Report INP No 1557/PH

INP--1557/PH.

ON GALILEAN COVARIANT QUANTUM MECHANICS

Andrzej Horzela, Edward Kapuścik and Jarosław Kempczyński

Department of Theoretical Physics, H. Niewodniczański Institute of Nuclear Physics, ul. Radzikowskiego 152, 31 342 Kraków, Poland

and

Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 101 000 Moscow, USSR, Head Post Office, P.O. Box 79.

Abstract:

Formalism exhibiting the Galilean covariance of wave mechanics is proposed. A new notion of quantum mechanical forces is introduced. The formalism is illustrated on the example of the harmonic oscillator.

Kraków, August 1991

I. Introduction

Galilean covariance is the most fundamental property of low energy physics. In spite of that there are many branches of low energy physics which lack from Galilean covariance formulation. This, highly undesirable, situation happened for example in classical mechanics of a single interacting particle where the usual way of defining the acting forces through some force laws breaks down the Galilean covariance. In order to prevent this catastrophe we have recently proposed [1] to determine the acting forces from Galilean covariant differential equations derived from the non-covariant force laws. The most important result of Part I consists in finding the correct transformation rule for the total energy of the interacting particle which is different from the known rule for its kinetic energy.

Part I of our paper, apart from its pedagogical value, opened also a new way of understanding the Galilean covariance in quantum mechanics. There exists a general proof [2] that quantum mechanics based on the Schrödinger equations is Galilean covariant but this proof contains the assumptions that the Hamiltonian of the interacting particle is constructed from a scalar potential. In view of Part I such an assumption must be considered as wrong because already in classical mechanics the potentials are not scalars with respect to the Galilean transformation and the question of a Galilean covariant formulation of quantum mechanics becomes open for a new discussion.

Our paper provides a new approach to Galilean covariant quantum mechanics. A central role in it, in analogy to classical mechanics, is played by the acting forces. As a matter of fact, in our approach we introduce a new notion of a quantum mechanical force which is independent from the classical force laws. Our formalism is not a quantization of the classical canonical formalism in which the non-covariant force laws are used at the very beginning. This fact implies some essential differences between our approach to quantum mechanics and the standard one. In particular, it turns out that the classical forces can be realized only on a restricted class of states and this provides a new insight on the role of constitutive relations in quantum physics.

We shall work in the framework of a five-dimensional model of the Galilean space-time [3] because, in contradistinction to the usual four dimensional models, it admits a fundamental non-singular metric form. The amazing property of this metric is the fact that its maximal symmetry group is not the original Galilei group but de Sitter group SO(4,1) [4] which contains as subgroups both Galilei and Lorentz groups. Therefore, having constructed explicitly Galilean covariant wave mechanics for the interacting particle in a five-dimensional space-time we obtain a unitary representation of the Galilei group which incorporates all the information on the dynamics of the particle and which may be used to induce [5] an irreducible representation of the whole de Sitter group. The representation of the de Sitter group obtained in this way may then be restricted to the subgroup S0(3,1) which is isomorphic to the Lorentz group and all resulting representations of the Lorentz group will contain the information on the assumed original low energy dynamics of the interacting particle but this time in a relativistic form. We consider this way as one of the possible ways of constructing relativistic dynamics of particles without an explicit use of the notions of fields. Our present paper is the first step in that direction.

II. The non-covariance of classical standard wave mechanics.

Through this paper we shall consider only the simplest case of the wave mechanics for scalar particles. The generalization to higher spins is straightforward and does not change our results.

The standard wave mechanics works with the following projective unitary representation of the Galilei group

$$(U(R,\vec{u},\vec{a},b)\Psi)(\vec{r}\,',t') = exp\left[\frac{im}{\hbar}(R\vec{r}\cdot\vec{u}+\frac{1}{2}\vec{u}^{\,2})\right]\Psi(\vec{r},t) \tag{2.1}$$

where m is the mass of the particle described by the wave function $\Psi(\vec{r},t)$ and we have used the standard notation for the unitary operators $U(R,\vec{u},\vec{a},b)$ implementing the representation in which R denotes the 3×3 orthogonal matrix describing the rotation of the coordinate axes used in two inertial reference frames, $\vec{a}$ and b the space and time translations, respectively. The primed coordinates $\vec{r}\,',t'$ are connected with the unprimed ones by the usual Galilei transformations

$$\begin{aligned} \vec{r}\,' &= R\vec{r}+\vec{u}t+\vec{a} \\ t' &= t+b \end{aligned} \tag{2.2}$$

The representation (2.1) is projective because

$$\begin{aligned} &U(R_1,\vec{u}_1,\vec{a}_1,b_1)U(R_2,\vec{u}_2,\vec{a}_2,b_2) = \\ &\quad = exp\left[\frac{im}{\hbar}(\vec{u}_1\cdot R_1\vec{a}_2)+\frac{1}{2}\vec{u}_1^{\,2}b_2\right]U(R_{12},\vec{u}_{12},\vec{a}_{12},b_{12}) \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} R_{12} &= R_1R_2 \\ \vec{u}_{12} &= \vec{u}_1+R_1\vec{u}_2 \\ \vec{a}_{12} &= \vec{a}_1+R_1\vec{u}_2+\vec{u}_1b_2 \\ b_{12} &= b_1+b_2 \end{aligned} \tag{2.4}$$

express the composition law for the Galilean transformations. The phase factor in (2.3) cannot be removed by a redefinition because it is an intrinsic property of the Galilei group.

