a string should always be finite with charges at the ends – infinite strings are unphysical objects (they are not gauge-invariant, they have infinite energies). The Abelian case is more tricky though, as we already mentioned, the total electric charge of the Universe in this case should also be zero [7]. We conclude that weak confinement takes place in every gauge theory. But in contrast to confinement in the strict sense, the static forces in electrodynamics may decrease with distance (*quasiconfinement*).

2. *Strong confinement* – when static forces between opposite charges (colors) are described by linearly rising potentials, i.e. when charges are connected by a string which cannot be broken. It is the case of pure gluodynamics with classical sources (heavy quarks). This possibility is not forbidden in the Abelian theories too (Secs.2-4).

3. *Normal confinement* – when strings break at a certain finite distance as it is the case in the hadron physics. Evidently, this presumes the existence of colored particles of finite masses. Of course, normal confinement does not prevent from observation of confined colored objects. It is evident for

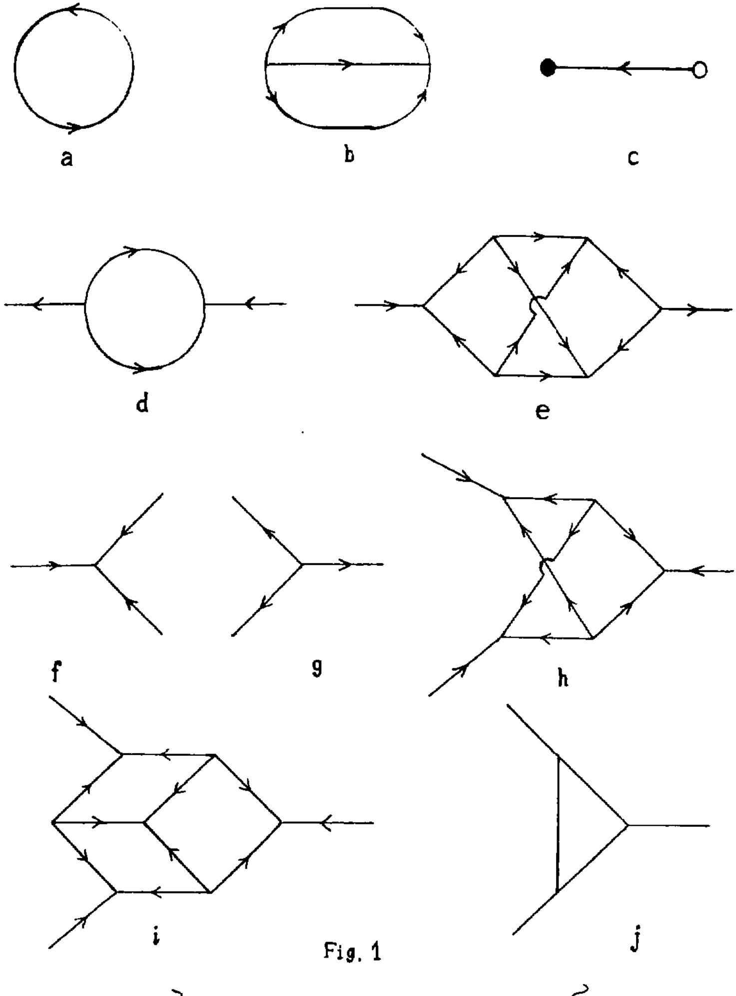

Fig. 1

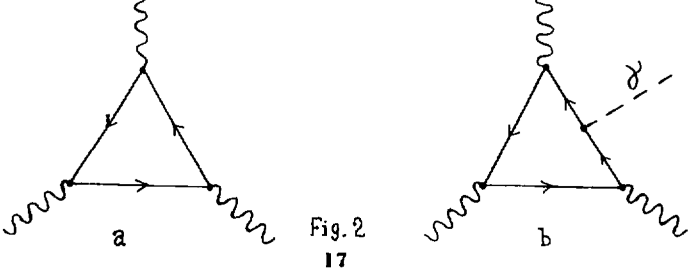

Fig. 2

electrically charged particles, but even being neutral they still interact with gravitons, they may have nonzero magnetic moments and hence, may be detected (in principle) by macroscopic tools. Confinement prevents only from independent propagation of qaurks.

4.*Screening.* There are two scales in these theories: the critical length of the string r_c at which it breaks, and the hadron size r_h. The former depends on the lowest masses of quarks, on the string tension, and on the spin of the hadron in question, while the latter comes as a result of dynamics. If $r_c \to 0$, strings do not exist, and all the charges are screened; this phenomenon takes place in the Schwinger model [18],[19]. Analogous behaviour should be expected for the QCD strings in the adjoint representation.

