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**SUPERCONFORMAL STRUCTURES  
AND HOLOMORPHIC  $\frac{1}{2}$ -SUPERDIFFERENTIALS  
ON  $N=1$  SUPER RIEMANN SURFACES**

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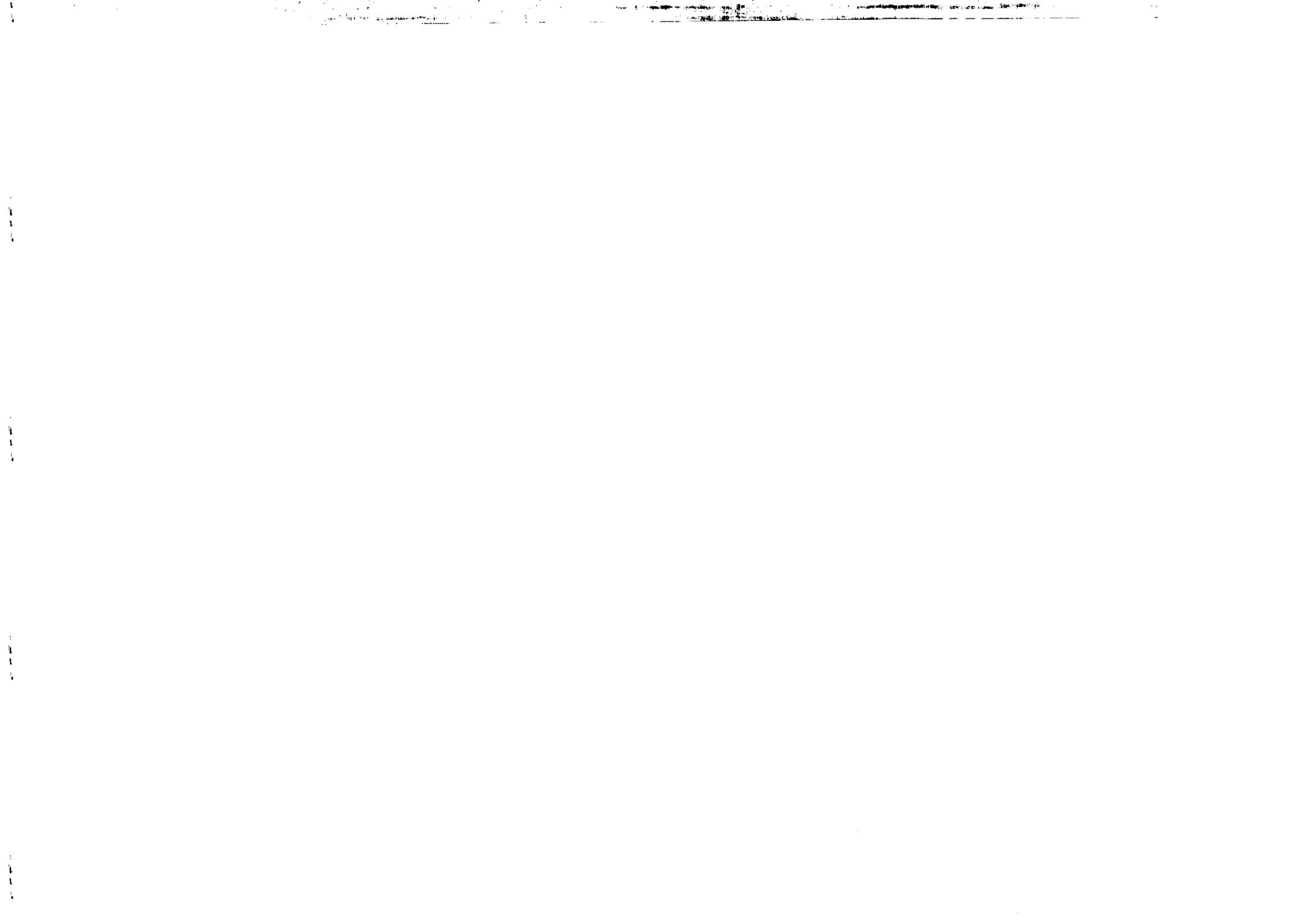


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**ABSTRACT**

Using the Super Riemann–Roch theorem we give a local expression for a holomorphic  $\frac{1}{2}$ -superdifferential in a superconformal structure parametrized by special isothermal coordinates on an  $N = 1$  Super Riemann Surface (SRS). This construction is done by choosing a suitable origin for these coordinates. The holomorphy of the latter with respect to super Beltrami differentials is proven.

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**1. INTRODUCTION**

For more than a decade, a great deal of literature has been devoted to the study of superconformal structures on super Riemann surfaces [1–3]. This interest is motivated by the fact that the geometrical approach to Super Riemann Surfaces has proved to be more suitable for the formulation of superconformal field theories thereon [2, 4, 5].

The computation of fermion string amplitudes in the Polyakov formalism [2, 6, 7] reduces to integration over a finite-dimensional superspace, the space of classes of superconformal manifolds, or equivalently superconformal structures parametrized by super Beltrami differentials. This parametrization makes the holomorphic factorization property manifest at all levels [8–11].

The use of holomorphic (super) differentials, among other geometrical objects defined on a (super) Riemann surface, turned out to be a powerful means to construct well-defined Lagrangian densities. In the bosonic case, these are (1,1)-differential forms [12] while in the supercase these become  $(\frac{1}{2}, \frac{1}{2})$ -superdifferentials [4, 13, 14]. The properties of a holomorphic differential on a Riemann surface of genus  $g \geq 1$  are well known [15–17] whereas a complete study of the same issues in the supercase is still lacking.

Accordingly, our work is a step in this direction. More precisely, we choose a superconformal structure on a generic  $N = 1$  SRS represented by special isothermal coordinates, i.e. solution to a constraint which fixes the second super Beltrami differential  $\nu$  to  $\theta$ . This condition, is of course, compatible with the superconformal invariance [4]. Then we define an origin for this coordinate system after eliminating the  $\bar{\theta}$ -dependence [18]. We prove, however, that this choice of origin does not break the holomorphy of the isothermal coordinates corresponding to this restriction w.r.t. the super Beltrami differentials. Finally, using the super Riemann–Roch theorem [13, 14], we give a local expression around the fixed origin for a holomorphic  $\frac{1}{2}$ -superdifferential in our special superconformal structure parametrized by the two remaining free super Beltrami differentials.

