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ABSTRACT. Lie bialgebras originated in a triangular decomposition of the underlying Lie algebra are discussed. The explicit formulas for the quantization of the Heisenberg Lie algebra and some motion Lie algebras are given, as well as the algebra of rational functions on the quantum Heisenberg group and the formula of the universal R-matrix.

§0. Introduction. One of the main points of the "philosophy of quantum groups" is to consider, as a subject of the "quantization" or deformation, an algebra of functions on a variety with a fixed, perhaps degenerated, Poisson bracket, in such a way that the Poisson bracket provided by the quantization coincides with the original one. According to this general principle, Drinfeld introduced the notion of Poisson-Lie groups: Lie groups which are Poisson manifolds, both structures related by imposing the multiplication to be a Poisson mapping. The infinitesimal version of this notion is that of Lie bialgebras: a vector space which is simultaneously a Lie algebra and a Lie coalgebra, both structures connected by a cocycle condition (cf. [D1]).

This paper is concerned with a family of Lie bialgebras whose cobracket has origin in a triangular decomposition of the Lie algebra. Examples of Lie algebras with triangular decomposition are, besides the Kac-Moody algebras, the Drinfeld's example that served us as inspiration, the (extended) Heisenberg algebras and some special motion Lie algebras. (We also benefited from the notes [MP] where a similar notion was discussed. Unfortunately, the Virasoro algebra does not fit in our scheme). Our first basic result states that these Lie bialgebras are quasitriangular. This enables us to provide new examples of classical r-matrices, see §2.

One of the central problems of the theory is: given an arbitrary Lie bialgebra, to quantize it (in the sense of [D2, §2, §6]). As far as we know, this problem was not solved yet. Partial answers are: quantization of triangular Lie bialgebras [D3]; quantization of simple Lie algebras and other important examples [D2]; and quantization of quasitriangular and coboundary Lie bialgebras, but in the more relaxed context of quasi-Hopf algebras [D4], [D5].

We give the explicit formula for the quantization -in the setting of Hopf (not quasi-Hopf) algebras - of the Heisenberg and (some particular) motion Lie algebras, §3. We also construct the ring of "rational functions" on the quantum Heisenberg group §4 and provide the explicit formula for the quantum R-matrix §6.

A Poisson structure on an arbitrary manifold provides a foliation of it by the so-called "symplectic leaves", cf. [Ki], [W], [S]. They are related to the representations of the algebra on functions on a quantum group, at least in the compact case [LS]. We classify the symplectic leaves of the real Heisenberg group in §5.

§1. Lie algebras with triangular decomposition.

For simplicity we shall work on the field $\mathbb{C}$ of complex numbers, unless explicitly stated.

Definition. Let $\mathfrak{g}$ be a Lie algebra. We shall say that the data $(\mathfrak{g}_0, \mathfrak{g}_+, \mathfrak{g}_-, k(\,|\,))$ is a *triangular decomposition* (TD), of $\mathfrak{g}$ if $\mathfrak{g}_-, \mathfrak{g}_0, \mathfrak{g}_+$ are subalgebras of $\mathfrak{g}$, $\mathfrak{g}_0$ abelian, and

$$k(\,|\,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

is a $\mathfrak{g}$-invariant, non degenerate, bilinear form such that $[\mathfrak{g}_\pm, \mathfrak{g}_0] \subset \mathfrak{g}_\pm$, $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, direct sum of subspaces, and

$$0 = k(\mathfrak{g}_+|\mathfrak{g}_+) = k(\mathfrak{g}_-|\mathfrak{g}_-) = k(\mathfrak{g}_+|\mathfrak{g}_0) = k(\mathfrak{g}_0|\mathfrak{g}_-).$$

In what follows, we shall simply say "$\mathfrak{g}$ is a Lie algebra with triangular decomposition", without mentioning the data defining it.

We shall use the notation $x = x_+ + x_0 + x_-$ for $x \in \mathfrak{g}$, if $x_j \in \mathfrak{g}_j$, $j \in \{+, 0, -\}$.

Example (a). Let $\mathfrak{g}$ be an abelian Lie algebra and let $<\,,\,>$ be a non degenerate bilinear form on $\mathfrak{g}$. Then $(\mathfrak{g}, 0, 0, <\,,\,>)$ is a TD.

Example (b). Let A be a symmetrizable complex matrix and let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding contragradient Lie algebra [K, Ch. 1]. Then $\mathfrak{g}$ has a well-known triangular decomposition, cf [K, 1.2, 2.2].

Example (c). Let A be a symmetrizable generalized Cartan matrix and let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding Kac-Moody algebra. Let $\mathfrak{l} = \mathfrak{g} \times \mathfrak{g}$ be the motion Lie algebra with respect to the adjoint representation, i.e.

$$[(x, y), (u, v)] = ([x, u], [x, v] - [u, y]).$$

Consider the usual decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{h}$ is the Cartan subalgebra, $\mathfrak{n}_+$ is the span of the positive root vectors, etc. Take

$$\mathfrak{l}_0 = \mathfrak{h} \times \mathfrak{h}, \quad \mathfrak{l}_+ = \mathfrak{n}_+ \times \mathfrak{n}_+, \quad \text{and} \quad \mathfrak{l}_- = \mathfrak{n}_- \times \mathfrak{n}_-.$$

Thus $\mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_-$. Let $k\,(\,|\,) : \mathfrak{l} \times \mathfrak{l} \to \mathbb{C}$ be defined by

$$k\,((x, y)|(u, v)) = K(x|u) + K(y|u) + K(x|v).$$

where $K(\,|\,)$ is the invariant non-degenerate bilinear form of $\mathfrak{g}$ given by [K, 2.2]. Then $(\mathfrak{l}_0, \mathfrak{l}_+, \mathfrak{l}_-, k\,(|))$ is a TD of $\mathfrak{l}$.

Remark 1. Let V be a $\mathfrak{g}$-module and consider the motion Lie algebra $\mathfrak{g} \oplus V$, i. e. with the Lie bracket given by $[(x, y), (u, v)] = ([x, u], xv - uy)$. Suppose that $\mathfrak{g} \oplus V$ admits a non degenerate invariant bilinear form $(\,|\,)$. Obviously $(V|\mathfrak{g}V) = 0$. If V is irreducible and non trivial, $(V|V) = 0$ and we obtain a monomorphism of $\mathfrak{g}$-modules $V \hookrightarrow \mathfrak{g}^*$. Assume that $\mathfrak{g}$ is simple: then $V \simeq \mathfrak{g}^*$. If in addition $\mathfrak{g}$ is finite dimensional, identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form. Then any invariant non-degenerate bilinear form on $\mathfrak{g} \oplus V$ is $aK(x|u) + bK(y|u) + bK(x|v)$, for some scalars a, b. Let c be a scalar and let T_c be the Lie algebra automorphism of $\mathfrak{g} \oplus V$, $T_c((x, v)) = (x, cv)$. By using an appropriate T_c, we may assume that an invariant non-degenerate bilinear form on $\mathfrak{g} \oplus V$ is a multiple of the one considered in Example (c).

Example (d). Extended Heisenberg algebras. Let $\mathfrak{H}^n$ be the Lie algebra generated by x_i, y_i, $i = 1, \dots, n, z, d$ with the following relations:

$$[x_i, y_j] = \delta_{ij} z, \quad [d, x_i] = x_i, \quad z \text{ is central and } \quad [d, y_i] = -y_i.$$

$B = \{x_i, y_i, z, d\}$ is a basis of $\mathfrak{H}^n$ and the bilinear form given by

$$k(x_i|y_j) = \delta_{ij}, \quad k(d|z) = 1,$$

the other products between generators equal to 0, is clearly invariant. Take

$$\mathfrak{H}_0^n = \text{span}\{z, d\}, \quad \mathfrak{H}_+^n = \text{span}\{x_i : i = 1, \dots, n\}, \quad \mathfrak{H}_-^n = \text{span}\{y_i : i = 1, \dots, n\}.$$

Then $(\mathfrak{H}_0^n, \mathfrak{H}_+^n, \mathfrak{H}_-^n, k(\,|\,))$ is a TD of $\mathfrak{H}^n$.

Example (e). The construction of Example (c) is valid replacing the Kac-Moody algebra $\mathfrak{g}(A)$ by an arbitrary $\mathfrak{g}$ with TD.

Example (f). The Virasoro algebra is not a Lie algebra with triangular decomposition in our sense: as is well-known, it lacks a non-zero invariant bilinear form.

Example (g). A Lie algebra $\mathfrak{g}$ with TD such that $\mathfrak{g}_0 = 0$ is equivalent to a Manin triple.

We recall that a *Manin triple* is a data (P, P_1, P_2), where P is a Lie algebra and P_1, P_2 are Lie subalgebras of P, together with a bilinear form $<\,|\,>: P \times P \to \mathbb{C}$, such that

(a) $P_1 \oplus P_2 = P$ as vector spaces,
(b) $<\,|\,>$ is P-invariant, non degenerate and $<P_i|P_i> = 0$.

Assume that P is finite dimensional. Then of course $P_1 \simeq P_2^*$ and $P_1 \otimes P_1 \simeq (P_2 \otimes P_2)^*$. Thus there is a bijection between Manin triples with a fixed P_1 and Lie bialgebra structures on P_1: the cobracket $\delta : P_1 \to P_1 \otimes P_1$ is the transpose of the bracket on P_2; i. e. δ is given by

$$<\delta(x)|u \otimes v> = <x|[u, v]>, \quad \forall\, x \in P_1, \quad \forall\, u, v \in P_2.$$

Let $\{x_i\}$ be a basis of P_1 and $\{x^i\}$ be its dual basis in P_2. If $[x^j, x^k] = b_l^{jk} x^l$, then

$$\delta(x_i) = b_i^{jk} x_j \otimes x_k. \tag{1}$$

Now let us allow the dimension of P to be infinite and let $\delta : P_2^* \to (P_2 \otimes P_2)^*$ still denote the transpose of the bracket on P_2. We have inclusions $P_1 \hookrightarrow P_2^*$ and $P_1 \otimes P_1 \hookrightarrow (P_2 \otimes P_2)^*$. We want to find conditions insuring that $\delta(P_1) \subset P_1 \otimes P_1$, with those identifications. Let $\{x_i : i \in I\}$ be a basis of P_1 and assume that P_2 admits a basis $\{x^i : i \in I\}$ such that $\langle x_i, x^j \rangle = \delta_{ij}$. Assume further that the support of the family $(b_l^{jk})_{j,k \in I}$ is finite for each l, where as before $[x^j, x^k] = b_l^{jk} x^l$. Then $\delta(P_1) \subset P_1 \otimes P_1$ and P_1 is a Lie bialgebra. On the other hand, let $\{x_i : i \in I\}$ be now only a set of generators of the Lie algebra P_1 and assume $\delta(x_i) \in P_1 \otimes P_1$ for all i. Again, $\delta(P_1) \subset P_1 \otimes P_1$ because of the cocycle condition $\delta[x,y] = [x \otimes 1 + 1 \otimes x, \delta(y)] - [y \otimes 1 + 1 \otimes y, \delta(x)]$.

