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A Polyakov Action On Riemann surfaces

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Abstract

We present a calculation of the effective action for induced conformal gravity on higher genus Riemann surfaces. Our expression, generalizing Polyakov's formula, depends holomorphically on the Beltrami and integrates the diffeomorphism anomaly.

1 Introduction

In the past few years a large body of literature has been devoted to the study of 2-dimensional conformal field theories on Riemann surfaces without boundary [1,2]. The relevance of these models in string perturbation theory and in the analysis of 2-dimensional statistical system obeying certain periodic boundary conditions at criticality is well-known and needs no further mention.

Most of the studies on the subject are concerned with Lagrangian field theories on 2-dimensional Riemannian manifolds which are both diffeomorphism and Weyl invariant at the classical level [3]. The quantization program is carried out by means of a diffeomorphism invariant renormalization scheme, typically the ζ function scheme [4,5]. In general, a Weyl anomaly is produced in this way. This can be eliminated by either *i*) constraining the field content of the theory so that the total central charge vanishes, as in the case of string theory [4], or *ii*) subtracting from the effective action a suitable local counterterm that absorbs the Weyl anomaly at the cost of creating a diffeomorphism anomaly [6]. In either way, conformal invariance emerges, but in an indirect way, which is obscured by the insistence on the use of metric data extraneous to the underlying conformal geometry and narrowed in scope by the restriction to a Lagrangian framework.

It would be desirable to have a scheme for the study of 2-dimensional conformal field theory on Riemann surfaces that relies *ab initio* and exclusively on conformal geometry [7]-[9]. It would also be preferable if such a scheme were non Lagrangian in nature, in order to allow for a wider range of applications and not bias our understanding of the subject [10]. This is, in outline, the point of view advocated in this paper.

Let us now illustrate in greater detail the essential features of the above program. Consider a compact surface Σ without boundary of genus $g \geq 2$. An arbitrary conformal structure s on Σ is represented by a Beltrami differential μ, a conformal field of conformal weights $(-1,1)$ in some fixed reference conformal structure s_0 satisfying the condition $|\mu| < 1$. s_0 corresponds to $\mu = 0$. In this way, the dependence on the conformal structure s is translated in field theoretic terms as a dependence on the Beltrami differential μ while all quantities can be written by using the holomorphic coordinates $z, \bar{z}$ of the reference conformal structure s_0 [8].

Consider now the effective action of an arbitrary conformal field theory on Σ. To simplify the analysis, we shall set all matter field excitations to zero. The inclusion of matter fields is straightforward and does not alter in any essential way the conceptual framework expounded below. In this manner, the effective action will be a functional $\Gamma[\mu, \bar{\mu}]$ of μ and $\bar{\mu}$ only. We shall adopt a somewhat axiomatic point of view and postulate a few general properties of $\Gamma[\mu, \bar{\mu}]$ which have been shown in a number of examples but are expected to hold in general. Then, we shall try to prove general theorems by relying exclusively on these properties.

The first postulate concerning $\Gamma[\mu, \bar{\mu}]$ is commonly called holomorphic factorization [11,12]. It states that:

$$\Gamma[\mu, \bar{\mu}] = \Gamma[\mu] + \Gamma^*[\bar{\mu}], \tag{1.1}$$

where the two contributions in the right hand side correspond to the chiral and antichiral sectors of the theory. The second postulate is the conformal Ward identity expressing the anomalous breakdown of the diffeomorphism symmetry [8,9]:

$$\left(\bar\partial - \mu\partial - 2(\partial\mu)\right)\frac{\delta\Gamma[\mu]}{\delta\mu} = -\frac{k}{24\pi}\left(\partial^3\mu + 2\mathcal{R}_0\partial\mu + \partial\mathcal{R}_0\mu\right),$$
$$\left(\partial - \bar\mu\bar\partial - 2(\bar\partial\bar\mu)\right)\frac{\delta\Gamma^*[\bar\mu]}{\delta\bar\mu} = -\frac{k^*}{24\pi}\left(\bar\partial^3\bar\mu + 2\mathcal{R}_0^*\bar\partial\bar\mu + \bar\partial\mathcal{R}_0^*\bar\mu\right). \tag{1.2}$$

Here, k (k^*) is the central charge of chiral (antichiral) sector. $\mathcal{R}_0$ ($\mathcal{R}_0^*$) is a holomorphic (antiholomorphic) projective connection in the reference conformal structure and ensures the correct conformal covariance of the right hand side of (1.2).

In a chiral conformal field theory $\Gamma^*[\bar\mu]$ needs not equal $\overline{\Gamma[\mu]}$. Nevertheless, the formal properties of $\Gamma^*[\bar\mu]$ and $\overline{\Gamma[\mu]}$ are the same. Hence, below we shall consider only the chiral component $\Gamma[\mu]$.