This work is organized as follows. In Section 2 we give an overview of the basic definitions and conformal structures of (super) Riemann surfaces with a special emphasis on the (super) Beltrami parametrization [4, 11]. In Section 3 we show an important identity which yields the section of the supercanonical line bundle over a SRS in terms of the covariant Vierbein field. Then using this we derive the right action of a superdiffeomorphism on the super Beltrami differentials. Section 4 is devoted to the construction of a holomorphic  $\frac{1}{2}$ -superdifferential on a SRS. Our conclusion and some further comments appear in Section 5.

**2. GENERAL FRAMEWORK**

In this section we start by presenting the main features of (super) Riemann surfaces.

An equivalent class of complex analytic coordinate coverings with holomorphic transition functions of a surface  $\Sigma$  is called a complex analytic structure (or simply a complex structure) on  $\Sigma$ . This surface with a fixed complex structure defines a Riemann surface (RS) [15–17].

Let us then consider a compact RS  $\Sigma$  of genus  $g$  (without boundary) with a reference structure represented by the coordinate charts  $\{(U, z)\}$ ,  $U$  is an open subset of  $\Sigma$ . Then by a quasiconformal transformation parametrized by the Beltrami differential  $\mu$  one obtains a projective atlas  $\{(U, Z)\}$ , where  $Z$  is the isothermal coordinate satisfying the Beltrami equation [12, 19]

$$(\bar{\partial} - \mu\partial)Z = 0. \quad (1)$$

in  $U$ .

Now let  $\omega^\tau(\mu)$ ,  $\tau = 1, 2, \dots, g$  be the basis of monodromic (single-valued) holomorphic 1-differentials in the  $\mu$ -structure. The holomorphy of  $\omega^\tau(\mu)$  in the reference structure is expressed by the differential equation [12, 20]

$$(\bar{\partial} - \mu\partial - \partial\mu)\omega^\tau = 0. \quad (2)$$

For a generic  $j$ -differential  $\omega$  this equation becomes [20]

$$(\bar{\partial} - \mu\partial - j\partial\mu)\omega = 0. \quad (3)$$

In the supercase we consider a compact SRS  $\hat{\Sigma}$  of genus  $g$  with  $\Sigma$  as its body. By definition,  $\hat{\Sigma}$  is a manifold which is composed of  $(1|1)$ -dimensional<sup>1</sup> superdomains glued up together by means of superconformal transition functions [13, 18, 25]. These are transformations between the coordinate charts  $\{(U, (z, \theta))\}$  and  $\{(V, (\bar{z}, \bar{\theta}))\}$  satisfying

$$\begin{cases} \bar{z} = f(z) + \theta\chi(z)\sqrt{\partial\bar{f}} \\ \bar{\theta} = \chi(z) + \theta\sqrt{\partial\bar{f} + \chi\partial\chi} \end{cases} \quad (4)$$

where  $f$  and  $\chi$  are respectively even and odd functions with values in the Grassmann algebra [13, 14, 18, 25]. Under the coordinate transformations (4), the spinor derivative  $D_\theta = \partial/\partial\theta + \theta\partial/\partial z \equiv \partial_\theta + \theta\partial$  becomes

$$\bar{D}_\theta = (D_\theta\bar{\theta})^{-1} D_\theta. \quad (5)$$

Furthermore, a SRS can be provided with another superconformal structure which is represented by a set of isothermal supercoordinates  $(\hat{Z}, \hat{\Theta}, \bar{\hat{Z}}, \bar{\hat{\Theta}})$ <sup>2</sup> together with superconformal transition functions [4]. The coordinates  $(\hat{Z}, \hat{\Theta})$  satisfy, in the reference structure  $\{(z, \theta, \bar{z}, \bar{\theta})\}$ , the super Beltrami equations [18, 26],

$$\begin{aligned} \bar{\partial}\hat{Z} + \hat{\Theta}\bar{\partial}\hat{\Theta} &= \hat{\mu}(\partial\hat{Z} + \hat{\Theta}\partial\hat{\Theta}) \\ -\partial_\theta\hat{Z} + \hat{\Theta}\partial_\theta\hat{\Theta} &= \nu(\partial\hat{Z} + \hat{\Theta}\partial\hat{\Theta}) \\ -\bar{\partial}_\theta\hat{Z} + \hat{\Theta}\bar{\partial}_\theta\hat{\Theta} &= \sigma(\partial\hat{Z} + \hat{\Theta}\partial\hat{\Theta}), \end{aligned} \quad (6)$$

where  $\hat{\mu}$ ,  $\nu$  and  $\sigma$  are the super Beltrami differentials which parametrizes the superconformal structure  $\{(\hat{Z}, \hat{\Theta})\}$  [4, 11, 13, 26]. Though, there is another formalism<sup>3</sup> which makes use of the super Beltrami

<sup>1</sup> The vertical line will separate even from odd variables.

<sup>2</sup> Here the caret is used to distinguish the super isothermal coordinates from the isothermal ones.

<sup>3</sup> In the sequel we will use both of the formalisms alternatively depending on the convenience each one of them might present.

differentials  $H_{\hat{Z}}, H_{\hat{\Theta}}, H_{\bar{\hat{Z}}}$  and in which these equations become [4]

$$\begin{aligned} \bar{\partial}\hat{Z} + \hat{\Theta}\bar{\partial}\hat{\Theta} &= H_{\hat{Z}}(\partial\hat{Z} + \hat{\Theta}\partial\hat{\Theta}) \\ D_\theta\hat{Z} - \hat{\Theta}D_\theta\hat{\Theta} &= H_{\hat{\Theta}}(\partial\hat{Z} + \hat{\Theta}\partial\hat{\Theta}) \\ \bar{D}_\theta\hat{Z} - \hat{\Theta}\bar{D}_\theta\hat{\Theta} &= H_{\bar{\hat{Z}}}(\partial\hat{Z} + \hat{\Theta}\partial\hat{\Theta}). \end{aligned} \quad (7)$$

It is easy to see that these two sets of equations are related by the following identification

$$\begin{aligned} H_{\hat{Z}} &= \hat{\mu} \\ H_{\hat{\Theta}} &= \theta - \nu \\ H_{\bar{\hat{Z}}} &= -\sigma + \bar{\theta}\hat{\mu}. \end{aligned} \quad (8)$$

One should note here that  $\hat{\mu}$  is the supersymmetric extension of the bosonic Beltrami differential  $\mu$ . This will be discussed later.