The wave functions $\psi(\vec{x},t)$ are determined from the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=\hat{H}\psi \tag{2.5}$$

where

$$\hat{H}=-\frac{\hbar^2}{2m}\Delta+V(\vec{x}) \tag{2.6}$$

is the Hamiltonian of the particle. According to ref. [2] the Schrödinger equation is Galilean covariant provided the potential V transforms like a scalar. We however have shown [1] that in classical mechanics the potential energy cannot be a scalar because such property of the potential enforces the same Galilean transformation rules for the total and kinetic energies and this in turn contradicts the conservation law for the total energy. The contradiction disappears only for free particles and consequently the proof in the ref. [2] applies only to this case.

There is also another argument in favour of this statement. In fact, Schrödinger equation (2.5) expresses the equality of two expressions for the generator of time translations: the first one given by the representation (2.1) and equal to $i\frac{\partial}{\partial t}$ and the second one given by the operator $\hbar^{-1}\hat{H}$. The generator of time translations $i\frac{\partial}{\partial t}$ commutes however with the generators of space translations $i\frac{\partial}{\partial x_j}$. Since all physical states satisfy the equality (2.5) it follows that the Hamiltonian $\hat{H}$ must also commute with the operators of space translations and if $\hat{H}$ is of the form of (2.6) this means that $V(\vec{x})=0$.

The commutativity of $\hat{H}$ and $i\frac{\partial}{\partial x_j}$ creates however a new problem. If we identify the operators $-i\hbar\frac{\partial}{\partial x_j}$ with the operators for the components of momentum we must conclude that

$$\left[\hat{\vec{p}}_j, \hat{H}\right] = 0 \tag{2.7}$$

and this means that the operators $\hat{\vec{p}}_j$ independently from the form of $\hat{H}$ are constant in time. This is compatible with the equation of motion

$$\left[\hat{\vec{p}}_j, \hat{H}\right] = i\hbar\hat{\vec{F}} \tag{2.8}$$

where $\hat{\vec{F}}_j$ are the operators which correspond to the acting force, only provided

$$\hat{\vec{F}}_j = 0 \tag{2.9}$$

and this again means that the Galilean covariance of the standard wave mechanics can be achieved only for the case of free particles. Since this conclusion is independent from the form of the Hamiltonian and does not depend on the transformation rules of the potential energy it means that in a Galilean covariant wave mechanics of an interacting particle we must resign from the identification of the momentum operators with the operators of space translations. Such an identification is consistent with the structure of the translation group only for the case of free particles.

In the next section we shall show that our approach to wave mechanics is free from the above mentioned defects and for the free particles it reduces to the usual formulation of wave mechanics.

III. Galilean covariant formulation of wave mechanics.

We shall develop our new approach to wave mechanics in the framework of the five-dimensional model of the Galilean space-time [3,6]. In this model, in addition to the usual space and time coordinates $\vec{x}$ and t, with the transformation rules (2.2), we use also a fifth coordinate θ with the transformation rule

$$\theta \to \theta' = \theta + R\vec{x}\cdot\vec{u} + \frac{1}{2}\vec{u}^2 t + \varphi \tag{3.1}$$

where φ realizes the one-parameter central extension of the Galilei group [6]. The advantage of using the eleven parametric extended Galilei group instead of the usual ten parametric Galilei group primarily consists in the fact that the unitary projective representation (2.1) for the latter is replaced by the ordinary unitary representation

$$(U(R,\vec{u},\vec{a},b,\varphi)\Psi)(\vec{x}',t',\theta') = \Psi(\vec{x},t,\theta) \tag{3.2}$$

for the former but now the new wave functions depend on five coordinates $(\vec{x},t,\theta)$. It has been shown in ref. [7] that the five-dimensional approach to the Galilean covariant wave equations reproduces the results of the four-dimensional approach. Unfortunately all these discussion in the case of wave mechanics again applies only in the case of free particles.

Another advantage of the five-dimensional formulation of wave mechanics is connected with the Galilean transformation rules for quantum mechanical observables. In particular, identifying the operators $-i\hbar\frac{\partial}{\partial x_j}$ with the components of the momentum operator $\hat{\vec{p}}_j$ of the free particle, in the four-dimensional formulation of wave mechanics we get the transformations rule

$$\hat{\vec{p}}_j \to \hat{\vec{p}}_j{}' = \sum_{j=1}^{3} R_{jk}\hat{\vec{p}}_k \tag{3.3}$$

and to bring this in agreement with the usual classical Galilean transformation rule for the momentum according to which

$$\hat{\vec{p}}_j \to \hat{\vec{p}}_j{}' = \sum_{j=1}^{3} R_{jk}\hat{\vec{p}}_k + m\vec{u}_j \tag{3.4}$$

where m is the mass of the particle, we must use the representation (2.1) and instead of (3.3) write

$$(\hat{\vec{p}}_j{}'\Psi')(\vec{x}',t') = exp\left[\frac{im}{\hbar}(R\vec{x}\cdot\vec{u} + \frac{1}{2}\vec{u}^{\,2}t\right]\left(\sum_{j=1}^{3} R_{jk}\hat{\vec{p}}_k + m\vec{u}_j\right)\Psi(\vec{x},t) \tag{3.5}$$

