IC/92/150

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

# BRST COHOMOLOGY RING IN 2D GRAVITY COUPLED TO MINIMAL MODELS

H. Kanno

and

M.H. Sarmadi

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

MIRAMARE-TRIESTE

IC/92/150

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

# BRST COHOMOLOGY RING IN 2D GRAVITY COUPLED TO MINIMAL MODELS

H. Kanno *
International Centre for Theoretical Physics, Trieste, Italy

and

M.H. Sarmadi
International Centre for Theoretical Physics, Trieste, Italy
and
INFN, Sezione di Trieste, Trieste, Italy.

ABSTRACT

The ring structure of Lian–Zuckerman states for $(q,p)$ minimal models coupled to gravity is shown to be $\mathcal{R} = \mathcal{R}_0 \otimes \mathrm{C}[w, w^{-1}]$ where $\mathcal{R}_0$ is the ring of ghost number zero operators generated by two elements and $w$ is an operator of ghost number $-1$. Some examples are discussed in detail. For these models the currents are also discussed and their algebra is shown to contain the Virasoro algebra.

MIRAMARE - TRIESTE
August 1992

* Address after July 1992: DAMTP, University of Cambridge.

## 1. Introduction

Two dimensional gravity coupled to conformal matter has been investigated in various approaches as a toy model of quantum gravity. The matrix model approach has perhaps uncovered some non-perturbative aspects and in any case gives a powerful computational method combined with the theory of integrable systems [1]. There is a more conventional method called continuum approach: in the conformal gauge the induced gravity sector is described by the Liouville theory [2]. Though the interacting quantum Liouville dynamics itself is very complicated [3], calculations based on a free field realization have given some results in agreement with the matrix model approach [4,5,6], in particular for $c = 1$.

In the BRST quantization framework, if one takes as matter sector the minimal model of BPZ, there exists an infinite tower of physical states (BRST cohomologies) for each conformal block [7]. This result is very remarkable in the sense that we have a physical state with any ghost number in contrast with the usual situation. Hence, it is crucial to clarify the origin and the implications of this infinite tower structure for a better understanding of 2D quantum gravity coupled to minimal models. Using the state-operator correspondence, we can construct physical operators (observables) from physical states (the Lian -Zuckerman states). The short distance behavior of the operator product defines a ring structure of BRST cohomologies. Thus we obtain an associative ring of observables.

For $c = 1$ matter, the existence of such discrete states or extra states is observed in several ways [8-15]. The significance of the ring structure, the so-called ground ring, and the symmetry currents acting on it was pointed out by Witten [16] . It turns out this algebraic structure is very useful for both practical calculations of correlation functions and physical interpretation of the $c = 1$ matrix model [16-19].

In this paper, we will discuss the ring structure for minimal models. An important difference arises from the existence of the infinite tower of BRST cohomologies which is absent in the $c = 1$ case. We will propose a ring structure behind this infinite tower. We show the existence of a generator $w$ with ghost number $-1$ together with its 'inverse' $w^{-1}$. By proving any power of $w$ gives a non-trivial BRST cohomology, we find the ghost number $n$ sector is given by

$$\mathcal{R}_n = w^{-n}\, \mathcal{R}_0 , \tag{1}$$

where $\mathcal{R}_0$ is the ring of ghost number zero physical operators. The ring structure of $\mathcal{R}_0$ has been discussed in [17] and will be explained below. Consequently the full ring structure is

$$\mathcal{R} = \mathcal{R}_0 \otimes \mathrm{C}[w, w^{-1}] , \tag{2}$$

which is one of main results in the present paper. As we will see below, this ring is non-commutative; the generator $w$ anti-commutes with the generators of $\mathcal{R}_0$. The infinite tower structure is essentially generated by the single element $w$. We will prove the ring structure (2) for the (2,3), (2,5), (2,7) and (3,4) minimal models by an explicit calculation of the operator product expansion. In general cases we will give an argument based on the calculation of relevant 3-point functions.

The organization of the paper is as follows. In Section 2 we briefly review the result of Lian and Zuckerman on physical states. In Section 3 we discuss how the above ring structure can follow by considering the Liouville momenta. Section 4 contains some examples for which the ring structure is proved, and also the algebra of the vector fields obtained from the currents is discussed. In Section 5 we give a discussion of the general $(q,p)$ models.

## 2. Lian-Zuckerman States and Physical Operators

In the BRST quantization procedure the physical states are defined to be BRST cohomology classes. For any conformal field theory (CFT) with the total central charge $c^{tot} = 26$, we can introduce the (Virasoro) BRST complex with coefficients in the Virasoro module $\mathcal{M}^{c=26}$. The BRST operator

$$Q_B = \oint \frac{dz}{2\pi i} : (T(z) + \frac{1}{2}T^G(z))c(z) :, \tag{3}$$

acts on the cochain space

$$C^*(Vir, \mathcal{M}) = \mathcal{M} \otimes \Lambda^{bc} , \tag{4}$$

where $T(z)$ is the energy momentum tensor for $\mathcal{M}$, and $\Lambda^{bc}$ is the Fock module of the reparametrization ghosts $(b,c)$ with energy momentum tensor $T^G(z)$. The $Z$-gradation of $C^*(Vir,\mathcal{M})$ is called ghost number. Let us consider the case

$$\mathcal{M} = L(c_{q,p}, \Delta_{m',m}) \otimes \mathcal{F}^L(Q^L, p^L) , \tag{5}$$