### 3. THE VIERBEIN FIELD AS THE SECTION OF A LINE BUNDLE AND THE SUPERDIFFEOMORPHISM TRANSFORMATION OF $\hat{\mu}, \nu, \sigma$

Here we show that the Vierbein field [1, 4, 21, 22]  $E^{\hat{Z}} = d\hat{Z} + \hat{\Theta}d\hat{\Theta}$  is a section of the square of the supercanonical fibre bundle generated by the  $\frac{1}{2}$ -form  $(d\hat{Z}|d\hat{\Theta})$  over the SRS  $\hat{\Sigma}$ . This identification will allow us to express the superconformal invariance equation of a  $(\frac{p}{2}, \frac{q}{2})$ -superdifferential in terms of the super Beltrami differentials  $\hat{\mu}, \nu, \sigma$ . Then using this we derive the right action of a superdiffeomorphism on them. To perform this task we proceed as follows.

On the SRS  $\hat{\Sigma}$ , a  $(\frac{p}{2}, \frac{q}{2})$ -superdifferential  $\Psi$  is a collection of locally smooth functions  $\{\psi(\hat{Z}, \hat{\Theta}, \bar{\hat{Z}}, \bar{\hat{\Theta}})\}$  satisfying the gluing rule

$$\bar{\psi}(\bar{\hat{Z}}, \bar{\hat{\Theta}}, \hat{\hat{Z}}, \hat{\hat{\Theta}}) = (D_{\hat{\Theta}}\bar{\hat{\Theta}})^{-p} (\bar{D}_{\hat{\Theta}}\bar{\hat{\Theta}})^{-q} \psi(\hat{Z}, \hat{\Theta}, \bar{\hat{Z}}, \bar{\hat{\Theta}}) \quad (9)$$

in the overlapping  $(U, (\hat{Z}, \hat{\Theta})) \cap (V, (\bar{\hat{Z}}, \bar{\hat{\Theta}}))$ .

Then  $\Psi = \psi(\hat{Z}, \hat{\Theta}, \bar{\hat{Z}}, \bar{\hat{\Theta}}) (d\hat{Z}|d\hat{\Theta})^p \otimes (d\bar{\hat{Z}}|d\bar{\hat{\Theta}})^q$  is a smooth section of the cross fibre bundle  $\hat{\omega}^{\otimes p} \otimes \bar{\omega}^{\otimes q}$ , where  $\hat{\omega}$  is the supercanonical line bundle over  $\hat{\Sigma}$  [13, 14] whose generator is  $(d\hat{Z}|d\hat{\Theta})$ .

The superdifferential  $\Psi$  is invariant under a superconformal change of coordinates  $(\hat{Z}, \hat{\Theta})^4$  i.e.,

$$\psi(d\hat{Z}|d\hat{\Theta})^p \otimes (d\bar{\hat{Z}}|d\bar{\hat{\Theta}})^q = \bar{\psi}(d\bar{\hat{Z}}|d\bar{\hat{\Theta}})^p (d\hat{\hat{Z}}|\hat{\hat{\Theta}})^q. \quad (10)$$

The next step is to rewrite this invariance equation in terms of super Beltrami differentials. For this purpose it is crucial to use the abovementioned identity, namely

$$\begin{aligned} E^{\hat{Z}} &\equiv d\hat{Z} + \hat{\Theta}d\hat{\Theta} \\ &= (d\hat{Z}|d\hat{\Theta})^2. \end{aligned} \quad (11)$$

<sup>4</sup> Unless otherwise stated, we will drop the barred variables for simplicity, knowing that everything goes along similar lines in the "bar sector".

Now let us prove this identity.

Let  $M$  be a manifold with spin structure glued up with  $\bar{z} = f(z)$ . This spin structure defines a line bundle  $\omega_0$  by the following (conformal) rules of gluing

$$(z, \ell) \rightarrow (\bar{z}, \bar{\ell}) = \left( f(z), \left( \frac{\partial f}{\partial z} \right)^{-1/2} \ell \right).$$

In addition, let  $L_0$  be a holomorphic line bundle over  $M$ . Then a split superconformal manifold  $\hat{M}$  and a bundle  $E$  over it are constructed by means of the gluings

$$\begin{aligned} (\bar{z}, \bar{\theta}) &= (f(z), \theta \sqrt{\partial f(z)}) \\ (\bar{z}, \bar{\theta}, \bar{\ell}) &= (f(z), \theta \sqrt{\partial f(z)}, g(z)\ell). \end{aligned}$$

According to Rosly *et al.* [14] a holomorphic section  $s$  of the bundle  $E$  over  $\hat{M}$  can be written in components as follows

$$s(z, \theta) = s_0(z) + \theta s_1(z), \quad (12)$$

where  $s_0$  is an (even) section of  $L_0$  and  $s_1$  an (odd) section of  $L_0 \otimes \omega_0$ . Formally one has the direct sum

$$E = L_0 \oplus \theta(L_0 \otimes \omega_0). \quad (13)$$

Now put  $L_0 = K$ , the canonical line bundle with section  $s_0 = dz$  which transforms as  $d\bar{z} = \left( \frac{\partial f}{\partial z} \right) dz$ ; and note that a section  $\ell$  of  $\omega_0$  transforms as  $\bar{\ell} = \left( \frac{\partial f}{\partial z} \right)^{-1/2} \ell$ , then the section  $s_1$  of  $L_0 \otimes \omega_0$  glues with the transition function  $\left( \frac{\partial f}{\partial z} \right)^{1/2}$ . On the other hand, this is also the transition function of  $d\theta$  as the local section of  $\omega_0^*$  (dual of  $\omega_0$ ) [13]. Thus we identify  $s_1$  with  $d\theta$  and hence Eq.(12) becomes<sup>5</sup>

$$s(z, \theta) = dz + \theta d\theta \equiv E^s \quad (14)$$

In fibre bundle terms we have

$$E = K \oplus \theta(K \otimes \omega_0).$$

This is just the "projection in components" of the square of the supercanonical bundle  $\hat{\omega}$ . Equivalently, in terms of the first cohomological groups one has,

$$H^1(\hat{\Sigma}, E) = H^1(\Sigma, K) \oplus \theta \pi H^1(\Sigma, K \otimes \omega_0) = H^1(\hat{\Sigma}, \hat{\omega}^2) \quad (15)$$

$\pi$  is the parity of elements in  $H^1(\Sigma, K \otimes \omega_0)$ . Therefore,

$$E^s = dz + \theta d\theta = (dz|d\theta)^2, \quad (16)$$

independently of the coordinate system.