In the five dimensional formulation of wave mechanics we have

$$\frac{\partial}{\partial x_j'} = \sum_{j=1}^{3} R_{jk}\frac{\partial}{\partial x_k} - \vec{u}_j\frac{\partial}{\partial\theta} \tag{3.6}$$

and instead of (3.3) we may write

$$\hat{\vec{p}}_j{}' = \sum_{j=1}^{3} R_{jk}\hat{\vec{p}}_k + i\hbar\vec{u}_j\frac{\partial}{\partial\theta} \tag{3.7}$$

This will agree with (3.4) provided we identify the operator $i\hbar\frac{\partial}{\partial\theta}$ with the mass operator. For states $\psi(\vec{x},t,\theta)$ which are eigenstates of this operator with the eigenvalue m, the transformation rule (3.7) for the quantum mechanical momentum coincides with the corresponding transformation rule for the classical momentum.

For more general states the eigenvalue of the mass m has to be replaced by the mass operator.

In ref. [1] we have shown that the requirement of the Galilean covariance in classical mechanics of a single particle is easy to satisfy if we treat the acting forces as basic mechanical quantities which are independent from other quantities like the position and velocity. In accordance with that we shall construct the Galilean covariant formalism of wave mechanics from wave functions $\psi(\vec{x}, t, \theta, \vec{f})$ which carry the following unitary representation of the extended Galilei group.

$$(U(R, \vec{u}, \vec{a}, b, \varphi)\Psi)(\vec{x}', t', \theta', \vec{f}') = \Psi(\vec{x}, t, \theta, \vec{f}) \tag{3.8}$$

where the primed variables $\vec{x}', t'$ and θ' are related to the unprimed ones by the Galilean transformations (2.2) and (3.1), respectively, while

$$\vec{f}' = R\vec{f} \tag{3.9}$$

The new variables $\vec{f} = (f_1, f_2, f_3)$ possess the transformation property of the force and will indeed serve to realize our new notion of quantum mechanical force which is independent from the notion of position and velocity. For more complicated cases more than one triple of variables with the property (3.9) may be used for the realization of quantum mechanical forces but at this early stage of our approach we restrict the discussion to the simplest case of just one triple.

In general, the wave functions apart from carrying representation of all symmetry groups of the physical system must also provide a tool for expressing the content of all quantum mechanical observables. This is usually achieved by expressing all observables as differential operators acting on the wave functions. In mechanics of a single particle we have four basic observables loaned from classical physics; the position $\hat{X} = (\hat{X}_1, \hat{X}_2, \hat{X}_3)$, the velocity $\hat{V} = (\hat{V}_1, \hat{V}_2, \hat{V}_3)$, the momentum $\hat{P} = (\hat{P}_1, \hat{P}_2, \hat{P}_3)$ and the force $\hat{F} = (\hat{F}_1, \hat{F}_2, \hat{F}_3)$ acting on the particle. Taking from classical mechanics the usual relation between momentum and velocity and a specific force law we may reduce the set of basic observables to $\hat{X}$ and $\hat{P}$ only. In our approach we loan from classical mechanics only the usual relation between momentum and velocity but reject all possible force laws because they violate the Galilean covariance already on the classical level. For this reason the set of our basic observables consists at least from three elements: $\hat{X}$, $\hat{P}$ and $\hat{F}$. To this set we shall add as an independent observable the Hamiltonian $\hat{H}$, because, as it was shown in ref. [8] this is necessary for expressing the time evolution of all observables. Of course, like any other observable, the Hamiltonian, at some later stage of the theory, may be related to other observables by a specific constitutive relation but from the fundamental point of view it is a priori an independent observable.

As a consequence of their definitions, the basic observables are interrelated by the following relations valid for $j = 1, 2, 3$

$$\left[\hat{H}, \hat{X}_j\right] = -i\hbar\hat{V}_j \tag{3.10}$$

$$\left[\hat{H}, \hat{P}_j\right] = -i\hbar\hat{F}_j \tag{3.11}$$

$$\hat{P}_j = m\hat{V}_j \tag{3.12}$$

and our task is to find explicit expressions for these observables.

The construction of the basic observables we shall start with the force operator $\hat{F}$ for which we adopt the following definition

$$\left(\hat{F}_j\Psi\right)(\vec{x}, t, \theta, \vec{f}) = F_j(\vec{f})\Psi(\vec{x}, t, \theta, \vec{f}) \tag{3.13}$$

where $F_j(\vec{f}) \quad (j = 1, 2, 3)$ form a vector valued function of the variables $\vec{f} =$

(f_1, f_2, f_3) which determines the particular shape of the acting force. The definition (3.13) uniquely follows from the transformation rules for the force and the variables of the wave function and from the assumption that the variables $\vec{f}$ are in some sense the canonical variables for the operators $\hat{F}_j$. The presence of the functions $F_j(\vec{f})$ in (3.13) needs however some explanation because in the usual definition of the canonical variable [9] it is assumed that the given operator acts simply as a multiplication operator. We shall see below that

$$F_j(\vec{f}) = f_j \tag{3.14}$$

only for the harmonic oscillator and consequently only in this case the variables f_j are canonical variables for the acting forces in the mathematical sense. For other forces we shall need nontrivial functions $F_j(\vec{f})$ to express their nonlinear character.