The following fact is a generalization of [D2, Ex. 3.2].

Proposition 1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra with TD. Then $\mathfrak{g}$ admits a "canonical" structure of Lie bialgebra.*

Proof. Let $P = \mathfrak{g} \times \mathfrak{g}$ with the product Lie algebra structure. Let $P_1 = \{(a,a) : a \in \mathfrak{g}\}$ and let $P_2 = \{(a_- + a_0, a_+ - a_0) : a_- \in \mathfrak{g}_-,\ a_+ \in \mathfrak{g}_+,\ a_0 \in \mathfrak{g}_0\}$. Then $P_1, P_2 \subset P$ are Lie subalgebras. Let $< \mid >: P \times P \to \mathbb{C}$ be the bilinear form defined by $<(x,y)|(u,v)> = k(x|u) - k(y|v)$. Then $< \mid >$ is P-invariant, non degenerate and obviously $< P_1|P_1 > = 0$. If $(x,y) = (x_- + x_0, x_+ - x_0)$ and $(u,v) = (u_- + u_0, u_+ - u_0)$ belongs to P_2, then

$$<(x,y)|(u,v)> = k(x,u) - k(y|v) = k(x_0|u_0) - k(-x_0| - u_0) = 0,$$

thus $< P_2|P_2 > = 0$. It follows that (P, P_1, P_2) is a Manin triple and then $P_1 \simeq \mathfrak{g}$ has a structure of Lie bialgebra. □

Remark 2. $\delta = 0$ if and only if $[\mathfrak{g}_+, \mathfrak{g}_+] = [\mathfrak{g}_-, \mathfrak{g}_-] = [\mathfrak{g}, \mathfrak{g}_0] = 0$.

The following statement will be useful later:

Lemma 1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra with TD, $\{x_j : j \in J\}$ a basis of $\mathfrak{g}_+$, $\{y_j\}$ its dual basis in $\mathfrak{g}_-$, $\{h_i : i \in I\}$ a basis of $\mathfrak{g}_0$ and $\{l_i\}$ its dual basis in $\mathfrak{g}_0$. Identify $\mathfrak{g}$ with P_1, and $\mathfrak{g}^*$ with P_2. Then the vectors*

$$(2) \qquad x_j^* = (y_j, 0), \quad y_j^* = (0, -x_j) \quad \text{and} \quad h_i^* = \frac{1}{2}(l_i, -l_i),$$

$j \in J$, $i \in I$ constitutes the dual basis of $B = \{x_j\} \cup \{y_j\} \cup \{h_i\}$.

The proof is trivial.

Remark 3. Let $\mathfrak{g}$ be a Lie algebra with TD; then the proof of Proposition 1 shows that it gives rise to a Manin triple. In our Examples, this Manin triple induces a Lie bialgebra structure on $\mathfrak{g}$, as can be shown with the help of the considerations before the Proposition 1.

Example (c) (*continued*). Assume further, for notational simplicity, that A is a Cartan matrix. Let Φ be the root system of $\mathfrak{g}$, Φ^+ the set of positive roots and

$\Pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots. We choose $a_\alpha \in \mathfrak{g}_\alpha - \{0\}$, $\alpha \in \Phi$, ($\mathfrak{g}_\alpha$ is the root space) and $H_i, m_i \in \mathfrak{h}$ such that:

$$K(H_i|H) = \alpha_i(H),\ \forall H \in \mathfrak{h}, \quad [a_{\alpha_i}, a_{-\alpha_i}] = H_i,$$
$$K(a_\alpha|a_{-\alpha}) = 1, \quad K(H_i|m_j) = \delta_{ij}.$$

Let us consider the following elements of $\mathfrak{l}$:

$$\begin{aligned} x_\alpha &= (a_\alpha, 0), & u_\alpha &= (0, a_\alpha) \\ y_\alpha &= (0, a_{-\alpha}), & v_\alpha &= (a_{-\alpha}, -a_{-\alpha}), & \alpha \in \Phi \\ h_i &= (H_i, 0), & r_i &= (0, H_i) \\ l_i &= (m_i, 0), & s_i &= (m_i, -m_i) & i = 1, \dots, n. \end{aligned}$$

Then it is clear that $\{x_\alpha, u_\alpha\}_{\alpha \in \Phi^+}$ (resp. $\{h_i, r_i\}_{1 \le i \le n}$) is a basis of $\mathfrak{l}_+$ (resp. $\mathfrak{l}_0$), whose dual basis is $\{y_\alpha, v_\alpha\}_{\alpha \in \Phi^+}$ (resp. $\{l_i, s_i\}_{1 \le i \le n}$).

Applying (1) and (2), we obtain the cobracket δ:

$$(3) \qquad \begin{aligned} \delta(x_{\pm\alpha_i}) &= \frac{1}{2} x_{\pm\alpha_i} \wedge r_i + \frac{1}{2} u_{\pm\alpha_i} \wedge (h_i - r_i) \\ \delta(u_{\pm\alpha_i}) &= \frac{1}{2} u_{\pm\alpha_i} \wedge r_i \\ \delta(h_i) &= \delta(r_i) = 0. \end{aligned}$$

(The formulas (3) are of course valid for a generalized Cartan matrix too.)

Example (d)(*continued*). The dual basis of B in P_2 is

$$x_i^* = (y_i, 0), \quad z^* = \frac{1}{2}(d, -d), \quad y_i^* = -(0, x_i), \quad d^* = \frac{1}{2}(z, -z).$$

Using (1) again, we obtain the formula for the cobracket δ:

$$(4) \qquad \delta(x_i) = \frac{1}{2} x_i \wedge z, \quad \delta(y_i) = \frac{1}{2} y_i \wedge z, \quad \delta(z) = \delta(d) = 0.$$

Remark 4. Let $\mathfrak{h}^n \subset \mathfrak{H}^n$ be the Heisenberg algebra, i.e. the subalgebra of $\mathfrak{H}^n$ generated by x_i, y_i, z. Then it is clear that the same cobracket gives to $\mathfrak{h}^n$ a structure of Lie bialgebra, i.e. $\mathfrak{h}^n$ is a subbialgebra of $\mathfrak{H}^n$. This also follows from the following general fact.

Proposition 2 (Semenov-Tian-Shansky [S]). *Let G be a Poisson-Lie group, $\mathfrak{g}$ its Lie algebra, H a Lie subgroup, $\mathfrak{h}$ the corresponding Lie subalgebra of $\mathfrak{g}$. Then H is a Poisson-Lie subgroup if and only if $\mathfrak{h}^\perp \subseteq \mathfrak{g}^*$ is an ideal. In such case, the Lie algebra structure on $\mathfrak{h}^*$ coincides with that of $\mathfrak{g}^*/\mathfrak{h}^\perp$.*

§2. The double and classical r-matrices. Let us recall some notions from [D2]. Let $\mathfrak{g}$ be a Lie algebra and $r = a_i \otimes b^i$ an element of $\mathfrak{g} \otimes \mathfrak{g}$; we say that r satisfies the classical Yang-Baxter equation (CYBE) if

$$(5) \qquad [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$$

Here, for instance $[r^{12}, r^{13}] := [a_i, a_j] \otimes b^i \otimes b^j$. A quasitriangular Lie bialgebra is a pair $(\mathfrak{g}, r)$, where $\mathfrak{g}$ is a Lie bialgebra, $r \in \mathfrak{g} \otimes \mathfrak{g}$, the coboundary of r is the cobracket of $\mathfrak{g}$, and r satisfies (5).

Let $(\mathfrak{g}, \delta)$ be a finite dimensional Lie bialgebra and let (P, P_1, P_2) be the corresponding Manin triple. The double of $\mathfrak{g}$, $D(\mathfrak{g})$, is the Lie bialgebra whose underlying Lie algebra is P and whose Lie cobracket is ∂r, where r is the image of the canonical element of $\mathfrak{g} \otimes \mathfrak{g}^*$ under the embedding $\mathfrak{g} \otimes \mathfrak{g}^* \hookrightarrow D(\mathfrak{g}) \otimes D(\mathfrak{g})$ (the canonical element is $e_i \otimes e^i$, where e_i is a basis of $\mathfrak{g}$ and e^i is the dual basis in $\mathfrak{g}^*$). Let (Q, Q_1, Q_2) be the Manin triple corresponding to the Lie bialgebra $D(\mathfrak{g})$ and identify Q_2 with P thanks to the bilinear form $(\,|\,)$; the Lie bracket in Q_2, denoted $[\,,\,]_*$, is

$$[u, v]_* = [v_1, u_1] + [u_2, v_2], \tag{6}$$

where u_i belongs to P_i, etc., and the bracket in the right hand side is that of P. Indeed, $\langle \delta(x) | u \otimes v \rangle = \sum_i (\langle [x, e_i] | u \rangle \langle e^i | v \rangle + \langle e_i | u \rangle \langle [x, e^i] | v \rangle) = \langle [x, \sum_i \langle e^i | v \rangle e_i] | u \rangle + \langle [x, \sum_i \langle e_i | u \rangle e^i] | v \rangle = \langle [x, v_1] | u \rangle + \langle [x, u_2] | v \rangle = \langle x | [v_1, u] \rangle + \langle x | [u_2, v] \rangle$. Note that in this way the double of a Manin triple makes sense even when the dimension of P_1 is not finite.