5.*Other possibilities.* All other possibilities result as an interplay of parameters r_c, r_h in the framework of normal confinement and the vacuum characteristics. For stable hadrons it should be $r_c > r_h$ (in the hadron physics they are of the same order). An opposite inequality $r_c < r_h$ leads to a contradiction. Indeed, it implies that hadrons are unstable, but they decay again into hadrons (colors connected by strings), so either there are stable hadrons of lower masses with $r_h < r_c$, or the theory is inconsistent.

Another possibility $r_c >> r_h$ looks more interesting, especially when r_c is of a macroscopic scale. In this case colors can be separated at macroscopic distances still being confined. This occurs in the case of small tension, such that macroscopically long strings have insignificant energies (*"soft" confinement*, contrary to the hadron physics $r_c \geq r_h$ (*hard confinement*)).

Still another possibility gives the model of electroweak interactions; but in this theory the ground state plays a principal role, so this phenomenon demands a special investigation.

*Remark.*Tension σ plays an important role in these theories ($E(r) = \sigma r$, where E is the string energy and r is its length). In Sec.4 we obtained (Eqs.(4.15), (4.23))

$$E(r) = \frac{g^2}{6} Tr\lambda_a^2 \, \delta^{(2)}(0) r \tag{5.3}$$

(introducing g explicitly). According to this equation, in gauge theories given by Eq.(2.1) the tension is infinite. It means that chromodynamics by itself is not a closed theory. In standard formulation (the Lagrangian (2.1)) it deals with infinitely thin strings, i.e. with strings of infinite tension. Of course these strings cannot be physical objects. Hence, to make the theory meaningful one has to introduce (by hand) a new parameter – the radius of the strings r_s. Experimentally one has $2\pi\sigma = 1/\alpha' \sim \mathrm{GeV}^2$, where $\alpha(s)$ is the Regge trajectory. Substituting $\delta^{(2)}(0) \to c/\pi r_s^2$, c is a constant, and passing to the standard λ-matrices $\lambda \to \lambda/\sqrt{2}$, $g \to g/\sqrt{2}$, we obtain

$$\sigma = \frac{g^2}{24} \, Tr\lambda_a^2 \, \frac{c}{\pi r_s^2} \, , \tag{5.4}$$

i.e. $r_s^2 = cg^2 Tr\lambda_a^2 \, \alpha'/12 \approx (4 \cdot 10^{-14})^2 \mathrm{cm}^2$ for $c \sim 1$ and $g^2/4\pi = \alpha_s \approx 0,2$. This new length r_s completely changes the theory. At distances less than r_s the Lagrangian (2.1) is not applicable, and one has to go into the problem of the string structure. We know, however, that the interquark forces are not due to a simple string. In fact the estimation (5.4) gives an average radius of an effective string.

6 Conclusion

The main statements of the paper are: 1) the P-exponent (string) is the only fundamental structure of the gauge theories (i.e. all the gauge-invariant structures are built of strings and local g-tensors); 2) in every gauge theory there exists a kind of confinement, i.e., in a sense, a "charge" is always confined; 3) in pure gluodynamics, the interaction of static external sources (heavy quarks) is given by the linear potential. All the investigation is based on the following (trivial) assumptions: in the classical theory only gauge-invariant configurations of fields are physically meaningful; in the quantum theory physical operators and state vectors are gauge-invariant. The gauge invariance manifests itself in constraints. The existence of the first-class constraints is the very feature of gauge theories that makes them so different from the standard non-gauge ones. Constraints do not contain time derivatives of canonical variables (in contrast with the equations of motion), they are conditions on instantaneous configurations of fields. As a result, the gauge field excitations may appear only in the form of some 1-dimensional structures with more or less complex topology - depending on the rank of the gauge group, and every "charge" is accompanied by a static external field. It is due to these external fields that the instantaneous interaction of static charged objects takes place: the Coulomb interaction in electrodynamics is a well-known example of the forces. In some theories (in those with gauge groups $SU(2)$ and (for some models) $U(1)$, and in pure gluodynamics) this immediately leads to linearly rising potentials, i.e. to confinement. It implies that confinement is a pure kinematical effect appearing as a consequence of the gauge invariance of a theory, i.e. following from the existence of the (secondary) first-class constraints.

Confinement may exhibit itself in quite different forms depending on parameters specifying the theory. These are the string critical length, or its tension σ and the lowest mass m of a charged (colored) particle ($r_c \sim m/\sigma$). When $r_c \to 0$, one speaks about screening (or dechromatization) of charge [18],[19]. The string does not exist in this case, i.e. it is not stable, it collapses into neutral infinitely small sections. On the contrary, when $r_c \to \infty$, one has in fact an absolute confinement. There are still two possibilities: (i) $\sigma \to 0$, m is fixed, (ii) σ is fixed, $m \to \infty$. In the case (i) charges can be moved aside from one another at an arbitrary large distance at the expense of a finite energy, while in the case (ii) this procedure takes an infinite energy. The Coulomb force is again a special case. According to Sec.2 opposite charges are connected by infinitely many strings, so we should expect that the case (i) takes place (see Sec.2 : $e = gN = fix$, $g \to 0, N \to \infty$). In models of the charge with a finite number of strings N [9],[10] one has very small (though finite) σ and very large r_c, so that at distances $r \approx r_c$ one should expect a modification of the Coulomb potential (it should be close to the linear one and very small).