Taking into account the transformation rules for the position operators $\hat{X}_j$ and that of the variables of the wave functions we define

$$\left(\hat{X}_j \Psi\right)(\vec{x}, t, \theta, \vec{f}) = \left(x_j + \alpha f_j + \beta \frac{\partial}{\partial f_j}\right) \Psi(\vec{x}, t, \theta, \vec{f}) \tag{3.15}$$

where α and β are two dimensional constants to be determined for each case from the fundamental and characteristic constants of the problem. We do not treat here the more general definition with scalar valued functions $\alpha(\vec{f}^{\,2})$ and $\beta(\vec{f}^{\,2})$ because our calculations with such functions showed that they do not introduce anything new to the theory. The definition (3.15) may be considered as a part of the definition of the canonical character of the variables f_j.

Assuming now that we shall consider only velocity independent forces, in addition to the relations (3.10) - (3.12), we may also require the relation

$$\left[\hat{X}_j, \hat{F}_k\right] = 0 \tag{3.16}$$

from which it follows that

$$\beta = 0 \tag{3.17}$$

On the contrary, we shall show below that for interacting particles α cannot vanish. It is, therefore, improper to interpret the variable $\vec{x}$ of the wave functions as a position of the particle. We may interpret the variable $\vec{x}$ only as a variable which describes the possible positions of the particle but not its actual one. The variable $\vec{x}$ is needed to create the space-time arena for defining the position operator with the right transformation rule and only for free particles it determines their actual positions.

Similarly, taking into account the transformation rules of the momentum operators $\hat{P}_j$ and of the variables of the wave function we may define

$$\left(\hat{P}_j \Psi\right)(\vec{x}, t, \theta, \vec{f}) = \left(-i\hbar \frac{\partial}{\partial x_j} + \tilde{\alpha} f_j + \tilde{\beta} \frac{\partial}{\partial f_j}\right) \Psi(\vec{x}, t, \theta, \vec{f}) \tag{3.18}$$

where again we have to do with two dimensional constants $\tilde{\alpha}$ and $\tilde{\beta}$ and we excluded the case when they are scalar valued functions of $\vec{f}^2$. But, by a gauge transformation

$$\Psi(\vec{x}, t, \theta, \vec{f}) \to \Psi(\vec{x}, t, \theta, \vec{f}) = e^{\ i\lambda \vec{f}^2} \Psi(\vec{x}, t, \theta, \vec{f}) \tag{3.19}$$

we may always remove in (3.18) the term proportional to f_j and, therefore, without lost of generality we may assume that

$$\hat{\alpha} = 0 \tag{3.20}$$

Quite similarly as for operators $\hat{X}_j$ we shall see that for the interacting particles $\hat{\beta}$ does not vanish. From that it follows that the operator $-i\hbar\frac{\partial}{\partial x_j}$ is the momentum operator $\hat{P}_j$ only for free particles.

To proceed further let us now concentrate our attention on the Hamiltonian $\hat{H}$. From the Schrödinger equation (2.5) it follows that under Galilean transformations we should have

$$\hat{H} \rightarrow \hat{H}' = \hat{H} - i\hbar\vec{u}\cdot R\vec{\nabla} + \frac{i\hbar}{2}\vec{v}^2\frac{\partial}{\partial\theta} \tag{3.21}$$

Comparing this with the transformation rule [1]

$$E \rightarrow E' = E + \vec{u}\cdot R\vec{P} + \frac{1}{2}m\vec{u}^2 \tag{3.22}$$

for the total energy of the particle, we see that all physical states must be eigenstates of the mass operator $i\hbar\frac{\partial}{\partial\theta}$ and that the operator $-i\hbar\vec{\nabla}$ has to be interpreted as the quantum mechanical counterpart of the time independent quantity $\vec{P}$ present in the transformation rule (3.22) and formed from the momentum of the particle $\vec{p}(t)$ and quantities describing the interaction. Since only for free particles

$$\vec{P} = \vec{p}(t) = \overrightarrow{\text{const}} \tag{3.23}$$

we must conclude that only for free particles the operator $-i\hbar\vec{\nabla}$ is the momentum operator.

In Galilean physics time has an absolute meaning and therefore the total classical time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} + \frac{1}{2}\vec{v}^2 \frac{\partial}{\partial \theta} \tag{3.24}$$

is invariant under the change of the reference frame. Translating this condition into the quantum mechanical language we must require that

$$\left[\hat{H}, \hat{H}'\right] = 0 \tag{3.25}$$

because otherwise the time derivatives defined by the operators $\hat{H}$ and $\hat{H}'$, respectively, will be incompatible. The time evolution of two observables in two different inertial reference frames should then have different physical meaning. Condition (3.25) is just that condition which is usually not taken into account in all discussion of the Galilean covariance of wave mechanics.