Example (g) (*continued*). The Manin triple obtained from a a Lie bialgebra $\mathfrak{g}$ with TD such that $\mathfrak{g}_0 = 0$ (Proposition 1) coincides with the double of the Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra with TD, and consider on $\mathfrak{g}$ the structure of Lie bialgebra provided by Proposition 1.

Lemma 2. (i) $\mathfrak{b}_+ = \mathfrak{g}_0 \oplus \mathfrak{g}_+$ *and* $\mathfrak{b}_- = \mathfrak{g}_0 \oplus \mathfrak{g}_-$ *are Lie subbialgebras of* $\mathfrak{g}$. *As Lie algebras,* $\mathfrak{b}_\pm^* \cong \mathfrak{b}_\mp$.
(ii) $D(\mathfrak{b}_+)$ *is isomorphic as a Lie algebra to the direct product* $\mathfrak{g} \times \mathfrak{g}_0$.

Proof. (i) The subspace orthogonal to $\mathfrak{b}_+$ (resp., $\mathfrak{b}_-$) in P_2 is $0 \times \mathfrak{n}_+$ (resp., $\mathfrak{n}_- \times 0$) which is clearly an ideal of P_2, and obviously $P_2/(0 \times \mathfrak{n}_+)$ (resp., $P_2/\mathfrak{n}_- \times 0)$) is isomorphic to $\mathfrak{b}_-$ (resp., $\mathfrak{b}_+$), as Lie algebras. Notice that the pairing $\langle\,,\,\rangle$ between $\mathfrak{b}_+$ and $\mathfrak{b}_-$ (with respect to the identification $\mathfrak{b}_+^* \cong \mathfrak{b}_-$) is

$$\langle x, y \rangle = k(x_0|y_0) + k(x|y). \tag{7}$$

(ii) Let $\Upsilon : D(\mathfrak{b}_+) \to \mathfrak{g} \times \mathfrak{g}_0$ be the linear isomorphism $\Upsilon(x_+ + x_0, y_0 + y_-) = (x_+ + x_0 + y_0 + y_-, x_0 - y_0)$. We want to show that $\Upsilon([x,y]) = [\Upsilon(x), \Upsilon(y)]$ and it suffices to consider $x \in \mathfrak{b}_+$, $y \in \mathfrak{b}_-$. Let us write $[x, y] = [x, y]_1 + [x, y]_2$, where $[x, y]_1 \in \mathfrak{b}_+$, $[x, y]_2 \in \mathfrak{b}_-$. We deduce easily from (7) that

$$[x, y]_1 = [x, y]_+ + \frac{1}{2}[x, y]_0, \quad [x, y]_2 = [x, y]_- + \frac{1}{2}[x, y]_0. \tag{8}$$

(Indeed, if $u \in \mathfrak{b}_-$, $\langle [x, y]_1, u \rangle = \langle x, [y, u] \rangle = k(x|[y, u]) = \langle [x, y]_+ + \frac{1}{2}[x, y]_0, u \rangle$.) Clearly, (8) implies our claim. $\square$

Our first basic important result is the following:

Theorem 1. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra with TD. Then the Lie cobracket on* $\mathfrak{g}$ *provided by Proposition 1 is* ∂r_0, *where, in the notation of Lemma 1,*

$$r_0 = \sum_{j \in J} x_j \otimes y_j + \frac{1}{2} \sum_{i,t \in I} k(l_t|l_i) h_i \otimes h_t. \tag{9}$$

That is, $\mathfrak{g}$ *is quasitriangular.*

Proof (cf. [D2]*).* Preserve the notation of the preceding proof. The orthogonal of the ideal $\Upsilon^{-1}(0 \times \mathfrak{g}_0)$ is $\{(u, v) \in D(\mathfrak{b}_+) : u_0 = v_0\}$, clearly a Lie subalgebra of the dual of $D(\mathfrak{b}_+)$. Then $D(\mathfrak{b}_+)/\Upsilon^{-1}(0 \times \mathfrak{g}_0) \simeq \mathfrak{g}$ inherits a Lie bialgebra structure and the canonical projection is a morphism of Lie bialgebras. We claim that it coincides with the defined in Proposition 1. Let $\{(u, v) \in D(\mathfrak{b}_+) : u_0 = v_0\} \to P_2$ be the application $(u, v) \mapsto (v, -u)$; it is easy to check that it is an isomorphism of Lie algebras. It is also straightforward to verify that the introduced isomorphisms preserve the corresponding dualities; the claim follows.

Let r be the canonical element of $D(\mathfrak{b}_+)$. An easy calculation shows that the image of r under the above projection is r_0; obviously, the later satisfies (5) because the former does. $\square$

Remark 5. Lemma 2 and Theorem 1 suggest the following method of constructing Lie algebras with TD. Let $\mathfrak{b}$ be a finite dimensional Lie bialgebra. Assume that

(a) there exists an abelian subalgebra $\mathfrak{h}$ such that, as vector spaces, $\mathfrak{b} = \mathfrak{h} \oplus [\mathfrak{b}, \mathfrak{b}]$;
(b) $\mathfrak{h}^\perp = [\mathfrak{b}^*, \mathfrak{b}^*]$; there exists an abelian subalgebra $\widetilde{\mathfrak{h}}$ such that, as vector spaces, $\mathfrak{b}^* = \widetilde{\mathfrak{h}} \oplus [\mathfrak{b}^*, \mathfrak{b}^*]$, and $\widetilde{\mathfrak{h}}^\perp = [\mathfrak{b}, \mathfrak{b}]$;
(c) for any $x \in \mathfrak{h}$, there exists a unique $\widetilde{x} \in \widetilde{\mathfrak{h}}$ such that $\operatorname{ad} \widetilde{x}$ coincides with the unique derivation T_x of $\mathfrak{b}^*$ satisfying $\langle [u, x], w \rangle = \langle u, T_x(w) \rangle$, for all $u \in \mathfrak{b}$, $w \in \mathfrak{b}^*$.

(To see that T_x is a derivation, proceed as follows. First, note that

$$[h, y]_1 = 0, \quad \forall h \in \mathfrak{h},\ y \in \mathfrak{b}^*, \tag{10}$$

and similarly,

$$[\widetilde{h}, x]_2 = 0, \quad \forall \widetilde{h} \in \widetilde{\mathfrak{h}},\ ,x \in \mathfrak{b}. \tag{11}$$

Then, using the cocycle condition on the cobracket of $\mathfrak{b}^*$, we obtain

$$\langle [u, x], [w, t] \rangle = \langle u, [w, T_x(t)] + [T_x(w), t] + [[x, w]_1, t] - [[x, t]_1, w] \rangle,$$

for all u, $x \in \mathfrak{b}$, w, $t \in \mathfrak{b}^*$, which proves that T_x is a derivation when $x \in \mathfrak{h}$.)

Now there exists an isomorphism $h \mapsto \widehat{h}$ from $\mathfrak{h}$ onto $\widetilde{\mathfrak{h}}$ such that $\langle x, \widehat{h} \rangle = \langle \widetilde{x}, h \rangle$. Let $\mathfrak{r} = \{(x, -\widetilde{x}) : x \in \mathfrak{h}\}$, which is an ideal of $D(\mathfrak{b})$, by (c) and (11). Let $\mathfrak{g}_+ = [\mathfrak{b}, \mathfrak{b}]$, $\mathfrak{g}_- = [\mathfrak{b}^*, \mathfrak{b}^*]$, $\mathfrak{g}_0 = \{(h, \widehat{h}) : h \in \mathfrak{h}\}$. Then

$$\mathfrak{r}^\perp = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+.$$

We claim now that $\mathfrak{g} := \mathfrak{r}^{\perp}$ is a subalgebra of $D(\mathfrak{b})$. Let $u \in \mathfrak{b}$, $w \in \mathfrak{b}^*$, and write $[u,w]_1 = [u,w]_0 + [u,w]_+$, where $[u,w]_0 \in \mathfrak{h}$, $[u,w]_+ \in \mathfrak{g}_+$, $[u,w]_2 = [u,w]_\Delta + [u,w]_-$, where $[u,w]_\Delta \in \tilde{\mathfrak{h}}$, $[u,w]_- \in \mathfrak{g}_-$. Then we need to prove that

$$\widehat{[u,w]_0} = [u,w]_\Delta.$$

But if $z \in \mathfrak{h}$, $\langle z, \widehat{[u,w]_0}\rangle = \langle \tilde{z}, [u,w]_0\rangle = \langle \tilde{z}, [u,w]\rangle = \langle [\tilde{z},u], w\rangle = \langle u, [w,z]\rangle = \langle u, [w,z]_\Delta\rangle$. As $\mathfrak{g}$ inherits a non-degenerate invariant bilinear form from $D(\mathfrak{b})$, we deduce that it has a TD.

Example (b) (*continued*). Let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(A)$ be the Lie algebra defined in [K, §1.2]. We preserve the notation $\mathfrak{h}$, $\tilde{\mathfrak{n}}^{\pm}$, etc. from *loc cit*, but we denote X_i^+, X_i^-, H_i instead of $e_i, f_i, \alpha_i^\vee$. Let $D = (d_1, \dots d_n)$ be an invertible diagonal matrix such that $DA = A^tD$ and let $h_i = d_iH_i$. Let $\mathfrak{r}$ be the unique maximal ideal among the ideals intersecting in trivially; then $\mathfrak{g} \simeq \tilde{\mathfrak{g}}/\mathfrak{r}$. It is known that the triangular decomposition of $\mathfrak{g}$ gives rise to a Lie bialgebra structure [D2]; indeed, the cobracket is given by

$$\delta(h_i) = 0 \quad \text{and} \quad \delta(X_i^{\pm}) = \frac{1}{2}(X_i^{\pm} \otimes h_i - h_i \otimes X_i^{\pm}), \tag{12}$$

Where we still denote by $X_i^{\pm}, h_i$, their images in $\mathfrak{g}$. Now we claim that (12) extends to a Lie bialgebra structure on $\tilde{\mathfrak{g}}$. This follows from the following general fact.