It has been shown by Becchi that in the infinite plane the conformal Ward identity has a unique solution [8]. This has been found by Polyakov in ref. [13] eq. (5). To the best of our knowledge, the generalization of Polyakov's formula to a compact Riemann surface without boundary has never been worked out. In this case, moreover, the solution is certainly non unique as it is apparent from the fact that the operator $\bar\partial - \mu\partial - 2(\partial\mu)$ has globally defined zero modes, the holomorphic quadratic differentials in the conformal structure μ. In this paper, we shall work out a solution of the conformal Ward identity on an arbitrary compact Riemann surface without boundary which turns out to have remarkable properties.

2 Solution of the conformal Ward identity

2.1 Singlevalued and multivalued fields

Our solution of the conformal Ward identity (1.2) involves both singlevalued and multivalued fields on the relevant surface Σ. While the use of singlevalued fields poses no problem, the treatment of multivalued fields requires some care. For the analysis below relies in an essential way on Stokes' theorem, which is applicable only if the fields involved are smooth and singlevalued on the appropriate manifold. To circumvent this difficulty, one must choose for each multivalued field a branch which is a singlevalued field defined on a suitable dissection Σ^* of Σ.

There are two types of multivaluedness: *a*) multivaluedness around a non contractible loop of Σ and *b*) multivaluedness around a point of Σ. For a field suffering only type *a* multivaluedness, the dissection Σ^* is obtained by cutting Σ along the loops of a set of generators of the first homotopy group $\pi_1(\Sigma, p^*)$ of Σ with base point p^*. For simplicity, we shall use as generators the standard *a*- and *b*-loops of Σ based at p^*. In this way, one obtains the dissection $\tilde\Sigma$ shown in fig. 1. For a field exhibiting both type *a* and *b* multivaluedness, the dissection Σ^* is obtained by cutting along a set of generators of the first homotopy group $\pi_1(\Sigma \setminus \{p_1, ..., p_N\}, p^*)$ of $\Sigma \setminus \{p_1, ..., p_N\}$ with base point p^*, where $p_1, ..., p_N$ are the points of Σ around which the field is multivalued. It is convenient to choose the generators to be the standard a- and b-loops of Σ based at p^* plus the loops based at p^* that only wind once around one of the points p_i, call them c-loops. The loop c_i consists of two distinct but geometrically coinciding arcs and an infinitesimal circle drawn around p_i. The resulting dissection, denoted by $\tilde{\Sigma}_{p_1...p_N}$ or, more briefly, $\tilde{\Sigma}_p$, is shown in fig. 2.

The choice of the base point p^* of the relevant homotopy group is arbitrary. Further, for a given base point p^*, the a-, b- and c-loops are defined up to homotopic equivalence. We shall assume, however, that the base point p^* and the a- and b-loops of $\tilde{\Sigma}$ and $\tilde{\Sigma}_p$ coincide, so that the boundary $\partial\tilde{\Sigma}$ of $\tilde{\Sigma}$ is a proper subset of the boundary $\partial\tilde{\Sigma}_p$ of $\tilde{\Sigma}_p$. Eventually, it will be necessary to check how our solution of the conformal Ward identity depends on the choice of the base point and the loops, each in its respective homotopy class.

It is clear that the dissections $\tilde{\Sigma}$ and $\tilde{\Sigma}_p$ can be thought of as subsets of Σ. As to integral formulae, the set theoretic differences $\Sigma \setminus \tilde{\Sigma}$ and $\Sigma \setminus \tilde{\Sigma}_p$ have zero measure. Thus, when only singlevalued fields are involved, we are allowed to replace Σ by either $\tilde{\Sigma}$ or $\tilde{\Sigma}_p$. Similarly, when only fields with type a multivaluedness are involved, we may replace $\tilde{\Sigma}$ by $\tilde{\Sigma}_p$.

2.2 Holomorphic j-differentials and holomorphic sections of holomorphic j-differentials

A holomorphic j-differential in the conformal structure μ is a conformal field η_j of weights $(j, 0)$ in the reference conformal structure satisfying the equation [8]

$$\bar{\partial}\eta_j - \mu\partial\eta_j - j\partial\mu\eta_j = 0. \tag{2.1}$$

One considers a vector bundle $\mathcal{H}_j$ whose base manifold is the set of all conformal structures and whose fiber at a given conformal structure μ is the complex vector space of holomorphic j-differentials in μ. A holomorphic section of $\mathcal{H}_j$ is a map that assigns to any conformal structure μ a holomorphic j-differential $\eta_j[\mu]$ in μ depending holomorphically on μ, i.e.

$$\frac{\delta}{\delta\bar{\mu}}\,\eta_j[\mu] = 0. \tag{2.2}$$

We assume further that the dependence on μ is compatible with the action of the diffeomorphism group $\mathrm{Diff}_0(\Sigma)$ by pull-back, i.e. it is such that [8]

$$\eta_j[\mu^f] = \left(\partial f + \mu \circ f \partial\bar{f}\right)^j \eta_j[\mu] \circ f, \tag{2.3}$$

where $f \in \mathrm{Diff}_0(\Sigma)$ and μ^f is the transform of μ under f.