Moreover, this also follows from the identity  $\partial_s = (D_\theta)^2$  and the fact that  $E^s$  is dual to  $\partial_z$  and  $(dz|d\theta)$  is dual to  $D_\theta$ . This ends the proof of the identity (11).

<sup>5</sup> This identification is in fact up to a meromorphic function.

Now let us rewrite (10) in terms of  $E^{\hat{z}}$ 's

$$\psi(E^{\hat{z}})^{p/2} (E^{\hat{z}})^{q/2} = \bar{\psi}(E^{\hat{z}})^{p/2} (E^{\hat{z}})^{q/2}. \quad (17)$$

Then using the expression of  $E^{\hat{z}}$  in terms of  $\hat{\mu}, \nu, \sigma$

$$\begin{aligned} E^{\hat{z}} &= (\partial \hat{Z} + \hat{\Theta} \partial \hat{\Theta})(dz + \hat{\mu} d\bar{z} + \nu d\theta + \sigma d\bar{\theta}) \\ &\equiv \Lambda(dz + \hat{\mu} d\bar{z} + \nu d\theta + \sigma d\bar{\theta}) \end{aligned} \quad (18)$$

we rewrite  $\Psi$  as follows

$$\Psi = T(z, \theta, \bar{z}, \bar{\theta})(dz + \hat{\mu} d\bar{z} + \nu d\theta + \sigma d\bar{\theta})^{p/2} (d\bar{z} + \bar{\mu} dz + \bar{\nu} d\bar{\theta} + \bar{\sigma} d\theta)^{q/2} \quad (19)$$

where

$$T(z, \theta, \bar{z}, \bar{\theta}) = \Lambda^{p/2} \bar{\Lambda}^{q/2} \psi(\hat{Z}, \hat{\Theta}, \bar{\hat{Z}}, \bar{\hat{\Theta}}). \quad (20)$$

Eq.(20) expresses the transformation law of the supertensor  $T$  under the quasisuperconformal transformation  $(z, \theta) \rightarrow (\hat{Z}, \hat{\Theta})$ . Now following the work of Lazzarini in the bosonic case [12], we find that a superconformal change of coordinates in the SB-structure<sup>6</sup>  $(\hat{Z}, \hat{\Theta})$  induces a local change of coordinates in the reference structure via a sensepreserving superdiffeomorphism  $\varphi \in S\text{Diff}^*(\hat{\Sigma})$ , that is, the following diagram commutes

$$\begin{array}{ccc} (z, \theta) & \xrightarrow{(\hat{\mu}, \nu, \sigma)} & (\hat{Z}, \hat{\Theta}) \\ \varphi \downarrow & & \downarrow \text{superholomorphic} \\ (\bar{z}, \bar{\theta}) & \xrightarrow{(\bar{\mu}, \bar{\nu}, \bar{\sigma})} & (\bar{\hat{Z}}, \bar{\hat{\Theta}}) \end{array} \quad \text{transformation}$$

This defines a new complex structure on  $\hat{\Sigma}$  by the new SB  $\bar{\mu}, \bar{\nu}, \bar{\sigma}$ .

Finally, the invariance of  $\Psi$  in the reference structure under the superdiffeomorphism  $\varphi$  is obtained by rewriting Eq.(17) using Eq.(19).

Then if we denote this superdiffeomorphism by

$$\varphi : (z, \theta) \rightarrow (\varphi_s(z, \theta, \bar{z}, \bar{\theta}), \varphi_f(z, \theta, \bar{z}, \bar{\theta})),$$

we obtain the right action  $\text{SR}_\varphi$  of  $S\text{Diff}^*(\hat{\Sigma})$  on the SB as follows.

$$\begin{aligned} \bar{\mu} &\equiv \text{SR}_\varphi(\hat{\mu}) = \frac{\bar{\partial} \varphi_s + (\hat{\mu} \circ \varphi) \bar{\partial} \bar{\varphi}_s + (\nu \circ \varphi) \bar{\partial} \varphi_f + (\sigma \circ \varphi) \bar{\partial} \bar{\varphi}_f}{\partial \varphi_s + (\hat{\mu} \circ \varphi) \partial \bar{\varphi}_s + (\nu \circ \varphi) \partial \varphi_f + (\sigma \circ \varphi) \partial \bar{\varphi}_f} \\ \bar{\nu} &\equiv \text{SR}_\varphi(\nu) = \frac{-\partial_\theta \varphi_s - (\hat{\mu} \circ \varphi) \partial_\theta \bar{\varphi}_s + (\nu \circ \varphi) \partial_\theta \varphi_f + (\sigma \circ \varphi) \partial_\theta \bar{\varphi}_f}{\partial \varphi_s + (\hat{\mu} \circ \varphi) \partial \bar{\varphi}_s + (\nu \circ \varphi) \partial \varphi_f + (\sigma \circ \varphi) \partial \bar{\varphi}_f} \\ \bar{\sigma} &\equiv \text{SR}_\varphi(\sigma) = \frac{-\bar{\partial}_\theta \varphi_s - (\hat{\mu} \circ \varphi) \bar{\partial}_\theta \bar{\varphi}_s + (\nu \circ \varphi) \bar{\partial}_\theta \varphi_f + (\sigma \circ \varphi) \bar{\partial}_\theta \bar{\varphi}_f}{\partial \varphi_s + (\hat{\mu} \circ \varphi) \partial \bar{\varphi}_s + (\nu \circ \varphi) \partial \varphi_f + (\sigma \circ \varphi) \partial \bar{\varphi}_f}. \end{aligned} \quad (21)$$

<sup>6</sup> SB stands for super Beltrami differentials, and SB-structures for the structure defined by these differentials.

Now let us consider the special case of superconformal transformations, i.e.,

$$\varphi^{SC} : (z, \theta) \rightarrow (\tilde{z}(z, \theta), \tilde{\theta}(z, \theta))$$

with

$$D_{\tilde{\theta}} \tilde{z} = \tilde{\theta} D_{\theta} \tilde{\theta}. \quad (22)$$

Then using the fact that  $\tilde{\mu}, \nu, \sigma$  are dumb variables in Eqs.(21), these become respectively <sup>7</sup>

$$\tilde{\mu} = \frac{(\tilde{\mu}_{SC} \circ \varphi^{SC}) \partial \tilde{z} + (\sigma_{SC} \circ \varphi^{SC}) \partial \tilde{\theta}}{\partial \tilde{z} + (\nu_{SC} \circ \varphi^{SC}) \partial \tilde{\theta}}. \quad (23a)$$

Note that this reduces in the bosonic case to the transformation of the Beltrami differential  $\mu$  under a conformal change of coordinates [12]. Then,

$$\nu = \frac{-\partial_{\theta} \tilde{z} + (\nu_{SC} \circ \varphi^{SC}) \partial_{\theta} \tilde{\theta}}{\partial \tilde{z} + (\nu_{SC} \circ \varphi^{SC}) \partial \tilde{\theta}}. \quad (23b)$$

Furthermore, considering the case of superconformal transformations implies the constraint  $\nu = \theta$  since this is nothing than the relation (22).