From (3.21) and (3.25) it once again follows that

$$\left[\frac{\partial}{\partial x_j}, \hat{H}\right] = 0 \tag{3.26}$$

and therefore the Hamiltonian $\hat{H}$ may be of the following general form

$$\hat{H} = \sum_{j=1}^{3} \left\{ A(f^2)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} + B(f^2)\frac{\partial}{\partial x_j}\frac{\partial}{\partial f_j} + C(f^2)f_j\frac{\partial}{\partial x_j} + \right.$$
$$\left. + D(f^2)f_j\frac{\partial}{\partial f_j} + E(f^2)\frac{\partial}{\partial f_j}\frac{\partial}{\partial f_j} \right\} + V(f^2) \tag{3.27}$$

where all coefficients are arbitrary functions of the indicated argument. Under the Galilean transformation this expression behaves according to the rule

$$\hat{H} \rightarrow \hat{H}' = \hat{H} = A(f^2)\left(-2\vec{u}\cdot R\vec{\nabla} + \vec{u}^2\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial\theta} - \\ - \left[B(f^2)\vec{u}\cdot R\frac{\partial}{\partial\vec{f}} + C(f^2)\vec{u}\cdot R\vec{f}\right] \tag{3.28}$$

Comparing this with the required transformation rule (3.21) and taking into account that all states must be eigenstates of the mass operator $i\hbar\frac{\partial}{\partial\theta}$ we get

$$A\left(f^2\right) = -\frac{\hbar^2}{2m} \tag{3.29}$$

$$B\left(\vec{f}^2\right) = C\left(\vec{f}^2\right) = 0 \tag{3.30}$$

From (3.10), (3.12), (3.18) and (3.27) we get the conditions

$$D\left(\vec{f}^2\right) = 0 \tag{3.31}$$

$$\beta = \frac{2imE\left(\vec{f}^2\right)\alpha}{\hbar} \tag{3.32}$$

from which it follows that the function $E\left(f^2\right)$ is a constant. From (3.11), (3.13) and (3.27) we finally get the relation

$$F_j\left(\vec{f}^2\right) = \frac{2mE\alpha}{\hbar^2}\frac{\partial V\left(\vec{f}^2\right)}{\partial f_j} = \frac{4mE\alpha}{\hbar^2}V\left(\vec{f}^2\right)f_j \tag{3.33}$$

Summarizing up and changing the notation from E to β we get the following representation for our basic quantum mechanical observables

$$\hat{X}_j = x_j + \alpha f_j \tag{3.34}$$

$$\hat{P}_j = -i\hbar\frac{\partial}{\partial x_j} + \frac{2im\alpha\beta}{\hbar}\frac{\partial}{\partial f_j} \tag{3.35}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + \beta\sum_{j=1}^{3}\frac{\partial}{\partial f_j}\frac{\partial}{\partial f_j} + V\left(\vec{f}^2\right) \tag{3.36}$$

$$\hat{F}_j = \frac{4m\alpha\beta}{\hbar^2}V\left(\vec{f}^2\right)f_j \tag{3.37}$$

where α and β are two dimensional constants which in each particular case have to be determined from the characteristic constants of the problem and the function $V\left(\vec{f}^2\right)$ plays the role of "potential' which determine the shape of the acting forces.

Since for the free particle there are no constants which may give the right dimension to the constants α and β we must conclude that in this case we have

$$\alpha = \beta = 0 \tag{3.38}$$

and our formalism reduces to the standard formalism of wave mechanics for free particles.

In part I we have shown that the non-covariant force laws may be replaced by covariant differential equations for the acting forces. Similarly, we may form such equations also in quantum mechanics where they will have the form

$$-\frac{1}{\hbar^2}\left[\hat{H},\left[\hat{H},\hat{F}_j\right]\right] = \varrho_j(\hat{F}) \tag{3.39}$$

where the right-hand side is determined from the corresponding differential equation for the classical force. In this way we get the correspondence between the quantum and classical problems.

IV. The physical interpretation of the formalism.

Before going to a particular example we would like to discuss some aspects of the physical interpretation of the new wave functions.

From the shape of the Hamiltonian (3.36) we get the following representation of the wave functions

$$\psi(\vec{x}, t, \theta, \vec{f}\,) = \left(exp\, \frac{-im\theta}{\hbar} \right) \int d^3x\, exp \left[\frac{i\vec{q}\cdot\vec{x}}{\hbar} - i\frac{q^2}{2m\hbar}t \right] \psi(\vec{q}, t, \vec{f}\,) \qquad (4.1)$$

where the function $\psi(\vec{q}, t, \vec{f}\,)$ satisfies the wave equation

$$i\hbar \frac{\partial \psi(\vec{q}, t, \vec{f}\,)}{\partial t} = \left[\beta \Delta_f + V(\vec{f}\,) \right] \psi(\vec{q}, t, \vec{f}\,) \qquad (4.2)$$

with

$$\Delta_f = \sum_{j=1}^{3} \frac{\partial^2}{\partial f_j^2} \qquad (4.3)$$

The dependence of $\psi(\vec{q}, t, \vec{f}\,)$ on the argument $\vec{q}$ is not determined by the dynamical equation (4.2) but by the initial form of the wave packet represented by the wave function $\psi(\vec{x}, t, \theta, \vec{f}\,)$. During the time this wave packet as usually is spreading and after a long period of time it may be quite different from the initial packet.