2.3 Diffeomorphism invariance and the Slavnov operator

It is convenient to write the conformal Ward identity in an alternative form in which its meaning as a manifestation of the anomalous breakdown of the diffeomrophism symmetry is manifest. To this end, we need the expression of the infinitesimal action of the diffeomorphism group $\mathrm{Diff}_0(\Sigma)$ on the relevant fields. This is given by the Slavnov operator s, which is nilpotent [8,9]:

$$s^2 = 0. \tag{2.4}$$

The action of the Slavnov operator s on the relevant fields can be expressed in terms of the diffeomorphism ghost field C associated to the complex analytic structure on Lie $\mathrm{Diff}_0(\Sigma)$ induced by the conformal structure μ on Σ [9]:

$$C = \mathrm{c} + \mu\bar{c}, \tag{2.5}$$

where c and $\bar{c}$ are the standard diffeomorphism ghosts. C is a conformal field of weights $(-1, 0)$ in the reference conformal structure and satisfies [9]

$$sC = C\partial C. \tag{2.6}$$

The action of s on the Beltrami differential μ is given by [8,9]:

$$s\mu = \bar{\partial}C - \mu\partial C + \partial\mu C. \tag{2.7}$$

If $\eta_j \equiv \eta_j[\mu]$ is a holomorphic section of holomorphic j-differentials, then

$$s\eta_j = C\partial\eta_j + j\partial C\eta_j. \tag{2.8}$$

This important relation follows directly from (2.3) upon taking (2.1) into account.

All objects referring to the reference conformal structure $\mu = 0$ are assumed to be invariant under the diffeomorphism group $\mathrm{Diff}_0(\Sigma)$. In particular, if η_{j0} is a holomorphic j-differential in $\mu = 0$, then

$$s\eta_{j0} = 0. \tag{2.9}$$

2.4 Construction of the solution of the conformal Ward identity

By using (2.7), it is straightforward to check that the conformal Ward identity (1.2) can be written equivalently as:

$$s\Gamma[\mu] = \frac{k}{48\pi}\int_\Sigma d^2z\left[C\partial^3\mu - \mu\partial^3 C + 2\mathcal{R}_0(C\partial\mu - \mu\partial C)\right]. \tag{2.10}$$

This relation expresses the noninvariance of $\Gamma[\mu]$ under the diffeomorphism group, the right hand side being precisely the diffeomorphism anomaly. The conformal Ward identity is most easily solved in the form (2.10). The solution contains three parts:

$$\Gamma[\mu] = \Gamma^{I}[\mu] + \Gamma^{II}[\mu] + \Gamma^{III}[\mu], \tag{2.11}$$

$$\Gamma^{I}[\mu] = \frac{k}{48\pi}\int_{\tilde{\Sigma}} d^2z \Big[2\mathcal{R}_0\mu + 2\partial^2\mu + \partial\ln\Omega\partial\mu + \partial\ln\Omega_0\partial\mu + \mu\partial\ln\Omega\partial\ln\Omega_0\Big], \tag{2.12}$$

$$\Gamma^{II}[\mu] = \frac{k}{96\pi i}\oint_{\partial\tilde{\Sigma}_{p_0}} \ln(\Omega/\omega_0)(dz\partial + d\bar{z}\bar{\partial})\ln(\omega/\Omega_0), \tag{2.13}$$

$$\Gamma^{III}[\mu] = \frac{k}{48}\Sigma_i\nu_{0i}\ln(\omega/\Omega_0)(p_{0i}). \tag{2.14}$$