Unfortunately, the above analysis says little about the real interquark potential. We learned that in QCD it cannot be a simple linear function of distance. To find the potential, one has to sum contributions of all the multistring graphs discussed in Sec.2 (like those c,d,e in Fig.1). This aspect of the string physics is usually omitted in the hadron model construction.

A standard tool in the study of confinement is the Wilson P-exponent (P_W, the Wilson loop [20]). There is a principal difference between P_W and the P-exponents $P_{xx'}$ used in the text. The Wilson gauge-invariant P-exponent emerges from the QCD Lagrangian for massive quarks, and its integration contour has time-like sectors, while the path-ordered exponents $P_{xx'}$ are universal objects of gauge field theories irrespective of the masses of quarks and the orientation of the integration contour – time-like or space-like. Physically these two cases differ considerably. For instance, in the Minkowski space the space-like loops correspond to some instantaneous gluonic excitations, while the "time-like" ones are meaningful only for heavy quarks (external sources); of course, in the Euclidean approach these differences disappear. We state also that the open space-like P-exponents with quarks at the ends are responsible for the static interquark forces giving in pure gluodynamics linear potentials. Hence, P-exponents in the Wilson criterion and in the present paper differ physically and play different roles.

One may find in literature statements that "in five and higher dimensions we have no confinement" [21]. We see from the above consideration that linear potentials appear in gauge theories in any space of non-zero dimension.

The point of view that strings in QCD are built of chromoelectric lines of force squeezed into a tube due to the special structure of vacuum ("the monopole - antimonopole vacuum" [22]) is rather popular among physicists [23]. We see (Secs. 2,3) that gauge theories do not need this hypothesis for getting string-like objects. They are inborn entities of the gauge field theories. We have seen that the existence of strings follows from the first principles. Nevertheless, the vacuum structure plays an important role in QCD, partly because the theory is not closed, and some its physical parameters specifying the hadron physics (like quark masses) depend on the ground state of the real dynamical system.

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Прохоров Л.В., Шабанов С.В. E2-91-195
Инвариантные структуры в калибровочных теориях
и конфайнмент

Рассматривается проблема построения всех калибровочных инвариантов в связи с проблемой конфайнмента. Вводятся и изучаются полилокальные калибровочные тензоры. Показано /в физическом и чисто геометрическом подходах/, что упорядоченная экспонента есть единственный фундаментальный билокальный калибровочный тензор, последнее означает, что любой нередуцируемый полилокальный калибровочный тензор может быть построен из P-экспонент и локальных тензоров /полей материи/. Отдельно рассматриваются простейшие инвариантные структуры в электродинамике, хромодинамике и теории с калибровочной группой SU(2). Как следствие калибровочной инвариантности любой "элементарный" заряд окружен внешним статическим полем, локализованным на контуре интегрирования P-экспоненты, т.е. струной. С этой точки зрения анализируется кулоновское поле; демонстрируется, что оно может также быть построено из упорядоченных вдоль линии интегралов. В КХД струны могут ветвиться - это означает, что межкварковое статистическое поле не может быть связано с простой P-экспонентой. Напротив, в чистой глюодинамике струны не ветвятся. Это позволяет показать, что в этом случае статическое межкварковое взаимодействие дается линейно растущим потенциалом, т.е. в этом случае для массивных кварков имеет место конфайнмент. Кратко рассматриваются различные формы конфайнмента.

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Prokhorov L.V., Shabanov S.V. E2-91-195
Invariant Structures in Gauge Theories and Confinement

The problem of finding all gauge invariants is considered in connection with the problem of confinement. Polylocal gauge tensors are introduced and studied. It is shown /both in physical and pure geometrical approaches/ that the path-ordered exponent is the only fundamental bilocal gauge tensor, which means that any irreducible polylocal gauge tensor is built of P-exponents and local tensors /matter fields/. The simplest invariant structures in electrodynamics, chromodynamics and a theory with the gauge group SU(2) are considered separately. As a consequence of gauge invariance any "elementary" charge is accompanied by an external static field located on the integration contour of a P-exponent, i.e. by a string. The Coulomb field is analyzed from this point of view; it is demonstrated that it can be also considered as made of exponential line integrals. In QCD strings can branch - it means that the interquark static field cannot be associated with a simple P-exponent. On the contrary, in pure gluodynamics strings do not branch. It allows to show that in this case the static quark interaction is given by a linearly rising potential, i.e. that in this case massive quarks are confined. Different forms of confinement are briefly reviewed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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