Lemma 3. (a) *Let I be a set and L the free Lie algebra generated by I. Then any function $\phi : I \to \Lambda^2 L$ extends to a 1-cocycle $L \to \Lambda^2 L$.*

(b) *Let g be a Lie algebra, $r \subset g$ an ideal, $\delta : g \to \Lambda^2 g$ a 1-cocycle. Then δ induces a 1-cocycle $\delta' : g/r \to \Lambda^2 g/r$ if and only if $\delta(r) \subseteq g \otimes r + r \otimes g$, if and only if $\delta(B) \subseteq g \otimes r + r \otimes g$, for some system of generators B of the ideal r.*

Proof. (a) Apply [B, Ch. II §2 Prop 8] to $M = \Lambda^2 L(I)$. (b) is obvious. □

Now let L be the free Lie algebra generated by $\mathfrak{h}$, $X_i^{\pm}$, R the ideal generated by $[H,H']$, $H,H' \in \mathfrak{h}$, $[H, X_i^{\pm}] \pm \alpha_i(H)X_i^{\pm}$, $[X_i^+, X_j^-] - \delta_{ij}H_i$, $\delta : L \to \Lambda^2 L$ the cocycle defined by (12). Then

$$\begin{aligned}
\delta([H, X_i^{\pm}] \mp \alpha_i(H)X_i^{\pm}) &= ([H, X_i^{\pm}] \mp \alpha_i(H)X_i^{\pm}) \otimes h_i + X_i^{\pm} \otimes [H, h_i] - \\
&\quad [H,h_i] \otimes X_i^{\pm} - h_i \otimes ([H, X_i^{\pm}] \mp \alpha_i(H)X_i^{\pm}); \\
\delta([X_i^+, X_j^-] - \delta_{ij}H_i) &= ([X_i^+, X_j^-] - \delta_{ij}H_i) \otimes (h_j + h_i) \\
&\quad - (h_i + h_j) \otimes ([X_i^+, X_j^-] - \delta_{ij}H_i) \\
&\quad + X_j^- \wedge ([X_i^+, h_j] + \alpha_i(h_j)X_i^+) - X_j^- \wedge \alpha_i(h_j)X_i^+ \\
&\quad + ([X_j^-, h_i] - \alpha_j(h_i)X_j^-) \wedge X_i^+ + \alpha_i(h_j)X_j^- \wedge X_i^+.
\end{aligned}$$

This shows that $\delta(R) \subset L \otimes R + R \otimes L$ and hence (12) defines a 1-cocycle on $\tilde{\mathfrak{g}}$. The co-Jacobi identity is also easy to check; indeed it suffices to verify it on generators, because of the formula

$$\begin{aligned}
(\delta \otimes id)\delta([x,y]) &= \operatorname{ad} x(\delta \otimes id)\delta(y) - \operatorname{ad} y(\delta \otimes id)\delta(x) + [y_i, x_j] \otimes x^j \otimes y^i \\
&\quad + x_j \otimes [y_i, x^j] \otimes y^i - [x_j, y_i] \otimes y^i \otimes x^j - y_i \otimes [x_j, y^i] \otimes x^j,
\end{aligned} \tag{13}$$

where $\delta(x) = x_i \otimes x^i = -x^i \otimes x_i$, etc. An elementary computation shows that (12) provides L (and *a fortiori* $\tilde{\mathfrak{g}}$) with a Lie bialgebra structure.

Example (c) (*continued*). We take h_i, r_i a basis of $\mathfrak{l}_0$, and the dual basis $s_i, t_i - s_i$. We have also that $m_i = \sum_t (m_t|m_i) l_t$, then $s_i = \sum_t (m_t|m_i) h_i$, and $t_i - s_i = \sum_t (m_t|m_i)(h_i - r_i)$. Then

$$r_0 = \sum_{\alpha \in \Phi^+} (a_\alpha, 0) \otimes (0, a_{-\alpha}) + (0, a_\alpha) \otimes (a_{-\alpha}, -a_{-\alpha}) + \frac{1}{2}\sum_{i,t} (m_t|m_i)(l_i, 0) \otimes (l_t, 0) + (0, l_i) \otimes (0, l_t). \tag{14}$$

Example (d) (*continued*). Here we choose the basis $\{x_i, \frac{1}{\sqrt{2}}(d+z), \frac{1}{\sqrt{-2}}(d-z), y_i\}$. Then the dual basis is $\{y_i, \frac{1}{\sqrt{2}}(d+z), \frac{1}{\sqrt{-2}}(d-z), x_i\}$. So

$$r_0 = \sum_i x_i \otimes y_i + \frac{1}{2} d \otimes z + \frac{1}{2} z \otimes d. \tag{15}$$

Note that (14) and (15) are new examples of classical r-matrices.

§3. Quantizations.

We recall that a *quantization* of a Lie bialgebra $(\mathfrak{g}, \delta)$ is a topological Hopf algebra A over $\mathbb{C}[[\hbar]]$ such that $A/\hbar A \simeq \mathcal{U}\mathfrak{g}$ (as Hopf algebras), A is topologically free over $\mathbb{C}[[\hbar]]$ and satisfies

$$\delta(\overline{a}) = \frac{\Delta(a) - \Delta'(a)}{\hbar} \mod \hbar$$

where Δ denotes the coproduct in A, $\Delta' = \tau \circ \Delta$, $\tau(x \otimes y) = (y \otimes x)$.

Let $q = exp(\hbar/4)$. Given $u \in \mathbb{N}_0$, we denote (as usual)

$$[u]_q^! = \prod_{1 \le t \le u} \frac{q^t - q^{-t}}{q - q^{-1}}.$$

Example (b) (*continued*). The quantized universal enveloping algebra of $\tilde{\mathfrak{g}}$ is the $\mathbb{C}[[\hbar]]$-algebra $\tilde{\mathcal{U}}_\hbar$ generated (in the $\hbar$-adic sense) by $\mathfrak{h}$ and $X_i^{\pm}$ with the relations:

$$[H, H'] = 0 \quad \forall\, H, H' \in \mathfrak{h} \qquad [H, X_i^{\pm}] = \pm\alpha_i(H)X_i^{\pm} \quad \forall\, H \in \mathfrak{h},$$

and also

$$[X_i^+, X_i^-] = 2\delta_{ij}(d_i\hbar)^{-1}\mathrm{sh}(\hbar h_i/2).$$

It can be shown that $\tilde{\mathcal{U}}_\hbar$ is a topologically free $\mathbb{C}[[\hbar]]$-module and that there exists a homomorphism $\tilde{\Delta} : \tilde{\mathcal{U}}_\hbar \to \tilde{\mathcal{U}}_\hbar \otimes \tilde{\mathcal{U}}_\hbar$ such that

$$\tilde{\Delta}(X_i^{\pm}) = X_i^{\pm} \otimes q^{h_i} + q^{-h_i} \otimes X_i^{\pm},$$

$\tilde{\Delta}(H) = H \otimes 1 + 1 \otimes H$ for $H \in \mathfrak{h}$. $(\tilde{\mathcal{U}}_\hbar, \Delta)$ is a quantization of $(\tilde{\mathfrak{g}}, \delta)$. Let $\tilde{\mathcal{U}}_\hbar^{\pm}$ (resp., $\tilde{\mathcal{U}}_\hbar^0$) be the subalgebra of $\tilde{\mathcal{U}}_\hbar$ generated by $X_i^{\pm}$ (resp., $\mathfrak{h}$). The Diamond Lemma [Be] implies an isomorphism $\tilde{\mathcal{U}}_\hbar \simeq \tilde{\mathcal{U}}_\hbar^+ \otimes \tilde{\mathcal{U}}_\hbar^0 \otimes \tilde{\mathcal{U}}_\hbar^-$ provided by the multiplication.

Example (c) (*continued*). Let $\mathcal{U}_\hbar\tilde{\mathfrak{l}}$ be the associative algebra over $\mathbb{C}[[\hbar]]$ generated by $\mathfrak{l}_0$, $x_{\pm\alpha_i}$, $u_{\pm\alpha_i}$, ($\alpha_i \in \Pi$), with the following relations:

$$
\begin{aligned}
[a_1, a_2] &= 0 \quad \forall a_1, a_2 \in \mathfrak{l}_0\\
[u_{\pm\alpha_i}, u_{\alpha_j}] &= 0 \quad \forall i,j\\
[(a,b), x_{\pm\alpha_i}] &= \pm\alpha_i(a)x_{\pm\alpha_i} \pm \alpha_i(b)u_{\pm\alpha_i} \quad \forall a,b \in \mathfrak{g}_0\\
[(a,b), u_{\pm\alpha_i}] &= \pm\alpha_i(a)u_{\pm\alpha_i} \quad \forall a,b \in \mathfrak{g}_0\\
[x_{\alpha_i}, x_{-\alpha_j}] &= 2\delta_{ij}(d_i\hbar)^{-1}\,\mathrm{sh}(\hbar r_i/2) + \delta_{ij}d_i^{-1}\,\mathrm{ch}(\hbar r_i/2)(h_i - r_i)\\
[x_{\alpha_i}, u_{-\alpha_j}] &= [u_{\alpha_i}, x_{-\alpha_j}] = 2\delta_{ij}(d_i\hbar)^{-1}\,\mathrm{sh}(\hbar r_i/2)
\end{aligned}
$$

Let $\tilde{\mathfrak{l}}$ be the Lie bialgebra constructed as $\mathfrak{l}$ but with $\tilde{\mathfrak{g}}$ in the place of $\mathfrak{g}$ (use Lemma 3 instead of Proposition 1). In particular, $\tilde{\mathfrak{l}} = \mathfrak{l}$ if $A = (2)$.

Proposition 3. *$\mathcal{U}_\hbar\tilde{\mathfrak{l}}$ is a topological Hopf algebra, whose coproduct, antipode and counit are given by*

$$
\begin{aligned}
\Delta(a) &= a \otimes 1 + 1 \otimes a, \qquad \text{for } a \in \tilde{\mathfrak{l}}_0\\
\Delta(x_{\pm\alpha_i}) &= x_{\pm\alpha_i} \otimes q^{2r_i} + q^{-2r_i} \otimes x_{\pm\alpha_i} +\\
&\qquad \frac{\hbar}{4}u_{\pm\alpha_i} \otimes q^{2r_i}(h_i - r_i) - \frac{\hbar}{4}q^{-2r_i}(h_i - r_i) \otimes u_{\pm\alpha_i},\\
\Delta(u_{\pm\alpha_i}) &= u_{\pm\alpha_i} \otimes q^{2r_i} + q^{-2r_i} \otimes u_{\pm\alpha_i},
\end{aligned}
$$

$S(x) = -x$ and $\epsilon(x) = 0$ for all x generator and $\epsilon(1) = 1$. Moreover, $\mathcal{U}_\hbar\tilde{\mathfrak{l}}$ is a quantization of the "motion" Lie algebra $\tilde{\mathfrak{l}}$.