Here, ω and Ω are holomorphic sections of holomorphic 1-differentials and, by definition, satisfy the relations (2.1)-(2.3) and (2.8) with $j = 1$ and η_j replaced by ω and Ω, respectively. Similarly, ω_0 and Ω_0 are holomorphic 1-differentials in the reference conformal structure $\mu = 0$ and satisfy the relations (2.1) with $\mu = 0$ and (2.9) with η_{0j} replaced by ω_0 and Ω_0, respectively. ω and ω_0 are singlevalued and have zeros p_i of order ν_i and p_{0i} of order ν_{0i}, respectively. Ω and Ω_0, conversely, are multivalued on Σ and have no zeros. It is not obvious *a priori* that such objects do exist. For this reason, we have provided an explicit construction of Ω and Ω_0 in terms of the prime form in the appendix. The dissection $\tilde{\Sigma}$ and $\tilde{\Sigma}_{p_0}$ have been defined earlier. The points p_{oi} used to define $\tilde{\Sigma}_{p_0}$ are precisely the zeros of ω_0. $\tilde{\Sigma}$ and $\tilde{\Sigma}_{p_0}$ are chosen to be independent from μ. It is also assumed that all zeros of ω are contained in the interior of $\tilde{\Sigma}_{p_0}$. Finally, $\partial\tilde{\Sigma}_{p_0}$ has the orientation induced by that of Σ.

It is not difficult to verify that the integrand of the right hand side of (2.12) is a well-defined conformal field of weights (1,1) in the reference conformal structure $\mu = 0$. Upon choosing branches of the multivalued fields Ω and Ω_0 defined on the same dissection $\tilde{\Sigma}$, the surface integration can be carried out consistently. The result, of course, will depend on the branches chosen. Similarly, it is apparent that the integrand of the right hand side of (2.13) is a well-defined 1-form. To perform the line integration, one must choose branches of the multivalued fields Ω and Ω_0, which, for consistency, must be the same as the ones used in (2.12). The combination $\ln(\Omega/\omega_0)$ suffers an additional multivaluedness around the zeros p_{oi} of ω_0. Once the branch of Ω is fixed, a branch of $\ln(\Omega/\omega_0)$ must be chosen. This will be defined on the dissection $\tilde{\Sigma}_{p_0}$, hence the choice of the contour in (2.13). Again, the results of the line integration depends on the branches.

The property that Ω and Ω_0 have no zeros is crucial for the consistency of the solution. This is the ultimate reason why multivalued fields are involved. Indeed, singlevalued holomorphic 1-differential in any conformal structure always have $2g-2$ zeros counting multiplicity [14]. If Ω and Ω_0 were singlevalued, the integrand of the right hand side would develop singularities some of which would become non integrable upon application of the Slavnov operator and which would make problematic the application of Stokes' theorem. The occurence of zeros can be avoided by relaxing the too restrictive condition of singlevaluedness.

The proof that the functional $\Gamma[\mu]$ defined by (2.11)-(2.14) does indeed solve the conformal Ward identity (2.10) relies only on the relations (2.1)-(2.7),(2.9) and Stokes' theorem.

The algebraic details are rather lengthy and will be presented elesewhere [15]. We give the expressions of $s\Gamma^{I}[\mu], s\Gamma^{II}[\mu]$ and $s\Gamma^{III}[\mu]$ to show the interplay of the three contributions:

$$
\begin{aligned}
s\Gamma^{I}[\mu] &= \frac{k}{48\pi}\int_{\tilde{\Sigma}} d^2z\,\Big[C\partial^3\mu - \mu\partial^3 C + 2\mathcal{R}_0(C\partial\mu - \mu\partial C)\Big], \\
&\quad -\frac{k}{96\pi i}\oint_{\partial\tilde{\Sigma}}\Big[d\bar{z}\,\Big((\partial + (\partial\ln\Omega))(\bar{\partial}C - \mu\partial C + \partial\mu C) \\
&\quad +(\partial + (\partial\ln\Omega_0))\bar{\partial}C + C\partial\bar{\partial}\ln\Omega\Big) \\
&\quad +dz\,\Big(C(\partial^2\ln\Omega + \partial^2\ln\Omega_0 - \partial\ln\Omega\partial\ln\Omega_0 - 2\mathcal{R}_0)\Big)\Big], \qquad (2.15)\\
s\Gamma^{II}[\mu] &= \frac{k}{96\pi i}\oint_{\partial\tilde{\Sigma}_{p_o}}\Big[d\bar{z}\,\Big((\partial + (\partial\ln\Omega))(\bar{\partial}C - \mu\partial C + \partial\mu C) \\
&\quad +(\partial + (\partial\ln\Omega_0))\,\bar{\partial}C + C\partial\bar{\partial}\ln\Omega\Big) \\
&\quad +dz\,\Big(C(\partial^2\ln\Omega + \partial^2\ln\Omega_0 - \partial\ln\Omega\partial\ln\Omega_0 - 2\mathcal{R}_0)\Big)\Big] \\
&\quad -\frac{k}{48}\Sigma_i\nu_{0i}(\partial + (\partial\ln\omega))C(p_{0i}), \qquad (2.16)\\
s\Gamma^{III}[\mu] &= \frac{k}{48}\Sigma_i\nu_{0i}\,(\partial + (\partial\ln\omega))\,C(p_{0i}). \qquad (2.17)
\end{aligned}
$$

In the right hand side of (2.15), it is permissible to replace the surface integration on $\tilde{\Sigma}$ by one over Σ for reasons explained earlier. Likewise, in the right hand side of (2.16), the line integration $\partial\tilde{\Sigma}_{p_0}$ can be substituted by one over $\partial\tilde{\Sigma}$. Then, it is immediate to see that the functional $\Gamma[\mu]$ defined in (2.11)-(2.14) does indeed satisfy the conformal Ward identity (2.10).