Since the variable $\vec{x}$ does not represent the position of the particle described by the wave function $\psi(\vec{x}, t, \theta, \vec{f}\,)$ we cannot interpret the absolute square of the wave function as the probability density of finding the particle at some point. However, the wave functions properly normalized should serve for the evaluation of mean values of quantum mechanical observables. To see what the normalization condition should be let us define the mean value of any observable $O(\vec{x}, \vec{\nabla}, \vec{f}, \partial_{\vec{f}})$ in the state $\psi(\vec{x}, t, \theta, \vec{f})$ by the usual formula

$$\bar{O}(\psi) = \int d^3x\, d^3f\, \psi^*(\vec{x}, t, \theta, \vec{f}) O(\vec{x}, \vec{\nabla}, \vec{f}, \partial_{\vec{f}}) \psi(\vec{x}, t, \theta, \vec{f}) \tag{4.4}$$

In particular, the mean value of the j-th component of the acting force is given by

$$\bar{F}_j(\psi) = \int d^3x\, d^3f\, \psi^*(\vec{x}, t, \theta, \vec{f}) \hat{F}_j \psi(\vec{x}, t, \theta, \vec{f}) = \int d^3x\, w(\vec{f}, t) F_j(\vec{f}) \tag{4.5}$$

where $F_j(\vec{f})$ is the function in (3.13) and

$$w(\vec{f}, t) = \int d^3x |\psi(\vec{x}, t, \theta, \vec{f})|^2 = (2\pi)^3 \int d^3q |\psi(\vec{q}, t, \vec{f})|^2 \tag{4.6}$$

serves as a statistical weight in the integral (4.5). From that it follows that the integrals in (4.6) may be interpreted as the probability density that at time t on the considered particle acts the force $\vec{F} = (F_1(\vec{f}), F_2(\vec{f}), F_3(\vec{f}))$. For this purpose the density $w(\vec{f}, t)$ should be normalized to unity and we get the normalization condition for our new wave functions in the form

$$\int d^3x\, d^3f |\psi(\vec{x}, t, \theta, \vec{f})|^2 = 1 \tag{4.7}$$

As we have seen the wave functions, in general, do not have the probabilistic interpretation in the position space. The situation changes, however, if at the end of calculations we restrict our theory to the subspace of the space $R^3 \times R^3$, spanned by the arguments $\vec{x}$ and $\vec{f}$ defined by the classical force law

$$\vec{F}(t) = \vec{\mathcal{F}}(\vec{x}(t)) \tag{4.8}$$

i.e., to the subspace given by the equations

$$F_j(\vec{f}) = \mathcal{F}_j(\vec{x}) \tag{4.9}$$

where $F_j(\vec{f})$ is the function from (3.13). On this subspace, the functions $F_j(\vec{f})$ which realize our quantum notion of force are connected with the variable $\vec{x}$ by the same relation (4.8) which determines the classical force law and our "force picture" has an image in the $\vec{x}$-space. In this sense we may restore the usual space - time picture of the wave function. It must be however remembered that this procedure breaks down the Galilean covariance of the formalism because all classical force laws for a single particle are non-covariant.

It is also interesting to look on the restoration of the standard force laws in terms of observables $\hat{x}_j$ and $\hat{F}_j$. From (3.34) and (3.37) it is clear that a relation of the form

$$\hat{F}_j = \mathcal{F}_j(\hat{x}_1, \hat{x}_2, \hat{x}_3) \tag{4.10}$$

may be satisfied only for such states for which the multiplication by the variables x_j gives zero. Such states are described by wave functions $\psi(\vec{x}, t, \theta, \vec{f})$ which as functions of the variable $\vec{x}$ have the properties of Dirac δ-function. This can be however achieved only for the initial time because any wave packet is spreading in time. In fact, taking in (4.1) the functions $\psi(\vec{q}, t, \vec{f})$ as independent from the variable $\vec{q}$ we have

$$\psi(\vec{x}, 0, \theta, \vec{f}) = (2\pi\hbar)^3 \delta(\vec{x}) \psi(\vec{f}, t) \tag{4.11}$$

where $\Psi(\vec{f}, t)$ is some solution of the equation (4.2). Therefore, the classical force laws may be translated into quantum mechanics only as an initial relation between force and position.

In classical physics the force laws determine also the time derivative of the force. A different situation happens in wave mechanics. In fact, from (3.36) and (3.37) it is seen that the time derivative of the force is given by the operator

$$\frac{d}{dt}\hat{F}_j \equiv \hat{G}_j = \frac{i}{\hbar}[\hat{H}, \hat{f}_j] = \\ = \frac{i\beta}{\hbar}\sum_{k=1}^{3}\left(\frac{\partial^2 F_j}{\partial f_k^2} + 2\frac{\partial F_j}{\partial f_k}\frac{\partial}{\partial f_k}\right) \tag{4.12}$$

and this operator may be related to the operators $\hat{P}_j$ only on that states for which the wave functions do not depend on the variable $\vec{x}$. This may be achieved provided we choose the initial conditions in such a way that

$$\psi(\vec{q}, t, \vec{f}\,) \sim \delta(\vec{q})\tilde{\psi}(\vec{f}, t) \tag{4.13}$$

where again the function $\tilde{\psi}(\vec{f}, t)$ is a solution of the equation (4.2).