The proof is cumbersome but straightforward. We were not able to find the analogues of the Serre relations.

Example (d) (*continued*). Let $\mathcal{U}_\hbar\mathfrak{H}^n$ be the algebra over $\mathbb{C}[[\hbar]]$ generated in the $\hbar$-adic sense by X_i, Y_i, Z, D, with the following relations

$$
\begin{gathered}
[D, X_i] = X_i, \qquad [D, Y_i] = -Y_i,\\
[X_i, X_j] = 0 = [Y_i, Y_j]\\
[X_i, Y_j] = 2\delta_{ij}\hbar^{-1}\,\mathrm{sh}(\hbar Z/2), \qquad \text{and}\\
Z \text{ is central.}
\end{gathered}
$$

Proposition 4. *$\mathcal{U}_\hbar\mathfrak{H}^n$ is a quantization of the extended Heisenberg Lie algebra. The Hopf algebra structure is as follows: the coproduct is given by*

$$
\begin{aligned}
\Delta(X_i) &= X_i \otimes q^Z + q^{-Z} \otimes X_i \qquad \Delta(Y_i) = Y_i \otimes q^Z + q^{-Z} \otimes Y_i\\
\Delta(Z) &= Z \otimes 1 + 1 \otimes Z, \qquad \Delta(D) = D \otimes 1 + 1 \otimes D;
\end{aligned}
$$

the counit (resp., the antipode) is zero (resp., minus the identity) on the generators X_i, Y_i, Z, D.

Note that the subalgebra of $\mathcal{U}_\hbar\mathfrak{H}^n$ generated (in the $\hbar$-adic sense) by X_i, Y_i, Z is in fact a Hopf subalgebra; it is a quantization of $\mathfrak{h}^n$, cf. Remark 4.

Remark 6. Another interesting approach to the quantum Heisenberg algebra (different from ours) was given in [GF], cf. also [R]. The motivation there is noncommutative algebra (geometry) and q-analysis.

§4. The algebra of rational functions on the quantum Heisenberg group.

In this section we describe the algebra of rational functions on the quantum Heisenberg group. We shall follow the approach of [A]. Let $(A, m, 1, \delta, \epsilon, S)$ be a Hopf algebra over $\mathbb{C}$ and let (ρ, V) be a finite dimensional representation of A. We denote by $T(V \otimes V^*)$ the tensor algebra of $V \otimes V^*$. Let $\rho^d = (\rho.S)^t$ be the dual representation of ρ. Let $\phi_\rho : T(V \otimes V^*) \to A^*$ be the linear application induced by $< \phi_\rho(v \otimes \mu), x >=< \mu, \rho(x)v >$. The $\phi_\rho(v \otimes \mu)$'s are usually called the matrix coefficients. Assume that ρS^2 is isomorphic to ρ. Then the subalgebra of A^* spanned by $\phi_\rho(T(V \otimes V^*))$ and $\phi_{\rho^d}(T(V^* \otimes V))$ (i.e. the image of $\phi_\rho \oplus \phi_{\rho d} : T(V \otimes V^* \oplus V^* \otimes V) \to A^*$) is actually a Hopf algebra dual to A; we shall denote it by Coeff(ρ).

A $*$-Hopf algebra is a Hopf algebra provided with a star operation for the algebra structure, such that $\delta(v^*) = \delta(v)^*$, $S(S(v^*)^*) = v$. Assume further that A is a $*$-Hopf algebra and let $T : A \to A$, $T(a) = S(a)^*$ (note that T is an antilinear multiplicative antimultiplicative involution which satisfies $(TS)^2 = id$; conversely, any such T gives rise to a $*$-Hopf algebra structure on A). Assume that there exists an antilinear isomorphism $J : V \to V^*$ such that $J(xv) = T(x)J(v)$, $x \in A$, $v \in V$. Then Coeff(ρ) is a $*$-Hopf algebra, where the star is defined by $< \alpha^*, x >= \overline{< \alpha, T(x) >}$, ($\alpha \in$ Coeff(ρ), $x \in A$).

Now we turn our attention to $A = \mathcal{U}_\hbar\mathfrak{H}^n$. Let V be a $n+2$-dimensional complex vector space, fix a basis of $\{e_1, \ldots, e_{n+2}\}$ of V, and let $\{\mu_1, \ldots, \mu_{n+2}\}$ be its the dual basis. It is easy to see that the assignment

$$
\begin{aligned}
\rho(X_i).e_j &= \delta_{j,i+1}e_1, \qquad &\rho(Y_i).e_j &= \delta_{j,n+2}e_{i+1},\\
\rho(Z).e_j &= \delta_{j,n+2}e_1, \qquad &\rho(D).e_j &= \delta_{j,1}e_1 + \delta_{j,n+2}e_{n+2},
\end{aligned}
$$

extends to a representation ρ $U_\hbar\mathfrak{H}^n$.

(Note that if $\hbar = 0$ this is the fundamental representation of $\mathfrak{H}^n$, see §5.). Let $T : \mathcal{U}_\hbar\mathfrak{H}^n \to \mathcal{U}_\hbar\mathfrak{H}^n$ be the unique multiplicative antilinear involution such that

$$
T(X_i) = Y_i, \quad T(Y_i) = X_i, \quad T(Z) = -Z, \quad T(D) = -D;
$$

clearly, T is anticomultiplicative and $(TS)^2 = id$. Let $J : V \to V^*$ be the antilinear isomorphism given by

$$
J(e_1) = -\mu_{n+2}, \quad J(e_i) = \mu_i \ (1 < i < n+2), \quad J(e_{n+2}) = -\mu_1.
$$

From the above considerations, it follows that Coeff(ρ) is a $*$-Hopf algebra, generated by $\mu_{ij} = \phi_\rho(e_j \otimes \mu_i)$ and $\gamma_{ij} = \phi_{\rho^d}(\mu_i \otimes e_j)$. We want to give an explicit presentation of Coeff(ρ). Let B be an associative algebra and let $u_1, \ldots, u_n \in B$. If $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$, set $x^i = x_1^{i_1} \ldots x_n^{i_n}$, $|i| = i_1 + \ldots + i_n$, $i! = i_1! \ldots i_n!$ and $\binom{i}{j} = \binom{i_1}{j_1} \cdots \binom{i_n}{j_n}$. The next Lemma will be useful later.

Lemma 4. *Assume that*

(a) $\{u^i : i \in \mathbb{N}^n, n \geq 0\}$ *is a basis of* B.
(b) $u_k u_\ell - u_\ell u_k = \sum_{1\leq j\leq n} a^j_{k\ell} u_j$, *if* $1 \leq \ell < k \leq n$.

Then B *is isomorphic to the quotient of the free algebra* L *in generators* $U_1, \ldots, U_n$ *by the ideal* R *generated by* $U_k U_\ell - U_\ell U_k = \sum_{1\leq j\leq n} a^j_{k\ell} U_j$, $1 \leq \ell < k \leq n$.

Proof. The application $L/R \to B$ which sends $U_j + R \mapsto u_j$ sends a system of generators to a basis, hence it is an isomorphism. $\square$

Let $i, j \in \mathbb{N}^n$ and $m, l \in \mathbb{N}$. If $u = X^i Y^j D^m Z^l \neq 1 \in \mathcal{U}_h\mathfrak{H}^n$ and $2 \leq s, t \leq n+1$, then

$$ue_1 = \begin{cases} e_1 & \text{if } u = D^m \\ 0 & \text{otherwise}, \end{cases}$$

$$ue_s = \begin{cases} e_1 & \text{if } u = X_{s-1} \\ 0 & \text{otherwise}, \end{cases}$$

$$ue_{n+2} = \begin{cases} e_{n+2} & \text{if } u = D^m \\ e_s & \text{if } u = Y_{s-1}D^m \\ e_1 & \text{if } u = D^m Z, \text{ or } X_s Y_s D^m \\ 0 & \text{otherwise}. \end{cases}$$

Now $S(u) = (-1)^{l+m+|i|+|j|} Z^l D^m Y^j X^i$; hence

$$S(u)e_1 = \begin{cases} (-1)^m e_1 & \text{if } u = D^m \\ 0 & \text{otherwise}, \end{cases}$$

$$S(u)e_s = \begin{cases} (-1)^{m+1} e_1 & \text{if } u = X_{s-1}D^m \\ 0 & \text{otherwise}, \end{cases}$$

$$S(u)e_{n+2} = \begin{cases} -e_s & \text{if } u = Y_{s-1} \\ (-1)^{m+1} e_1 & \text{if } u = D^m Z \\ (-1)^m e_{n+2} & \text{if } u = D^m \\ 0 & \text{otherwise}. \end{cases}$$

Thus, we get

$$\mu_{1k}(u) = \begin{cases} 1 & \text{if } u = D^m \text{ and } k = 1, \text{ or } u = X_{s-1} \text{ and } k = s, \\ & \quad \text{or } u = D^m Z,\ X_s Y_s D^m \text{ and } k = n+2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{st}(u) = 0$$

$$\mu_{k,n+2}(u) = \begin{cases} 1 & \text{if } u = Y_{s-1}D^m \text{ and } k = s, \text{ or } u = D^m \text{and } k = n+2 \\ 0 & \text{otherwise}, \end{cases}$$

$$\gamma_{1k}(u) = \begin{cases} (-1)^m & \text{if } u = D^m, \text{ and } k = 1 \\ (-1)^{m+1} & \text{if } u = X_{s-1}D^m, \text{ and } k = s, \\ & \text{or } u = D^m Z \text{ and } k = n+2 \\ 0 & \text{otherwise}, \end{cases}$$

$$\gamma_{s,n+2}(u) = \begin{cases} -1 & \text{if } u = Y_{s-1} \\ 0 & \text{otherwise}, \end{cases}$$

Also, $\gamma_{s,t} = \mu_{s,t}$, $\gamma_{n+2,n+2} = \gamma_{11}$, and $\mu_{s,1} = \mu_{n+2,1} = \mu_{n+2,s} = \gamma_{s,1} = \gamma_{n+2,1} = \gamma_{n+2,s} = 0$.