3 Properties of the solution of the conformal Ward identity

The solution $\Gamma[\mu]$ of the conformal Ward identity (1.2), (2.10) worked out in the preceeding section depends on a variety of choices and parameters. Indeed, this reflects the nonuniqueness of the solution itself which was mentioned earlier. Now, it is important to study the dependence of $\Gamma[\mu]$ on these inputs. From (2.10), it is apparent that the variation $\Delta\Gamma[\mu]$ suffered by $\Gamma[\mu]$ when the inputs are varied must satisfy the basic identity

$$s\Delta\Gamma[\mu] = 0, \qquad (3.1)$$

a consistency condition which may be used to check the correctness of the expression of $\Gamma[\mu]$ we have provided.

3.1 Dependence of $\Gamma[\mu]$ on the branch of multivalued fields

There are essentially two ways of varying the branch of a multivalued field: either a) by changing the dissection on which the branch is defined while keeping the functional dependence of the branch on the intersection of the old and new dissection unchanged or b) by changing the functional dependence of the branch compatibly with the underlying multivalued field while keeping the dissection on which the branch is defined unchanged. In general, a variation of the branch amounts to a set of substitutions of the form

$$\begin{aligned} &\Omega \to \tilde{\Omega}, \\ &\Omega_0 \to \tilde{\Omega}_0, \\ &\ln(\Omega/\omega_0) \to \ln(\tilde{\Omega}/\omega_0) + 2\pi i n. \end{aligned} \tag{3.2}$$

where n is an integer which accounts for the multivaluedness of $\ln(\Omega/\omega_0)$ around the zeros of ω_0. The corresponding variation $\Delta\Gamma[\mu]$ of $\Gamma[\mu]$ is given by the general formula

$$\Delta\Gamma[\mu] = \frac{k}{48}\sum_{i\in\mathcal{I}}\nu_i \ln(\tilde{\Omega}/\Omega)(p_i) - \frac{k}{48}\sum_{i\in\mathcal{I}_0}\nu_{0i}\ln(\tilde{\Omega}_0/\Omega_0)(p_{0i}) + \frac{i\pi k}{24}N. \tag{3.3}$$

Here, $\mathcal{I}$ and $\mathcal{I}_0$ are certain sets of indices and N is an integer depending on the variation of the branch performed. Though the relation (3.3) is very simple, its proof is rather lenghty and will be provided elsewhere [15].

It is not difficult to check that the consistency condition (3.1) is satisfied by the expression (3.3). Indeed, under a diffeomorphism f sufficiently close to the identity, $\ln(\tilde{\Omega}/\Omega) \to \ln(\tilde{\Omega}/\Omega)\circ f$, $p_i \to f^{-1}(p_i)$ while $\ln(\tilde{\Omega}_0/\Omega_0)$, p_{0i}, $\mathcal{I}$, $\mathcal{I}_0$ and N are left unchanged, implying the diffeomorphism invariance of the expression (3.3).

Concerning branch variation of type a, we have proven the following. As a preliminary remark, we notice that any deformation of the dissection $\tilde{\Sigma}_{p_0}$ entails a corresponding deformation of the dissection $\tilde{\Sigma}$ since, by construction $\partial\tilde{\Sigma} \subseteq \partial\tilde{\Sigma}_{p_0}$. We notice also that a deformation of $\tilde{\Sigma}_{p_0}$ must leave the points p_{0i} fixed, for the p_{0i}'s are the zeros of ω_0 and ω_0 itself is not changed. Consider now a deformation of $\tilde{\Sigma}_{p_0}$ leaving the base point fixed and such that during the deformation $\partial\tilde{\Sigma}_{p_0}$ is never crossed by any zero p_i of ω. To this deformation there corresponds a type a branch variation for which $\mathcal{I} = \mathcal{I}_0 = \emptyset$ and $N = 0$ [15]. It then follows from (3.3) that $\Delta\Gamma[\mu] = 0$. This result is welcome! It shows that $\Gamma[\mu]$ is essentially independent from the 'shape' of the dissection $\tilde{\Sigma}_{p_0}$, a datum indeed extraneous to the underlying conformal geometry. For more general deformations of $\tilde{\Sigma}_{p_0}$ which leaves the base point fixed, it can be shown that $\mathcal{I}$ is the set of all indices i such that p_i crosses $\partial\tilde{\Sigma}$ during the deformation and that $\mathcal{I}_0 = \emptyset$ while N is a sum of products $\nu_i\nu_{0j}$ depending in a very complicated way on the details of the deformation of $\partial\tilde{\Sigma}_{p_0}\backslash\partial\tilde{\Sigma}$ [15].