Clearly, the class of states for which we may reproduce the classical force laws on the operator level being specified by (4.11) and by the class of states for which the time derivative of the force is fixed by the force law being specified by (4.13) are disjoint and therefore it is impossible reproduce simultaneously the force law and its time derivative. The operator versions of force laws carry therefore only part of the information contained in classical force laws and clearly this is a consequence of the quantum mechanical uncertainty relation.

Both classes of states specified either by (4.11) or (4.13) contain only non-normalizable states in the sense of (4.70 and therefore our probabilistic interpretation does not apply to them. For normalizable states the content of quantum mechanical force laws is different from their classical counterparts. The only common property they have is that they always break down the Galilean covariance.

Among all normalizable states a special role is played by the stationary states for which

$$\psi(\vec{q},t,\vec{f}) = e^{-i\frac{\Omega t}{\hbar}}\psi(\vec{q},\vec{f}) \tag{4.14}$$

where $\psi(\vec{q},\vec{f})$ satisfies the eigenvalue equation

$$\left[\beta\Delta_f + V(\vec{f}^2)\right]\psi(\vec{q},\vec{f}) = \Omega\psi(\vec{q},\vec{f}) \tag{4.15}$$

Substituting (4.14) into (4.1) we get the wave functions in the form

$$\psi(\vec{x},t,\theta,\vec{f}) = \int d^3x \;\; exp\frac{i}{\hbar}\left[\vec{q}\cdot\vec{x} - \left(\frac{q^2}{2m}+\Omega\right)t - m\theta\right]\psi(\vec{q},\vec{f}) \tag{4.16}$$

and we easily recognize them as elements of the carrier space of the unitary representation of the extended Galilei group. From the general theory of such representations [6] it is known that each of them is characterized by three numbers; the spin s, which we have assumed to be zero; the mass of the particle m, which we have fixed at the beginning and the value of the rest energy Ω, which is the only parameter not fixed a priori by our formalism. Since the eigenvalue problem (4.15) has solution only for restricted quantized values of Ω we see that our formalism produces representations of the Galilei group with quantized values of the rest energy, which are determined by the acting force. The picture just described shows the essential differences between classical and quantum Galilean covariant mechanics from one side and between our covariant formalism and the standard one from the other. In classical physics the acting forces influence the motion of the particle without changing its internal structure. In our formalism the acting forces change the internal structure of the particle because they determine its internal rest energy while the space-time picture is modelled by the initial form of wave packet and the spreading phenomenon. The standard formalism of quantum mechanics, like classical mechanics, does not allow to describe the changes of internal structure of the particle on which the force acts and restricts its role to the description of the influence of the acting forces on the spreading wave packets. As we have already explained, we can also describe this phenomenon by restricting our general covariant formalism to a non-covariant subspace in the combined $\vec{r}$ and $\vec{f}$ spaces which is selected by a given classical force law.

Since the $\vec{q}$-dependence of the functions $\psi(\vec{q},\vec{f})$ is determined by the initial conditions, the eigenfunctions $\psi_n(\vec{q},\vec{f})$ in (4.15) which belong to a non-degenerate eigenvalue Ω_n are of the form

$$\psi_n(\vec{q},\vec{f}) = \psi_n(\vec{q})\,\psi_n(\vec{f}) \tag{4.17}$$

where n denotes the whole collection of quantum numbers. For such states the probability density $w(\vec{f},t)$ defined by (4.6) is time independent and equal to

$$w_n(\vec{f},t) = c_n|\psi_n(\vec{f})|^2 \equiv w_n(\vec{f}) \tag{4.18}$$

where

$$c_n = (2\pi)^3 \int d^3q|\psi_n(\vec{q})|^2 \tag{4.19}$$

Because the functions $\psi_n(\vec{q})$ are determined by the initial shape of the wave packet we may always normalize the integrals in (4.19) to unity and then we simply have

$$w_n(\vec{f}) = |\psi_n(\vec{f})| \tag{4.20}$$

similarly to the standard wave mechanics probability density for the coordinate.

Among non-diagonal matrix elements of observables a physical meaning have only elements taken between states for which the eigenfunctions $\psi_n(\vec{f})$ are different but the functions $\psi_n(\vec{q})$ are the same because only transitions between such states initially prepared identically are connected with the interaction. For such states the matrix elements of the position operators (3.34) and momentum operators (3.35) have the form

$$x_j(n_1, n_2) = A_j\delta_{n_1\,n_2} + \alpha\int d^3f\,\psi^*_{n_1}(\vec{f})f_j\psi_{n_2}(\vec{f}) \tag{4.21}$$

and

$$p_j(n_1, n_2) = B_j\delta_{n_1\,n_2} + \beta\int d^3f\,\psi^*_{n_1}(\vec{f})\frac{\partial}{\partial f_j}\psi_{n_2}(\vec{f}) \tag{4.22}$$

where A_j and B_j are contributions from the first terms in (3.34) and (3.33), respectively, which obviously are diagonal in the space of the functions $\psi_n(\vec{f})$. From this it easily follows that in the space of $\psi_n(\vec{f})$ functions, the operators $\hat{x}_j$ and $\hat{p}_j$ satisfy the standard commutation relations.