We want to find the structure of Coeff(ρ). Now, if $n_1, n_2 \in \mathbb{N}^n$ and $n_3, n_4 \in \mathbb{N}$:

$$(16)\quad \Delta(X^{n_1}Y^{n_2}D^{n_3}Z^{n_4}) = \sum_{i_k+j_k=n_k} \binom{n_1}{i_1}\binom{n_2}{i_2}\binom{n_3}{i_3}\binom{n_4}{i_4} X^{i_1}Y^{i_2}D^{i_3}Z^{i_4}q^{-(|j_1|+|j_2|)Z} \otimes X^{j_1}Y^{j_2}Z^{j_3}D^{j_4}q^{(|i_1|+|i_2|)Z}.$$

If $\mu, \mu' \in (\mathcal{U}_h\mathfrak{H}^n)^*$ and $u \in \mathcal{U}_h\mathfrak{H}^n$, $(\mu\mu')(u) = (\mu \otimes \mu')(\delta(u))$, hence

$$\mu_{11}\gamma_{1,n+2} = \begin{cases} -1 & \text{if } u = Z \\ 0 & \text{otherwise}; \end{cases}$$

furthermore $\gamma_{11}\mu_{11} = \mu_{11}\gamma_{11} = 1$. We denote $\mathcal{X}_s = \mu_{1,s+1}, \mathcal{Y}_s = -\alpha_{s,n+2}$, $\mathcal{Z} = -\mu_{11}\gamma_{1,n+2}$, $\mathcal{D} = \mu_{11}$; clearly $\mathcal{D}^{-1} = \gamma_{11}$. If $t, l \in \mathbb{N}^n$, $k \in \mathbb{Z}$ and $r \in \mathbb{N}$:

$$(17)\quad \mathcal{X}^t\mathcal{Y}^l\mathcal{D}^k\mathcal{Z}^r(u) = \begin{cases} \dfrac{t!\,l!\,k^n\,r!}{(r-m)!}(|l|+|t|)^{r-m}(\dfrac{\hbar}{4})^{r-m} & \text{if } u = X^tY^lD^nZ^m,\ (m \in \mathbb{N}, 0 \leq m \leq r) \\ 0 & \text{otherwise} \end{cases}.$$

Then $\mu_{1,n+2} = \mathcal{D}(\mathcal{Z} + \sum_{i=1}^n \mathcal{X}_i\mathcal{Y}_i)$, $\mu_{s,n+2} = \mathcal{Y}_{s-1}\mathcal{D}$, $\gamma_{1,s} = -\mathcal{X}_{s-1}\mathcal{D}^{-1}$ and $\gamma_{1,n+2} = -\mathcal{D}^{-1}\mathcal{Z}$. So $\mathcal{X}_i, \mathcal{Y}_i, \mathcal{D}, \mathcal{D}^{-1}, \mathcal{Z}$ $(i = 1, \ldots, n)$ generate Coeff(ρ) as algebra. Now we shall see the defining relations. We have that $\mathcal{D}$ and $\mathcal{D}^{-1}$ are central, $[\mathcal{X}_i, \mathcal{X}_j] = [\mathcal{X}_i, \mathcal{Y}_j] = [\mathcal{Y}_i, \mathcal{Y}_j] = 0$ $(i, j = 1, \ldots, n)$, but

$$\mathcal{X}_i\mathcal{Z}(u) = \begin{cases} \frac{\hbar}{4} & \text{if } u = X_i \\ 1 & \text{if } u = X_iZ \\ 0 & \text{otherwise} \end{cases}, \qquad \mathcal{Z}\mathcal{X}_i(u) = \begin{cases} -\frac{\hbar}{4} & \text{if } u = X_i \\ 1 & \text{if } u = X_iZ \\ 0 & \text{otherwise} \end{cases};$$

that is $[\mathcal{X}_i, \mathcal{Z}] = \frac{\hbar}{2}\mathcal{X}_i$. Similarly, $[\mathcal{Y}_i, \mathcal{Z}] = \frac{\hbar}{2}\mathcal{Y}_i$.

Lemma 5. *Let* Coeff(ρ) *and* $\mathcal{X}_i, \mathcal{Y}_i, \mathcal{D}, \mathcal{D}^{-1}, \mathcal{Z}$, $(i = 1, \dots, n)$ *be as above. Then* $\mathcal{B} = \{\mathcal{X}^s\mathcal{Y}^l\mathcal{D}^k\mathcal{Z}^r : s, l \in \mathbb{N}^n, k \in \mathbb{Z}, r \in \mathbb{N}\}$ *is a basis of* Coeff(ρ).

Proof. As $\mathcal{X}_i, \mathcal{Y}_i, \mathcal{D}, \mathcal{D}^{-1}, \mathcal{Z}$ generate Coeff(ρ) as algebra ; one deduces from the above relations that the span of $\mathcal{B}$ is Coeff(ρ).

Let $s, l \in \mathbb{N}^n$, $r \in \mathbb{N}$, and $U = \sum_k a_k\mathcal{X}^s\mathcal{Y}^l\mathcal{D}^k\mathcal{Z}^r$. From (17) we have $U(X^sY^lD^nZ^r) = c\sum_k a_k k^n$ for all $n \in \mathbb{N}$, where c is a constant not equal to 0. If $U(X^sY^lD^nZ^r) = 0$ for all n, then $a_k = 0$ for all $k \in \mathbb{Z}$, so $U = 0$; hence $\{\mathcal{X}^s\mathcal{Y}^l\mathcal{D}^k\mathcal{Z}^r\}_{k\in\mathbb{Z}}$ are linearly independent.

Let $W_{slr} = \text{span}\{\mathcal{X}^s\mathcal{Y}^l\mathcal{D}^k\mathcal{Z}^r : k \in \mathbb{Z}\}$. Let $u_{twv} \in W_{twv}$ and suppose that $\sum u_{twv} = 0$. But then $\sum u_{twv}(\mathcal{X}^t\mathcal{Y}^l\mathcal{D}^n\mathcal{Z}^r) = 0$ and hence $\sum_{m\le r} u_{twm}(\mathcal{X}^t\mathcal{Y}^l\mathcal{D}^n\mathcal{Z}^r) = 0$. In particular, $u_{two}(\mathcal{X}^t\mathcal{Y}^l\mathcal{D}^n\mathcal{Z}^0) = 0$ and by the preceding, $u_{two} = 0$. By an inductive procedure, we deduce that $u_{t,w,r} = 0$ for all t, w, r; this implies the Lemma. □

Remark 7. To obtain the coproduct and the antipode in Coeff(ρ) it is useful to know the product in $\mathcal{U}_\hbar\mathfrak{H}^n$. If $t, k, l, p \in \mathbb{N}^n$ and $m, r, v, s \in \mathbb{N}$,

(18) $(X^tY^lD^mZ^v).(X^kY^pD^rZ^s) =$
$$\sum_{i=1}^{n}\sum_{j_i=0}^{\min\{l_i,k_i\}} (-1)^{|j|}\binom{l}{j}\binom{k}{j} j!X^{t+k-j}Y^{l+p-j}(D+|k|-|p|)^m D^r Z^{v+s+|j|}$$

From the preceding Lemmas and general facts in Hopf algebra theory, we obtain:

Theorem 2. Coeff(ρ) *is the Hopf algebra generated by* $\mathcal{X}_i, \mathcal{Y}_i, \mathcal{D}, \mathcal{D}^{-1}, \mathcal{Z}$ $(i = 1, \dots, n)$, *with the following relations:*

$$[\mathcal{X}_i, \mathcal{X}_j] = [\mathcal{X}_i, \mathcal{Y}_j] = [\mathcal{Y}_i, \mathcal{Y}_j] = 0, \quad [\mathcal{X}_i, \mathcal{Z}] = \frac{\hbar}{2}\mathcal{X}_i, \quad [\mathcal{Y}_i, \mathcal{Z}] = \frac{\hbar}{2}\mathcal{Y}_i, \quad \mathcal{D}\mathcal{D}^{-1} = 1,$$

$(i, j = 1, \dots, n)$ *and* $\mathcal{D}, \mathcal{D}^{-1}$ *are central elements. The comultiplication* Δ, *the antipode* S *and the counit* ϵ *are defined by:*

$$\Delta(\mathcal{X}_i) = \mathcal{D} \otimes \mathcal{X}_i + \mathcal{X}_i \otimes 1, \qquad \Delta(\mathcal{Y}_i) = \mathcal{D}^{-1} \otimes \mathcal{Y}_i + \mathcal{Y}_i \otimes 1,$$
$$\Delta(\mathcal{D}) = \mathcal{D} \otimes \mathcal{D}, \qquad \Delta(\mathcal{Z}) = -\sum_{i=1} \mathcal{Y}_i\mathcal{D} \otimes \mathcal{X}_i + \mathcal{Z} \otimes 1 + 1 \otimes \mathcal{Z},$$
$$S(\mathcal{X}_i) = -\mathcal{X}_i\mathcal{D}^{-1}, \quad S(\mathcal{Y}_i) = -\mathcal{Y}_i\mathcal{D} \qquad S(\mathcal{Z}) = -\mathcal{Z}, \quad S(\mathcal{D}) = \mathcal{D}^{-1},$$

and $\epsilon(u) = 0$ *for all* u *generator. Moreover,* Coeff(ρ) *is a* $*$*-Hopf algebra, where the* $*$ *is given by* $\mathcal{X}_i^* = \mathcal{Y}_i$, $\mathcal{Y}_i^* = \mathcal{X}_i$, $\mathcal{Z}^* = -\mathcal{Z}$ *and* $\mathcal{D}^* = \mathcal{D}^{-1}$.