Concerning branch variation of type b, it can be shown that $\mathcal{I}$ and $\mathcal{I}_0$ contain all indices and that $N = 2(g-1)n$ [15].

3.2 Dependence of $\Gamma[\mu]$ on the choice of the fields ω, Ω, ω_0 and Ω_0

Let us study first the dependence on the choice of Ω and Ω_0. A change of this choice is represented by a set of subsitutions formally coinciding with (3.2). For a fixed branch of

Ω and $\tilde{\Omega}$ there is no a priori relation between the branches of $\ln(\Omega/\omega_0)$ and $\ln(\tilde{\Omega}/\omega_0)$ since the latter are multivalued also around the zeros of ω_0. Thus, the integer n is arbitrary. Assuming that the branches of Ω and Ω_0 and $\tilde{\Omega}$ and $\tilde{\Omega}_0$ are defined on the same dissection $\tilde{\Sigma}$ and that, similarly, the branches of $\ln(\Omega/\omega_0)$ and $\ln(\tilde{\Omega}/\omega_0)$ are defined on the same dissection $\tilde{\Sigma}_{p_0}$, the variation $\Delta\Gamma[\mu]$ of $\Gamma[\mu]$ is given by eq. (3.3), $\mathcal{I}$ and $\mathcal{I}_0$ being the sets of all indices and N being $2(g-1)n$. This result is formally analogous to that of a branch variation of type b, but its meaning is completely different.

We have found no simple formula giving the variation $\Delta\Gamma[\mu]$ of $\Gamma[\mu]$ under a change of the choice of ω and ω_0.

4 Discussion and conclusions

Multiple functional derivatives of $\Gamma[\mu]$ with respect to μ give the connected Green functions of the energy momentum tensor

$$< T(z_1,\bar{z}_1)\dots T(z_n,\bar{z}_n) >^c_\mu \equiv \delta^n\Gamma[\mu]/\delta\mu(z,\bar{z}_1)...\delta\mu(z_n,\bar{z}_n). \tag{4.1}$$

By using the expression (2.11)-(2.14) of $\Gamma[\mu]$, we should be able in principle to compute these correlators. In practice, however, this may be a difficult task.

Let us analyze this point in greater detail. Consider the first order functional derivative $\delta\Gamma[\mu]/\delta\mu$. This is actually the combination through which $\Gamma[\mu]$ enters the conformal Ward identity in the form (1.2). From (1.2), it is possible to show that $\delta\Gamma[\mu]/\delta\mu$ has the following structure:

$$\frac{\delta\Gamma[\mu]}{\delta\mu} = \frac{k}{24\pi}\left[\mathcal{R}_0 - \partial^2\ln\theta + \frac{1}{2}(\partial\ln\theta)^2 - q[\theta]\right]. \tag{4.2}$$

Here, θ is a holomorphic section of singlevalued holomorphic 1-differentials, i.e. satisfies eqs.(2.1)-(2.3) with $\eta_j = \theta$ and $j = 1$. $q[\mu]$ is a holomorphic section of singlevalued meromorphic 2-differentials, i.e. it satisfies eqs (2.1)-(2.3) with $\eta_j = q[\theta]$ and $j = 2$ away from its singularities. Since θ has zeros, the combination $\partial^2\ln\theta - \frac{1}{2}(\partial\ln\theta)^2$ has singularities. It can be shown [15] that it is possible to choose $q[\theta]$ so that the right hand side of (4.2) is smooth. Of course, $q[\theta]$ is defined up to the addition of a holomorphic section of holomorphic 2-differentials. This, however, can not be fully arbitrary, since the integrability condition $\delta\Gamma[\mu]/\delta\mu(z_1,\bar{z}_1)\delta\mu(z_2,\bar{z}_2) = \delta\Gamma[\mu]/\delta\mu(z_2,\bar{z}_2)\delta\mu(z_1,\bar{z}_1)$ must be satisfied [8]. So far, the implications of such condition on $q[\theta]$ have not been worked out. However, by using the expression (2.11)-(2.14) of $\Gamma[\mu]$ we can in principle compute $q[\theta]$. If we choose $\theta = \omega$ for convenience, then $q[\theta]$ will be a functional of ω, ω_0, Ω and Ω_0 that automatically satisfies the integrability condition [15]. Similar consideration can be carried out for higher order functional derivatives of $\Gamma[\mu]$.