Now putting  $\nu = \theta$  in Eq.(23b) and using Eq.(22) we get

$$\nu_{SC} = \tilde{\theta}$$

which shows that the condition  $\nu = \theta$  is, as mentioned previously, invariant under superconformal transformations. Moreover, if we substitute  $\tilde{\theta}$  to  $\nu_{SC}$  in (23b) we get the following equation

$$-\partial_{\theta} \tilde{z} + \tilde{\theta} \partial_{\theta} \tilde{\theta} = \theta (\partial \tilde{z} + \tilde{\theta} \partial \tilde{\theta})$$

which is just the second super Beltrami equation satisfied by the conformal coordinates  $\tilde{z}(z, \theta), \tilde{\theta}(z, \theta)$  instead of  $\tilde{Z}, \tilde{\Theta}$  with  $\nu$  set to  $\theta$ . This is indeed related to the fact that quasisuperconformal transformations reduce to superconformal ones by imposing the conditions

$$\begin{aligned} \tilde{\mu} &= 0 \\ \nu &= \theta \\ \sigma &= 0 \end{aligned} \quad (24)$$

which are the analogue of the condition  $\mu = 0$  in the bosonic case. This follows from the results of Ref.4 and our identification in Eq.(8).

From the equation for  $\sigma$  (which we omit) and Eqs.(23a), (23b), one can see that also the other conditions in (24) are invariant under superconformal transformations.

Finally, demanding the invariance of a  $(\frac{f}{g}, \frac{g}{f})$ -superdifferential under a superconformal change of coordinates leads to the deformation of the super complex structure defined by  $(\tilde{\mu}, \nu, \sigma)$  under the right action of a superdiffeomorphism  $\varphi \in S\text{Diff}^+(\hat{\Sigma})$ , i.e.,

$$(\tilde{\nu}, \nu, \sigma) \rightarrow (\tilde{\mu}^{\varphi}, \nu^{\varphi}, \sigma^{\varphi}).$$

<sup>7</sup>  $f_{SC}$  refers to the superconformal transform of  $f$ .

Thus the set  $\{(\tilde{\mu}^{\varphi}, \nu^{\varphi}, \sigma^{\varphi}), \varphi \in S\text{Diff}^+(\hat{\Sigma})\}$  describes the  $S\text{Diff}^+(\hat{\Sigma})$ -orbit of the point  $(\tilde{\mu}, \nu, \sigma)$  in the supermoduli space  $\hat{\mathcal{M}}(\hat{\Sigma})$  of  $\hat{\Sigma}$ .

One should note that the right action  $\text{SR}_{\varphi}$  given above is holomorphic w.r.t. the SB. Moreover, it is straightforward to check that Eqs.(21) allow to recover those given in [4] through the identification (8) and that they reduce to the right action of  $\varphi \in \text{Diff}(\Sigma)$  on the bosonic Beltrami differential  $\mu$  [12].

#### 4. CONSTRUCTION OF A HOLOMORPHIC $\frac{1}{2}$ -SUPERDIFFERENTIAL ON A SRS

In this section we define a special system of coordinates which is a solution to the constraint  $\nu = \theta$ . We further restrict this solution to that without  $\tilde{\theta}$ -dependence [18]. Then we show how to choose an origin for the resulting coordinates. This will allow us to give a local expression for a holomorphic  $\frac{1}{2}$ -superdifferential on the SRS  $\hat{\Sigma}$  and then prove the holomorphy w.r.t. SB is preserved by this choice of origin.

Let  $(\hat{Z}, \hat{\Theta})$  be the isothermal coordinates representing a superconformal structure on  $\hat{\Sigma}$ . Now let us consider the condition  $\nu = \theta$  which is equivalent to  $D_{\theta} \hat{Z} = \hat{\Theta} D_{\theta} \hat{\Theta}$  which follows from the second Beltrami equation in (6). Applying this on  $\hat{Z}, \hat{\Theta}$  expressed (as superfunctions) in their respective components  $(\varphi_0, \dots, \varphi_3)$  and  $(\psi_1, \dots, \psi_3)$ , we get the general solution

$$\begin{aligned} \hat{Z}(z, \bar{z}, \theta, \bar{\theta}) &= \varphi_0(z, \bar{z}) + \theta \varepsilon(z, \bar{z}) \partial \varphi_0 \\ &\quad + \bar{\theta} \varphi_2(z, \bar{z}) + \theta \bar{\theta} \sqrt{\partial \varphi_0} \left( \frac{1}{2} \varepsilon \partial \varepsilon \psi_2(z, \bar{z}) - 2 \varepsilon \sqrt{\partial \varphi_0} \right) \\ \hat{\Theta}(z, \bar{z}, \theta, \bar{\theta}) &= \sqrt{\partial \varphi_0} \varepsilon + \theta \sqrt{\partial \varphi_0} \left( 1 + \frac{1}{2} \varepsilon \partial \varepsilon \right) + \bar{\theta} \psi_2 \\ &\quad + \theta \bar{\theta} \left( 2 \sqrt{\partial \varphi_0} \partial \varphi_2 - \sqrt{\partial \varphi_0} \partial \varphi_2 \varepsilon \partial \varepsilon + \psi_2 \left( 1 + \frac{1}{2} \sqrt{\partial \varphi_0} \varepsilon \partial^2 \varepsilon \right) \right. \\ &\quad \left. - 2 \partial \varphi_0 \partial \psi_2 \varepsilon \right) \end{aligned} \quad (25)$$

where  $\varphi_0(z, \bar{z})$  and  $\psi_2(z, \bar{z})$  are bosonic functions while  $\varepsilon(z, \bar{z})$  and  $\varphi_2(z, \bar{z})$  are fermionic ones.

Now if we insert (25) in the first super Beltrami differential in (6) we obtain the first term of the development of  $\tilde{\mu}$  in its components, that is

$$\frac{\partial \varphi_0}{\partial \varphi_0} (1 - \varepsilon \partial \varepsilon) + \varepsilon \bar{\theta} \varepsilon.$$

From this we see that requiring  $\tilde{\mu}$  to reduce in the bosonic case to the Beltrami differential  $\mu$  implies the identification of  $\varphi_0$  with the projective coordinate  $Z$  due to Eq.(1) [12].