Remark 8. Let (ρ', V) be the representation of $\mathcal{U}_\hbar\mathfrak{h}^n$ which is the restriction of ρ. In the same way as above we can obtain Coeff(ρ') as the $*$-Hopf algebra generated by $\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}$ $(i = 1, \dots, n)$, with the following relations:

$$[\mathcal{X}_i, \mathcal{X}_j] = [\mathcal{X}_i, \mathcal{Y}_j] = [\mathcal{Y}_i, \mathcal{Y}_j] = 0, \quad [\mathcal{X}_i, \mathcal{Z}] = \frac{\hbar}{2}\mathcal{X}_i, \quad [\mathcal{Y}_i, \mathcal{Z}] = \frac{\hbar}{2}\mathcal{Y}_i,$$

$(i, j = 1, \dots, n)$. The comultiplication Δ, the star, the antipode S and the counit ϵ are defined by:

$$\Delta(\mathcal{X}_i) = 1 \otimes \mathcal{X}_i + \mathcal{X}_i \otimes 1, \qquad \Delta(\mathcal{Y}_i) = 1 \otimes \mathcal{Y}_i + \mathcal{Y}_i \otimes 1,$$
$$\Delta(\mathcal{Z}) = -\sum_{i=1}^{n} \mathcal{Y}_i \otimes \mathcal{X}_i + \mathcal{Z} \otimes 1 + 1 \otimes \mathcal{Z},$$
$$S(\mathcal{X}_i) = -\mathcal{X}_i\mathcal{D}^{-1}, \quad S(\mathcal{Y}_i) = -\mathcal{Y}_i\mathcal{D} \qquad S(\mathcal{Z}) = -\mathcal{Z},$$

and $\epsilon(u) = 0$ for all u generator. Moreover, Coeff(ρ') is a $*$-Hopf algebra, where the $*$ is given by $\mathcal{X}_i^* = \mathcal{Y}_i$, $\mathcal{Y}_i^* = \mathcal{X}_i$ and $\mathcal{Z}^* = -\mathcal{Z}$.

§5. Symplectic leaves in the Heisenberg group. Throughout this section, we shall work on the field of real numbers $\mathbf{R}$. Our objective in this section is to compute the symplectic leaves of the Poisson structure on the real Heisenberg group. Let $\mathfrak{p}_1$ be a (finite dimensional) Lie bialgebra, $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ the corresponding Manin triple. Let P be the connected simply connected Lie group with Lie algebra $\mathfrak{p}$ and let P_1 (resp., P_2) be the connected subgroup of P with Lie algebra $\mathfrak{p}_1$ (resp., $\mathfrak{p}_2$). Suppose that the multiplication induces a diffeomorphism $P \simeq P_1 \times P_2$; then the symplectic leaves in P_1 are the orbits of the so-called dressing action of P_2 on P_1 and this action can be computed as follows. Let $x_1 \in P_1$, $x_2 \in P_2$ and express x_2x_1 as a product
$$x_2x_1 = y_1y_2, \quad y_1 \in P_1, \quad y_2 \in P_2.$$
Then the dressing action of x_2 on x_1 is y_1, see [S] and also [LW]. The "factorization problem" $P \simeq P_1 \times P_2$ has not always a positive answer. Fortunately, it does in our case.

For convenience, let us identify the Heisenberg algebra $\mathfrak{h}^n$ (resp., its extension $\mathfrak{H}^n$) with the Lie algebra of real $(n+2)\times(n+2)$-matrices of the form

$$\begin{pmatrix} 0 & t_1 & \dots & t_n & z \\ & 0 & \dots & & u_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 0 & u_n \\ & & & & 0 \end{pmatrix} \quad \text{(resp.,} \quad \begin{pmatrix} v & t_1 & \dots & t_n & z \\ & 0 & \dots & & u_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 0 & u_n \\ & & & & v \end{pmatrix}).$$

Then the Heisenberg group H^n (resp., its extension $\mathcal{H}^n$) can be taken as the Lie group of real $(n+2)\times(n+2)$-matrices of the form

$$\begin{pmatrix} 1 & X_1 & \dots & X_n & C \\ & 1 & \dots & & Y_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & Y_n \\ & & & & 1 \end{pmatrix} \quad \text{(resp.,} \quad \begin{pmatrix} D & X_1 & \dots & X_n & C \\ & 1 & \dots & & Y_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & Y_n \\ & & & & D \end{pmatrix}, \quad D > 0).$$

Let $\mathcal{D} = \mathcal{H}^n \times \mathcal{H}^n$; according to the proof of Proposition 1, $\mathcal{D}$ can be identified with the double group of $\mathcal{H}^n$, i. e. the group having Lie algebra $D(\mathfrak{H}^n)$. Let us identify

$\mathcal{H}^n$ with the diagonal subgroup of $\mathcal{D}$. The dual group $\mathcal{H}^n_d$ of $\mathcal{H}^n$ is the subgroup of $\mathcal{D}$ of elements of the form

$$\left(\begin{pmatrix} D & X_1 & \dots & X_n & C \\ & 1 & \dots & & 0 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & 0 \\ & & & & D \end{pmatrix}, \begin{pmatrix} D^{-1} & 0 & \dots & 0 & -CD^{-2} \\ & 1 & \dots & & Y_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & Y_n \\ & & & & D^{-1} \end{pmatrix}\right).$$

Lemma 6. *The multiplication provides a diffeomorphism $\mathcal{D} \simeq \mathcal{H}^n \times \mathcal{H}^n_d$.*

Proof. Let $(a, a') \in \mathcal{D}$, where

$$a = \begin{pmatrix} D & X_1 & \dots & X_n & C \\ & 1 & \dots & & Y_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & Y_n \\ & & & & D \end{pmatrix}, \quad a' = \begin{pmatrix} D' & X'_1 & \dots & X'_n & C' \\ & 1 & \dots & & Y'_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & Y'_n \\ & & & & D' \end{pmatrix}.$$

We denote $u = \frac{\sqrt{DD'}}{2DD'}(D'C + DC' - \sum X'_i(DY'_i - D'Y_i))$ and $u' = \frac{\sqrt{DD'}}{2DD'}(D'C - DC' + \sum X'_i(DY'_i - D'Y_i))$. Then $(a, a') = (b, b)(c, c')$, where

$$b = \begin{pmatrix} \sqrt{DD'} & X'_1 & \dots & X'_n & u \\ & 1 & \dots & & \frac{\sqrt{DD'}}{D}Y_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & \frac{\sqrt{DD'}}{D}Y_n \\ & & & & \sqrt{DD'} \end{pmatrix},$$

$$c = \begin{pmatrix} \frac{\sqrt{DD'}}{D'} & \frac{1}{\sqrt{DD'}}(X_1 - X'_1) & \dots & \frac{1}{\sqrt{DD'}}(X_n - X'_n) & u'/D' \\ & 1 & \dots & & 0 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & 0 \\ & & & & \frac{\sqrt{DD'}}{D'} \end{pmatrix},$$

$$c' = \begin{pmatrix} \frac{\sqrt{DD'}}{D} & 0 & \dots & 0 & -u'/D \\ & 1 & \dots & & Y'_1 - \frac{D'}{D}Y_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & Y'_n - \frac{D'}{D}Y_n \\ & & & & \frac{\sqrt{DD'}}{D} \end{pmatrix},$$

and this decomposition is unique. □

Remark 9. The preceding Lemma fails of course if the base field is $\mathbf{C}$.

It follows that the dressing action of

$$\left(\begin{pmatrix} \omega & \tau_1 & \dots & \tau_n & \phi \\ & 1 & \dots & & 0 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & 0 \\ & & & & \omega \end{pmatrix}, \begin{pmatrix} \omega^{-1} & 0 & \dots & 0 & -\phi\omega^{-2} \\ & 1 & \dots & & \mu_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & \mu_n \\ & & & & \omega^{-1} \end{pmatrix}\right)$$

on

$$\begin{pmatrix} \delta & \alpha_1 & \dots & \alpha_n & \gamma \\ & 1 & \dots & & \beta_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & \beta_n \\ & & & & \delta \end{pmatrix}$$

is given by

$$\begin{pmatrix} \delta & \omega^{-1}\alpha_1 & \dots & \omega^{-1}\alpha_n & \gamma + \frac{1}{2}(\omega^{-1}\sum \tau_i\beta_i + (\omega^{-2} - 1)\sum \alpha_i\beta_i - \delta\sum \alpha_i\mu_i) \\ & 1 & \dots & & \omega^{-1}\beta_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & \omega^{-1}\beta_n \\ & & & & \delta \end{pmatrix}.$$

Proposition 5. *The symplectic leaves in the extended Heisenberg group are the one-point sets*

$$\begin{pmatrix} \delta & 0 & \dots & 0 & \gamma \\ & 1 & \dots & & 0 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & 0 \\ & & & & \delta \end{pmatrix}$$

and the 2-dimensional submanifolds

$$\left\{\begin{pmatrix} \delta & \omega\alpha_1 & \dots & \omega\alpha_n & \gamma \\ & 1 & \dots & & \omega\beta_1 \\ & & & & \dots \\ & & & & \dots \\ & & & 1 & \omega\beta_n \\ & & & & \delta \end{pmatrix} : \omega > 0, \quad \gamma \in \mathbf{R}\right\},$$

where δ, α_i, β_i are fixed and some α_i or some β_i is not 0.

§6. The universal R-matrix of the quantum Heisenberg group. In this section we prove that the quantum Heisenberg group is a quasitriangular Hopf algebra by showing explicitly the corresponding universal R-matrix. We shall follow the approach of [D2]. Let A be a Hopf algebra and R be an invertible element of $A \otimes A$. We say that the pair (A, R) is a quasitriangular Hopf algebra if $\Delta^0(a) =$

$R\Delta(a)R^{-1}$ $(a \in A)$, $(\Delta \otimes 1)R = R^{13}R^{23}$ and $(1 \otimes \Delta)R = R^{13}R^{12}$ where Δ^0 is the opposite co-multiplication. The symbols R^{12}, R^{13}, R^{23} have the usual meaning: if $R = a_i \otimes a^i$ then $R^{12} = a_i \otimes a^i \otimes 1$,, $R^{13} = a_i \otimes 1 \otimes a^i$ and $R^{23} = 1 \otimes a_i \otimes a^i$. We call R the universal R- matrix of A.