Now, we shall argue that the solution (2.11)-(2.14) of the conformal Ward identity (2.10) is in fact the most general one. Indeed, from (2.10), it is apparent that the most general solution is of the form $\Gamma[\mu]+\Phi[\mu]$, where $\Phi[\mu]$ is an arbitrary holomorphic functional of μ such that $s\Phi[\mu] = 0$. However, from (3.3), it is easily checked that the replacement $\Omega \to \tilde{\Omega} = \exp(-24\Phi[\mu]/k(g-1))\Omega$ reabsorbes $\Phi[\mu]$ into $\Gamma[\mu]$, proving the statement.

Following Polyakov's philosophy [13], one may view $\Gamma[\mu]$ as an effective action for induced conformal gravity. From the above remark, it follows that $\Gamma[\mu]$ depends on the details of the underlying conformal field theory through the normalization of Ω. It is possible, though, that there are choices of Ω which do not correspond to any conformal field theory. The open problem remains of selecting the solutions of (2.10) which do arise from some conformal field theory and to find the map that associates to any given conformal model the corresponding field Ω [10].

Our construction of $\Gamma[\mu]$ has been local in the space of conformal structures and it is not clear at this stage if the expression (2.11)-(2.14) of $\Gamma[\mu]$ can be extended to a global one. This is a further issue that should be investigated [10].

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A Appendix

In this appendix, we shall outline briefly an explicit construction of the multivalued holomorphic 1-differential Ω in the conformal structure μ.

Define

$$\sigma(Z,Z') = \frac{\theta\left(\mathcal{I}(Z - \sum_{i=1}^{g} P_i + \Delta)|\tau\right) \prod_{i=1}^{g} E(Z',P_i)}{\theta\left(\mathcal{I}(Z' - \sum_{i=1}^{g} P_i + \Delta)|\tau\right) \prod_{i=1}^{g} E(Z,P_i)}. \tag{A.1}$$

Here, the P_i's are g arbitrary points of Σ. $\theta(\zeta|\tau)$ is Riemann's theta function. τ is the period matrix of the conformal structure μ. $\mathcal{I}$ is the Jacobi map of μ. Δ is the Riemann class of μ. $E(Z,Z')$ is the prime form of μ [15,16].

From general theorems concerning the theta function and the prime form it can be proven that: *a*) $\sigma(Z,Z')$ is a holomorphic $g/2$ differential is Z and a holomorphic $-g/2$ differential in Z' in the conformal structure μ, *b*) it is independent from the points P_i, *c*) it is multivalued around the b-loops of Σ, *d*) it is nowhere vanishing and *e*) the dependence on μ of $\sigma(Z,Z')$ is holomorphic and compatible with pull-back by diffeomorphisms [15,16].

Ω is expressed in terms of $\sigma(Z,Z')$ as follows

$$\Omega(Z) = \oint_a (dz' + \mu' d\bar{z}') q' \sigma(Z,Z')^{2/g}, \tag{A.2}$$

where a is a linear combination of the a-loops and q is a holomorphic section of singlevalued holomorphic 2-differentials. It is a simple matter to show that Ω is a holomorphic 1-differential in the conformal structure μ and that Ω is multivalued around the b-loops. From the identity $\Omega(Z) = \sigma(Z,Z')^{2/g}\Omega(Z')$, it follows that Ω is nowhere vanishing. Finally, it is apparent that the dependence on μ of Ω is holomorphic and compatible with pull-back by diffeomorphisms.

If we replace the loop a by a new loop $\tilde{a}$ and the section q of 2-differentials by a new section $\tilde{q}$, we construct another 1-differential section $\tilde{\Omega}$ by (A.2). It is straightforward to check that the ratio $\tilde{\Omega}/\Omega$ is singlevalued, holomorphic and nowhere vanishing, hence a non zero constant. Thus, (A.2) defines Ω completely up to a normalization depending holomorphically on μ and invariant under diffeomorphisms.

Ω_0 is obtained simply by setting $\mu = 0$ in the above construction.