V. The example of the harmonic oscillator.

We shall now apply our general approach to the simple case of the harmonic oscillator. The more complicate cases, in particular the Kepler problem we plan to consider in separate papers.

Assuming that the function $F_j(\vec{f})$ in (3.13) is simply equal to f_j and neglecting the unessential integration constant in (3.37) we get the following expression for the function $V(\vec{f}^2)$ in (3.36):

$$V(\vec{f}^2) = \frac{\hbar^2}{4m\alpha\beta}\vec{f}^2 \tag{5.1}$$

Calculating the second time derivative of the force according to the formula

$$\ddot{\hat{F}}_j = -\frac{1}{\hbar^2}\left[\hat{H},\left[\hat{H},\hat{F}_j\right]\right] \tag{5.2}$$

we come to the equation of motion for the acting force

$$\ddot{\hat{F}}_j - \frac{1}{m\alpha}\hat{F}_j = 0 \tag{5.3}$$

Identifying now the constant α to

$$\alpha = -\frac{1}{m\omega^2} \tag{5.4}$$

we see that equation (5.3) coincides with the equation for the harmonic oscillator force [1] where ω is the frequency of the oscillator.

The total energy of the classical harmonic oscillator is given by [1]

$$E = \frac{1}{2m}\left(\vec{p}(t) + \frac{\vec{G}(t)}{\omega^2}\right)^2 + \frac{1}{2m\omega^2}\left(\vec{F}^2(t) + \frac{\vec{G}^2(t)}{\omega^2}\right) \tag{5.5}$$

and the quantity $\vec{P}$ in (3.22) is given by

$$\vec{P} = \vec{p}(t) + \frac{\vec{G}(t)}{\omega^2} \tag{5.6}$$

We have earlier explained that the quantum mechanical counterpart of this quantity is simply given by the operator $-i\hbar\vec{\nabla}$. From that and (3.35) we get the quantum mechanical counterpart of the quantity $\vec{G}$ in the form

$$\hat{G}_j = \frac{2i\beta}{\hbar}\frac{\partial}{\partial f_j} \tag{5.7}$$

Substituting this and (5.6) into (3.36) we come to the following form of the Hamiltonian

$$\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} + \frac{\hat{\vec{G}}}{\omega^2}\right)^2 - \frac{\hbar^2}{4\beta}\hat{\vec{G}}^2 - \frac{\hbar^2\omega^2}{4\beta}\hat{\vec{F}}^2 \tag{5.8}$$

This quantity will be the quantum mechanical counterpart of the classical total energy (5.5) provided

$$\beta = -\frac{\hbar^2 m\omega^4}{2} \tag{5.9}$$

In our formalism the harmonic oscillator is therefore determined by the following basic quantum mechanical observables

$$\hat{X}_j = x_j - \frac{1}{m\omega^2}f_j \tag{5.10}$$

$$\hat{p}_j = -i\hbar\frac{\partial}{\partial x_j} + i\hbar m\omega^2\frac{\partial}{\partial f_j} \tag{5.11}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta_{\vec{x}} - \frac{\hbar^2 m\omega^4}{2}\Delta_{\vec{f}} + \frac{1}{2m\omega^2}\vec{f}^2 \tag{5.12}$$

$$\hat{F}_j = f_j \tag{5.13}$$

It is easy to verify that these observables satisfy the basic quantum mechanical relations (3.10) - (3.12) and have the correct Galilean transformation properties.

It is also easy to see that on states for which the condition (4.11) is satisfied we have the operator relation

$$\hat{F}_j = -m\omega\hat{X}_j \tag{5.14}$$

while on states which satisfy (4.13) we have

$$\hat{G}_j = \omega^2\hat{p}_j \tag{5.15}$$

Obviously the relations (5.14) and (5.15) coincide with the classical force law for the harmonic oscillator.

Solving the Schrödinger equation (2.5) with the Hamiltonian (5.12) we get the following expressions for the functions $\psi(\vec{q},\vec{f},t)$ in (4.1)

$$\psi(\vec{q},\vec{f},t) = exp\left(-\frac{\vec{f}^2}{2\hbar m\omega^2}\right)$$
$$\sum_{n_1,n_2,n_3} \psi_{n_1,n_2,n_3}(\vec{q})e^{i\omega_{n_1,n_2,n_3}t}\prod_{j=1}^{3} H_{n_j}\left(-\frac{f_j}{\sqrt{\hbar m\omega^3}}\right) \tag{5.16}$$

where $\psi_{n_1,n_2,n_3}(\vec{q})$ are arbitrary functions to be determined from the initial shape of the wave packet, (n_1,n_2,n_3) is a triple of integers, H_n are Hermite polynomials and

$$\omega_{n_1,n_2,n_3} = (n_1+n_2+n_3+\frac{3}{2})\omega \tag{5.17}$$

are the usual frequencies of the quantum mechanical oscillator.

Obviously, restricting the theory on the subspace of the classical force law given by the equations