Let $(A, m, \Delta, \eta, \varepsilon, S)$ be a Hopf algebra and let $(A^*, m^*, \Delta^*, \eta^*, \varepsilon^*, S^*)$ be some Hopf algebra dual to it. Denote by A^0, A^* but with the Hopf algebra structure $(m^*, (\Delta^*)^0, \eta^*, \varepsilon^*, S^*)$, i.e. with the opposite co-multiplication. It is well known that there exists a unique quasitriangular Hopf algebra $(D(A), R)$ such that

(a) $D(A)$ contains A an A^0 as Hopf subalgebras,

(b) R is the image of the canonical element of $A \otimes A^0$ under the embedding $A \otimes A^0 \hookrightarrow D(A) \otimes D(A^0)$ (if e_i is a base of A and e^i is the dual base in A^0, then the canonical element is $e_i \otimes e^i$),

(c) the linear mapping $A \otimes A^0 \to D(A)$ given by $a \otimes b \to ab$ is bijective.

As a vector space, $D(A)$ can be identified with $A \otimes A^0$ (here the tensor product must be interpreted in an appropriate topological way). We can explicitly describe the Hopf algebra structure of $D(A)$: by (c) we may think the elements of $D(A)$ as linear combinations of elements of the form af, $a \in A, f \in A^0$. We want to know gb, $g \in A^0, b \in A$. Let Δ^0 be the co-product in A^0. If

$$(1 \otimes \Delta)\Delta(b) = \sum_i b_{i1} \otimes b_{i2} \otimes b_{i3}$$

and

$$(1 \otimes \Delta^0)\Delta^0(g) = \sum_i g_{i1} \otimes g_{i2} \otimes g_{i3},$$

then

$$gb = \sum_{i,j} g_{j1}(S(b_{i1}))g_{j3}(b_{i3})b_{i2}g_{j2}. \tag{19}$$

Now we start to work with the quantum Heisenberg group. We shall use the notation of §4. Let us denote by $\mathcal{U}_\hbar^+$ (resp. $\mathcal{U}_\hbar^-$) the Hopf subalgebra of $\mathcal{U}_\hbar\mathfrak{H}^n$ generated by X_i, D, Z (resp. $Y_i, D, Z)(i = 1, \dots, n)$. We are interested in the structure of $(\mathcal{U}_\hbar^+)^0$. Let $\mathcal{D}_i$ be the element in $(\mathcal{U}_\hbar^+)^0$ given by $\mathcal{D}_i(X^t D^m Z^p) = \delta_{0,t}\delta_{i,m}\delta_{0,p}$. Now, if $k \in \mathbb{N}^n$ and $l, r \in \mathbb{N}$, formula (16) and the definition of product in $(\mathcal{U}_\hbar^+)^0$ give us

$$\frac{\mathcal{X}^k \mathcal{D}_1^l \mathcal{Z}^r}{k!\,l!\,r!}(u) = \begin{cases} (\frac{|k|\hbar}{4})^{r-s}\frac{1}{(r-s)!} & \text{if } u = X^k D^l Z^s,\ (0 \le s \le r) \\ 0 & \text{otherwise} \end{cases}$$

Let $\{\mu_{klr}\} \subset (\mathcal{U}_\hbar^+)^0$ the dual basis of $\{X^k D^l Z^r\}$. We can deduce by induction that

$$\mu_{klr} = \sum_{t=0}^{r} (\frac{-\hbar|k|}{2})^t \frac{1}{t!} \frac{\mathcal{X}^k \mathcal{D}_1^l \mathcal{Z}^r}{k!\,l!\,r!}, \tag{20}$$

so $\{\mathcal{X}_i, \mathcal{D}_1, \mathcal{Z}\}$ generates $(\mathcal{U}_\hbar^+)^0$ as algebra. In an analogous way, as in §4, we obtain that: $[\mathcal{Z}, \mathcal{X}_i] = -\frac{\hbar}{2}\mathcal{X}_i$, $[\mathcal{X}_i, x_j] = 0$ and $\mathcal{D}_1$ is central. From (18) and the definition of coproduct in $(\mathcal{U}_\hbar^+)^0$ we have $\Delta^0(\mathcal{X}_i) = 1 \otimes \mathcal{X}_i + \mathcal{X}_i \otimes \sum_{v=1}^{\infty} \mathcal{D}_i$. Formula (20) says that $\frac{\mathcal{D}_1^t}{t!} = \mathcal{D}_t$; thus $\Delta^0(\mathcal{X}_i) = 1 \otimes \mathcal{X}_i + \mathcal{X}_i \otimes e^{\mathcal{D}_1}$. Then

Lemma 7. *$(\mathcal{U}_\hbar^+)^0$ is the Hopf algebra generated by $\mathcal{X}_i$, $\mathcal{D}_1$, $\mathcal{Z}$ $(i = 1, \dots, n)$; with the following relations:*

$$[\mathcal{Z}, \mathcal{X}_i] = -\frac{\hbar}{2}\mathcal{X}_i, \qquad [\mathcal{X}_i, \mathcal{X}_j] = \delta_{ij}$$

$(i, j = 1, \dots, n)$ and $\mathcal{D}_1$ is central. The comultiplication Δ^0, the antipode S and the counit ϵ are defined by:

$$\Delta^0(\mathcal{X}_i) = 1 \otimes \mathcal{X}_i + \mathcal{X}_i \otimes e_1^{\mathcal{D}},$$
$$\Delta^0(\mathcal{Z}) = \mathcal{Z} \otimes 1 + 1 \otimes \mathcal{Z}, \qquad \Delta^0(\mathcal{D}_1) = \mathcal{D}_1 \otimes 1 + 1 \otimes \mathcal{D}_1,$$
$$S(\mathcal{X}_i) = -\mathcal{X}_i e^{-\mathcal{D}_1}, \quad S(\mathcal{D}_1) = -\mathcal{D}_1, \quad S(\mathcal{Z}) = -\mathcal{Z},$$

and $\epsilon(u) = 0$ for all u generator.

Let A be a Hopf algebra on $\mathbb{C}[[\hbar]]$, we denote $A((\hbar)) := A \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar))$.

Corollary. *$(\mathcal{U}_\hbar^+)^0((\hbar))$ is isomorphic (as Hopf algebra) to $\mathcal{U}_\hbar^-((\hbar))$ and the isomorphism on the generators is given by*

$$\mathcal{X}_i \to \hbar Y q^{Z/2}, \quad \mathcal{D}_1 \to \frac{\hbar}{2}Z, \quad \mathcal{Z} \to \frac{\hbar}{2}D.$$

Remark 10. Let us define $Y_i' := \frac{\mathcal{X}_i}{\hbar}e^{-\mathcal{D}_1/2}$, $Z' := \frac{2}{\hbar}\mathcal{D}_1$ and $D' := \frac{2}{\hbar}\mathcal{Z}$; then it is clear that the isomorphism of above send $Y' \to Y$, $D' \to D$ and $Z' \to Z$.

Proposition 6. *There is a unique epimorphism of Hopf algebras $\theta : D(\mathcal{U}_\hbar^+)((\hbar)) \to \mathcal{U}_\hbar\mathfrak{H}^n((\hbar))$ defined on the generators by $\theta(X_i) = X_i$, $\theta(D) = \theta(D') = D$, $\theta(Z) = \theta(Z') = Z$ $\theta(Y_i') = Y_i$*

Proof. The definition of $\mathcal{U}_\hbar^+$ implies that we can define a morphism $\theta_1 : \mathcal{U}_\hbar^+((\hbar)) \to \mathcal{U}_\hbar\mathfrak{H}^n((\hbar))$ such that $\theta_1(X_i) = X_i$, $\theta_1(D) = D$ and $\theta_1(Z) = Z$. The previous Corollary implies that there exists a morphism $\theta_2 : (\mathcal{U}_\hbar^+)^0((\hbar)) \to \mathcal{U}_\hbar\mathfrak{H}^n((\hbar))$ such that $\theta_2(Y_i') = Y_i$, $\theta_2(D') = D$ and $\theta_2(Z') = Z$. Let $\theta := \theta_1 \otimes \theta_2$. Applying (19) we obtain : Z and Z' are central elements of $D(\mathcal{U}_\hbar^+)((\hbar))$, $[D, D'] = 0$, $[D', X_i] = X_i$, $[D, Y_i'] = -Y_i'$ and $[X_i, Y_j'] = \delta_{ij}\frac{1}{\hbar}(q^{Z+Z'} - q^{-Z-Z'})$. Thus θ is a morphism of Hopf algebras. □

Theorem 3. *$(\mathcal{U}_\hbar\mathfrak{H}^n, R)$ is a quasitriangular Hopf algebra, where R is equal to*

$$\sum_{l,k,r} (\frac{1}{2})^{l+r} \frac{\hbar^{l+k+r}}{k!\,l!\,r!} X^k D^l Z^r \otimes Y^k (D - \frac{k}{2})^r Z^l q^{kZ}. \tag{21}$$

Proof. From (20) we known that

$$\theta(\mu_{klr}) = (\frac{1}{2})^{l+r}\frac{\hbar^{l+r+|k|}}{k!\,l!} Y^k (\sum_{t=0}^{r} (\frac{-|k|}{2})^t \frac{1}{t!} \frac{1}{(r-t)!} D^{r-t}) Z^l q^{|k|z} = \tag{22}$$
$$(\frac{1}{2})^{l+r}\frac{\hbar^{l+r+|k|}}{k!\,l!\,r!} Y^k (D - \frac{|k|}{2})^r Z^l q^{|k|Z}.$$

Let $R' = \sum_{klr} X^k D^l Z^r \otimes \mu_{klr}$ be the universal R-matrix of $D(\mathcal{U}_\hbar^+)((\hbar))$, then from (22) we obtain that $\theta(R')$ is equal to (21), furthermore $R = \theta(R')$ is the universal R-matrix of $\mathcal{U}_\hbar\mathfrak{H}^n((\hbar))$. Now, the term of degree 0 (respect to $\hbar$) of $\theta(R')$ is $1 \otimes 1$, so $\theta(R')^{-1}$ also belongs to $\mathcal{U}_\hbar\mathfrak{H}^n$. We conclude that R satisfies all the requirements to be the universal R-matrix of $\mathcal{U}_\hbar\mathfrak{H}^n$. □

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