As it was discussed by Crane and Rabin in [18] one can always drop the third super Beltrami equation and thus eliminates the  $\tilde{\theta}$ -dependence. In this case the solution (25) reduces to [23]

$$\begin{aligned} \hat{Z}(z, \theta, \bar{z}) &= Z(z, \bar{z}) + \theta \varepsilon(z, \bar{z}) \partial Z \\ \hat{\Theta}(z, \theta, \bar{z}) &= \theta \sqrt{\partial Z} + \sqrt{\partial Z} \varepsilon \left( 1 - \frac{1}{2} \theta \partial \varepsilon \right). \end{aligned} \quad (26)$$

Now let us denote by  $\hat{P} = (z, \theta, \bar{z})$  the generic point of the  $\hat{\Sigma}$  in this case. In the sequel, we will use the notation

$$\hat{P} = (P, \theta) \in \hat{\Sigma}$$

where

$$P \equiv (z, \bar{z}) \in \Sigma.$$

Then let  $Z_j$  be the isothermal coordinate of the  $\mu$ -structure defined in the neighbourhood of  $P_j \in \Sigma$  taken as its origin, i.e.,

$$Z_j(P_j) = 0.$$

Note that  $(\partial Z_j)(P_j) \neq 0$  for all  $P_j \in \Sigma$  which implies that  $\hat{\Theta}$  in (26) never vanishes. This suggests to choose an origin for the local coordinate system  $(\hat{Z}_j, \hat{\Theta}_j)$  in Eqs.(26) as follows

$$\hat{Z}_j(\hat{P}_j) = 0 \quad (27a)$$

$$\hat{\Theta}_j(\hat{P}_j) = \theta_j \sqrt{\partial Z_j(P_j)}. \quad (27b)$$

Now evaluating Eqs.(26) at the point  $\hat{P}_j$  we get

$$\hat{Z}_j(\hat{P}_j) = \theta_j \varepsilon(P_j)(\partial Z)(P_j)$$

$$\hat{\Theta}_j(\hat{P}_j) = \theta_j \sqrt{(\partial Z)(P_j)} + \sqrt{(\partial Z)(P_j)} \varepsilon(P_j) \left(1 - \frac{1}{2} \theta_j (\partial \varepsilon)(P_j)\right).$$

Then to be consistent with Eqs.(27) one should choose the function  $\varepsilon(z, \bar{z})$  to vanish at the point  $P_j$  with an order (at least) equal to 1, i.e.  $\varepsilon$  can be given the following local expression using the coordinate  $Z_j$

$$\varepsilon(z, \bar{z}) = \lambda Z_j^{\alpha_j}(z, \bar{z}) \quad (28)$$

where  $\alpha_j \geq 1$  is the order of the zero  $P_j$  of  $\varepsilon$  and,  $\lambda$  is a Grassmann number, i.e.  $\lambda^2 = 0$ .

Here we wish to note that the origin we have chosen is in fact a point of a  $(0, 1)$ -dimensional odd manifold. Furthermore, this origin is also one for the coordinate system

$$\begin{aligned} \hat{Z}(z, \bar{z}) &= Z(z, \bar{z}) \\ \hat{\Theta}(z, \theta, \bar{z}) &= \theta \sqrt{\partial Z} \end{aligned}$$

defining a split supermanifold obtained from the reference structure by the "split" quasisuperconformal transformation  $(z, \theta, \bar{z}) \rightarrow (\hat{Z}(z, \bar{z}), \hat{\Theta}(z, \theta, \bar{z}))$ .

Now we come to the construction of a  $\frac{1}{2}$ -superdifferential on  $\hat{\Sigma}$ .

Let us first recall that every monodromic  $\mu$ -holomorphic 1-differential has precisely  $(2g - 2)$  zeros counting multiplicity [15-17, 20]. In other words, if  $P_j$  denotes the distinct zeros of  $\omega^r$  introduced in Section 2 and  $\nu_j$  its order, a local expression for  $\omega^r$  is [20]

$$\omega^r = \alpha^r(z, \bar{z}) Z_j^{\nu_j} \quad (29)$$

around  $P_j$ , where  $\alpha(P_j) \neq 0$ .

For the particular case of a torus ( $g = 1$ ), there is a unique (up to a multiplicative holomorphic function)  $\mu$ -holomorphic nowhere-vanishing 1-differential  $\omega$ . This can be written locally as<sup>8</sup>

$$\omega(z, \bar{z}) = \partial Z. \quad (30)$$

In the supercase, according to the super Riemann-Roch theorem [13, 14] there are  $g$  even holomorphic  $\frac{1}{2}$ -superdifferentials on a SRS, and each one of them possesses globally  $(g - 1)$  zeros counting multiplicity. On the supertorus ( $g = 1$ ) one has a unique (up to a superholomorphic function) holomorphic nowhere-vanishing even  $\frac{1}{2}$ -superdifferential. This is given locally by [23]

$$\eta = D_\theta \hat{\Theta}.$$

In the general case of a generic SRS of genus  $g > 1$ , this expression generalizes to

$$\eta = \beta(z, \theta, \bar{z}, \bar{\theta}) \left( \hat{Z}_j(\hat{P}) + \theta_j \hat{\Theta}_j(\hat{P}) \sqrt{(\partial Z_j)(P_j)} \right)^{\alpha_j/2} \quad (31)$$

where  $\hat{P}$  is a point in the neighbourhood of  $\hat{P}_j$  and  $\alpha_j/2$ ,  $\alpha_j \in \mathbf{N}$ , is the order of the zero  $\hat{P}_j$  of  $\eta$ .  $\beta$  is a superfunction which does not vanish at  $\hat{P}_j$  and can, in general, take the expression

$$\beta = (D_\theta \hat{\Theta}) G(\hat{Z}, \hat{\Theta})$$

with

$$G(\hat{P}_j) \neq 0.$$

By using Eqs.(27) one can check that  $\eta$  indeed vanishes at  $\hat{P}_j$  with order  $\alpha_j/2$ . Furthermore, if we replace  $\hat{Z}_j$  and  $\hat{\Theta}_j$  by their expressions (in Eqs.(26)) into Eq.(31) we get

$$\eta = \beta Z_j^{\alpha_j/2}(P) \left\{ 1 + \alpha_j/2 \theta_j \frac{\varepsilon(P) \gamma_j(P)}{Z_j(P)} \right\}$$

where

$$\gamma_j(P) = (\partial_j Z_j)(P) \left\{ 1 + \sqrt{\frac{(\partial_j Z_j)(P)}{(\partial_j Z_j)(P)}} \right\}.$$

Note that  $\gamma_j(P_j) = 2(\partial_j Z_j)(P_j) \neq 0$ .