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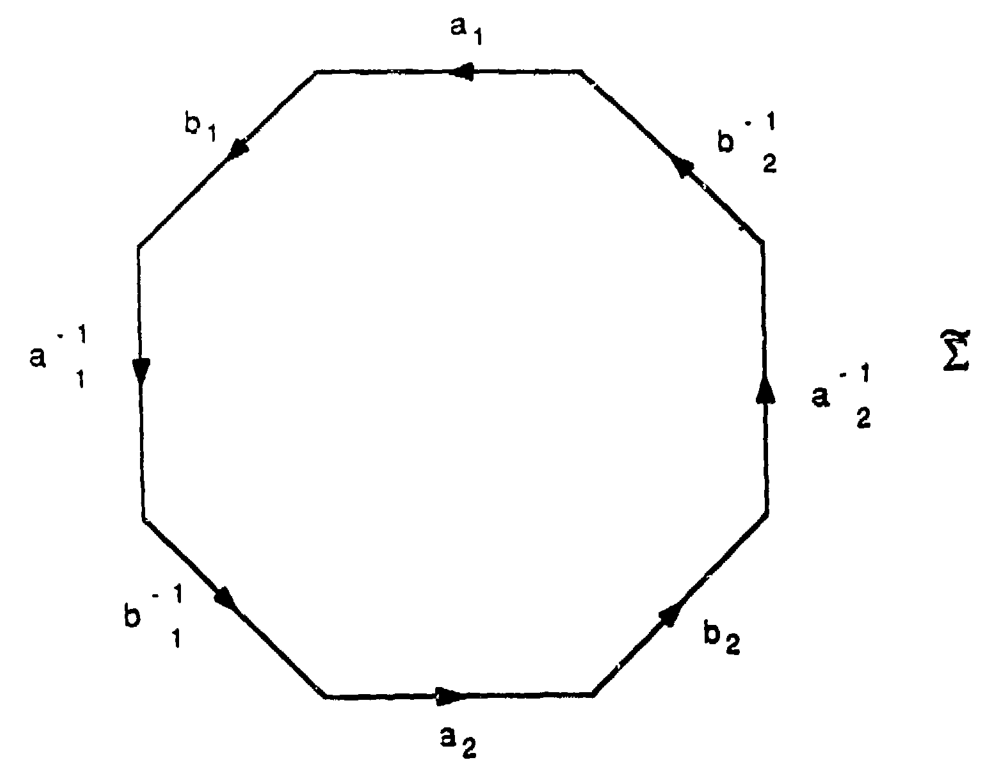

Fig.1 : The dissection $\tilde{\Sigma}$ for genus g=2

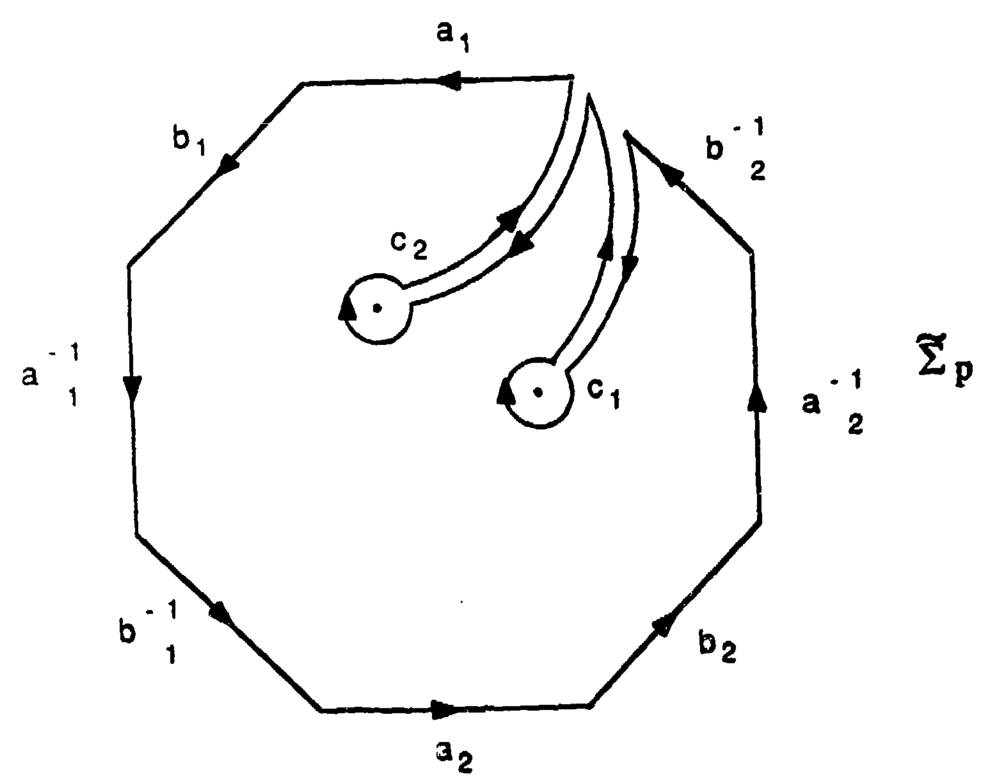

Fig.2 : The dissection $\tilde{\Sigma}$ for genus g=2 and two points p_1, p_2.