Then using the fact that  $P_j$  is also a zero of  $\varepsilon$  of order at least 1 (see Eq.(28)) we see that the second factor in  $\eta$  tends to 1 as  $P$  approaches  $P_j$ . Therefore,  $\eta$  behaves locally around  $\hat{P}_j$  as

$$\eta = \beta Z_j^{\alpha_j/2}(P).$$

One should note here that this expression is reminiscent of that of a 1-dimensional  $\omega^r$  on a RS (Eq.(29)), however with globally half the number of zeros. This can be seen by noticing that the first term (using  $\beta = \beta_0 + \theta \beta_1 + \dots$ ) is the square root of  $\omega^r$  according to Eq.(15) with  $\omega^r \in H^1(\Sigma, K)$  and  $\eta \in H^1(\hat{\Sigma}, \hat{\omega})$ . This is summarized as follows

$$\begin{array}{ccc} \eta \in H^1(\hat{\Sigma}, \hat{\omega}) & \xrightarrow{g-1} & \hat{P}_j = (P_j, \theta_j) \in \hat{\Sigma} \\ \text{"Projection relation" Eq. (15)} \downarrow & & \downarrow \text{projection} \\ \omega^r \in H^1(\Sigma, K) & \xrightarrow{2g-2} & P_j \in \Sigma. \end{array}$$

<sup>8</sup> Here and below it is understood that the differentials  $\omega$  in Eq.(29) and Eq.(31) are in fact the coefficients  $\omega_z$  of the 1-differential  $\omega = \omega_z dz$ . Similarly,  $\eta$  (in fact  $\eta_\theta$ ) is the coefficient of  $(dz|d\theta)$ .

The holomorphy of the superdifferential  $\eta = \eta_\theta (dz|d\theta)$  in the complex structure defined by the SB is expressed by

$$\bar{D}_\theta \eta_\theta = 0 \quad (32a)$$

and in the reference structure this reads

$$\left[ \bar{D}_\theta - H_\theta^\dagger \partial + \frac{1}{2} (D_\theta H_\theta^\dagger) D_\theta \right] \eta_\theta = \frac{1}{2} (\partial H_\theta^\dagger) \eta_\theta \quad (32b)$$

with  $H_\theta^\dagger = 0$ .

So far,  $\eta$  is a monodromic (single-valued) superdifferential on  $\hat{\Sigma}$  with globally  $(g-1)$  zeros. This means that the monodromy multiplier of  $\eta$  is trivial, i.e. equal to 1. However, a general  $\frac{1}{2}$ -superdifferential  $\zeta$  may have non-trivial multiplier and consequently, it either is nowhere-vanishing or vanishes identically. In this case  $\zeta$  is called a polydromic (multi-valued) superdifferential. It can be written locally as,

$$\zeta_\theta(z, \theta) = (D_\theta \hat{\Theta}) \xi_\theta(z, \theta)$$

around  $\hat{P}_j$ , where  $\xi(z, \theta)$  is a superholomorphic multi-valued function non-vanishing at  $\hat{P}_j$ . The monodromy of  $\zeta$  may be determined by using that of a bosonic polydromic differential given in [20] and the identity (15) [24].

### Holomorphy of $(\hat{Z}, \hat{\Theta})$ at the fixed origin

Now we proceed to the proof that our choice of origin which enables us to give a local expression for the superdifferential  $\eta$  does not break the holomorphy of the coordinates  $(\hat{Z}, \hat{\Theta})$ .

As it was shown by Zucchini the projective coordinate  $Z$  in the  $\mu$ -structure is a function of  $\mu$ , it has (as a coordinate of  $\Sigma$ ) a domain independent from  $\mu$  and it is holomorphic w.r.t.  $\mu$  [20]

$$\frac{\delta Z_j(\mu)}{\delta \bar{\mu}} = 0 \quad (33)$$

where  $\delta f$  is defined as an infinitesimal variation of the function  $f$  w.r.t. the complex conjugate Beltrami differential  $\bar{\mu}$ . Then applying its conjugate  $\delta$  on  $Z_j(P_j(\mu)) = 0$  and using the Beltrami equation one gets [20]

$$\delta Z_j(P_j) + (\delta z(P_j) + \mu(P_j) \delta \bar{z}(P_j)) (\partial Z_j)(P_j) = 0 \quad (34)$$

In the supercase we define  $\delta F(\bar{\delta}F)$  as an infinitesimal variation of the superfunction  $F$  w.r.t. the SB  $\bar{\mu}, \nu, \sigma(\bar{\mu}, \bar{\nu}, \bar{\sigma})$ .  $\delta$  is such that it reduces to  $\delta$  when it acts on a bosonic function, and similarly for  $\bar{\delta}$  related to  $\bar{\delta}$ . Now the action of  $\bar{\delta}$  on Eq.(27b) yields the identity

$$\begin{aligned} \bar{\delta} \left[ \hat{\Theta}_j(\hat{P}_j(\bar{\mu}, \bar{\nu}, \bar{\sigma})) \right] &= \bar{\delta} \hat{\Theta}_j(\hat{P}_j) + \bar{\delta} z \partial \hat{\Theta}_j(\hat{P}_j) + \bar{\delta} \theta \partial_\theta \hat{\Theta}_j(\hat{P}_j) + \bar{\delta} \bar{z} \bar{\partial} \hat{\Theta}_j(\hat{P}_j) \\ &\quad + \bar{\delta} \bar{\theta} \bar{\partial}_\theta \hat{\Theta}_j(\hat{P}_j) \\ &= 0 \end{aligned} \quad (35)$$

due to Eq.(34).

Furthermore, applying  $\bar{\delta}$  on Eq.(27a), using the super Beltrami equations and the identity (35) we get the supersymmetric generalization of Eq.(34), that is

$$(\bar{\delta} \hat{Z}_j + \hat{\Theta}_j \bar{\delta} \hat{\Theta}_j)(\hat{P}_j) + \left[ (\bar{\delta} z + \bar{\mu} \bar{\delta} \bar{z} + \nu \bar{\delta} \theta + \sigma \bar{\delta} \bar{\theta}) (\partial \hat{Z}_j + \hat{\Theta}_j \partial \hat{\Theta}_j) \right] (\hat{P}_j) = 0 \quad (36)$$

This identity is, of course, the same for  $\bar{\delta}$ .

Then the next step is to show the following identity

$$\bar{\delta} z + \bar{\mu} \bar{\delta} \bar{z} + \nu \bar{\delta} \theta + \sigma \bar{\delta} \bar{\theta} = 0 \quad (37)$$

For this purpose, we first express the fact that  $\hat{P}_j$  is a zero of  $\eta$  of order  $\alpha_j/2$  as follows,

$$(D_\theta^{\alpha_j-1} \eta)(\hat{P}_j) = 0 \quad (38a)$$

$$(D_\theta^{\alpha_j} \eta)(\hat{P}_j) \neq 0 \quad (38b)$$

Then applying the operator  $\bar{\delta}$  on Eq.(38a) and using the fact that  $\eta$  is holomorphic w.r.t.  $\hat{\mu}, \nu, \sigma$  and that  $\bar{\delta}$  commutes with  $D_\theta$  we get the following identity

$$\left[ D_\theta^{\alpha_j-1} (\bar{\delta} z \partial \eta + \bar{\delta} \theta \partial_\theta \eta + \bar{\delta} \bar{z} \bar{\partial} \eta + \bar{\delta} \bar{\theta} \bar{\partial}_\theta \eta) \right] (\hat{P}_j) = 0$$

This equation splits into two equations by distinguishing two cases, namely

$$D_\theta^k = \partial^{k/2} \quad \text{if } k \text{ is even}$$

$$D_\theta^k = \partial^{k/2} \partial_\theta + \theta \partial^{k/2} \quad \text{if } k \text{ is odd}$$

Then using the differential equation (32b) of  $\eta$  together with Eqs.(38) in the resulting identities we obtain two (rather lengthy) equations whose r.h.s.'s are equal to zero. Both of these equations are linear combinations of derivatives of  $\eta$  involving powers of  $\partial, D_\theta$  and mixed ones. Next, due to the fact that these derivatives are linearly independent, all their coefficients must vanish. Among these we get the following

$$\bar{\delta} \theta = 0$$

$$\bar{\delta} z + \bar{\mu} \bar{\delta} \bar{z} + \sigma \bar{\delta} \bar{\theta} = 0$$

In both cases of  $\alpha_j$  even or odd.

This actually ends the proof of the identity (37).

Now inserting (37) in (36) with  $\bar{\delta}$  substituted to  $\bar{\delta}$  we obtain

$$(\bar{\delta} \hat{Z}_j + \hat{\Theta}_j \bar{\delta} \hat{\Theta}_j)(\hat{P}_j) = 0 \quad (39)$$

Then applying  $\bar{\delta}$  on  $\hat{\Theta}_j$  in Eq.(26), using the fact that the function  $\varepsilon$  vanished at  $\hat{P}_j$  and the identity (33) we obtain

$$\bar{\delta} \hat{\Theta}_j(\hat{P}_j) = \sqrt{\partial Z(P_j)} \bar{\delta} \varepsilon(P_j) \left\{ 1 - \frac{1}{2} \theta_j \partial \varepsilon(P_j) \right\}$$

Moreover,  $\bar{\delta} \varepsilon(P_j) = 0$  due to the local expression in Eq.(28) and the identity (33). This finally yields the holomorphy of  $\hat{\Theta}_j$  and hence that of  $\hat{Z}_j$  at  $\hat{P}_j$ , that is

$$\bar{\delta} \hat{\Theta}_j(\hat{P}_j) = 0 \quad (40)$$

$$\bar{\delta} \hat{Z}_j(\hat{P}_j) = 0$$

In the general case (with the  $\bar{\theta}$ -dependence), the proof of the holomorphy of the projective coordinates in Eqs.(25) proceeds along similar lines as above. Indeed, this can be done by keeping the same origin as in Eqs.(27) but imposing further constraints, namely,  $\psi_2(P_j) = 0, \varphi_2(P_j) = 0, (\partial \varphi_2)(P_j) = 0$ .

## 5. CONCLUSIONS

We considered a SRS as a collection of superconformal structures parametrized by the super Beltrami differentials  $\hat{\mu}, \nu, \sigma$ . Then we have related these to the differentials  $H_x^s, H_\theta^s, H_y^s$  [4], a relation which holds throughout our development and is invariant under superconformal transformations. Next, we showed, by constructing a new fibre bundle, the identity which states that the generator of the supercanonical line bundle over a SRS is the square root of the Vierbein field which is the generator of the constructed fibre bundle. Using this together with the invariance under a superconformal transformation of a  $(\frac{1}{2}, \frac{1}{2})$ -superdifferential we changed the initial superconformal structure to a new one parametrized by new super Beltrami differentials. These are related to the initial ones by the right action of the induced superdiffeomorphism in the reference structure. By considering the special case of superconformal transformations and the condition  $\nu = \theta$ , we found the second super Beltrami equation of the superconformal coordinates. We explained this result as a consequence of the fact that quasi-superconformal transformations reduce to superconformal ones by imposing the superconformally invariant constraints  $\hat{\mu} = 0, \nu = \theta, \sigma = 0$ .

Afterwards, we defined an origin for the  $\bar{\theta}$ -independent solution to the constraint  $\nu = \theta$ . This allowed us to give a local expression for a holomorphic  $\frac{1}{2}$ -superdifferential  $\eta$ . Mainly, this expression reflects the fact that  $\eta$  has according to the super Riemann-Roch theorem [13, 14] globally  $(g - 1)$  zeros counting multiplicity. Consequently, we found only half of the global number of zeros of the corresponding bosonic differential by expounding  $\eta$  in its components. This was interpreted by the fact that the first term in the expansion of  $\eta$  is the square root of the holomorphic bosonic differential. In fact this shows that the theory of SRS's is not just trivial generalization of that of RS's, but rather a rich and elegant frame of many fascinating issues. Accordingly, we have suggested the general expression for a polydromic superdifferential without giving, however, its monodromy multiplier [24]. With all necessary tools at hand, we showed that our special isothermal supercoordinates are holomorphic w.r.t. the super Beltrami differentials at the fixed origin. This was performed by choosing the "parameter" function  $\varepsilon$  to have the same property due to the fact that it vanishes at the chosen origin.

Finally, the structure of SRS's promises many interesting features which need serious investigations.

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