corresponding to 2D gravity coupled to the $(q,p)$ minimal model. $L(c_{q,p}, \Delta_{m',m})$ is the Virasoro irreducible module with central charge

$$c_{q,p} = 1 - \frac{6(q-p)^2}{qp} , \qquad (q < p,\ coprime) \tag{6}$$

and highest weight

$$\Delta_{m',m} = \frac{1}{4qp}\left[(qm' - pm)^2 - (q-p)^2\right] , \tag{7}$$

$$(1 \le m \le q-1,\ \ 1 \le m' \le p-1) .$$

We assume a free field realization of the Liouville field. $\mathcal{F}^L(Q^L, p^L)$ is the Feigin-Fuchs module with the background charge $Q^L$ and vacuum momentum $p^L$. The central charge and the Virasoro highest weight are given by

$$c = 1 + 12(Q^L)^2 , \qquad h(p^L) = -\frac{1}{2}p^L(p^L - 2Q^L) . \tag{8}$$

The condition $c^{tot} = 26$ determines $Q^L$ up to sign. For a moment we will focus on the relative BRST cohomology $H^*_{rel}(Vir, \mathcal{M})$, in which we take the subcomplex satisfying $L_0^{tot}\psi = b_0\psi = 0,\ \ \psi \in C^*(Vir, \mathcal{M})$.

If we fix the matter sector, $H^*_{rel}(Vir,\mathcal{M})$ depends on the value of the Liouville momentum $p^L$. Lian and Zuckerman proved that $H^*_{rel}(Vir,\mathcal{M})$ is non-trivial for only special discrete values of $p^L$ and that there exists a physical state for an arbitrary ghost number (i.e. an infinite tower of physical states) for each conformal block. Let $E_{m',m}(q,p)$ denote the set of highest weights appearing in the embedding diagram of Verma modules :

$$E_{m',m}(q,p) = \{a_t,\ b_t\}_{t\in Z} , \tag{9}$$

$$\begin{aligned} a_t &= \frac{1}{4qp}\left[(2qpt + qm' + pm)^2 - (q-p)^2\right] , \\ b_t &= \frac{1}{4qp}\left[(2qpt + qm' - pm)^2 - (q-p)^2\right] . \end{aligned} \tag{10}$$

The relative cohomology $H^*_{rel}(Vir,\mathcal{M})$ is non-vanishing if and only if the Liouville momentum $p^L$ satisfies

$$1 - h(p^L) \in E_{m',m}(q,p) . \tag{11}$$

$H^*_{rel}(Vir,\mathcal{M})$ is at most one-dimensional. That is we have a unique physical state (Lian-

Zuckerman state) for each $p^L$ satisfying (11). The ghost number is given by

$$n_{gh} = \pi(p^L)\, d(p^L) \,, \tag{12}$$

where

$$d(p^L) := \text{ number of arrows leading from } \; b_0 = \Delta_{m',m} \; \text{ to } \; 1 - h(p^L) \; \text{ in the embedding diagram }, \tag{13}$$

and

$$\pi(p^L) := sign\ (p^L - Q^L) \,. \tag{14}$$

Note that for each weight $\Delta \in E_{m',m}(q,p)$ there exist two momenta such that $1 - h(p^L) = \Delta$, which are distinguished from each other by $\pi(p^L)$. Therefore, preparing two copies of the embedding diagram, one for the states with $\pi(p^L) > 0$ and the other for $\pi(p^L) < 0$, we can make a one-to-one correspondence between the Lian-Zuckerman states and the nodes of the diagrams .

One can construct physical operators (observables) $\mathcal{O}$ from the Lian-Zuckerman states by the standard prescription in CFT. The translation to the operators enables us to introduce a ring structure of BRST cohomology. In the coulomb gas description of the matter sector in terms of a free field $X(z)$, the matter Virasoro generators are expressed in terms of the oscillators $\partial^n X$. The Liouville vacuum is represented by the vertex operator $e^{\beta\phi(z)}$ $(\beta = p^L)$. Then the physical operator in general takes the form

$$\mathcal{O} = \mathcal{P}\big[\partial X, \partial\phi, b, c\big]\ \Phi_{m,m'}\ e^{\beta\phi} \,, \tag{15}$$

where $\mathcal{P}$ is a differential polynomial with a definite conformal weight and $\Phi_{m,m'}$ is a matter primary field. (In the following we give explicit examples.) Compared with the original states, the ghost number of the operators increases by one due to the gap between the $SL_2$-invariant ghost vacuum and the highest weight state of $\Lambda^{bc}$. In other words we will consider the observables in the 0-form version, not in the 2-form version. Let $H^*_{LZ}\ (Vir,(q,p))$ denote the set of observables in 2D gravity coupled to the $(q,p)$ model. Since the $(q,p)$ model consists of $\frac{1}{2}(q-1)(p-1)$ primaries or conformal blocks, we have

$$\dim\ H^n_{LZ}\ (Vir,(q,p)) = (q-1)(p-1) \,, \tag{16}$$

for any ghost number $n$. Note that in the above we only discussed the (chiral) relative cohomology. As we will see below, knowing the states in the relative cohomology it is easy to obtain the full set of states in the absolute cohomolgy which has twice as many states.



## 3. Ring Structure and the Spectrum of the Liouville Momentum

The short distance behavior of operator product (OPE) of two observables

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \cdots + (z-w)^{-n}A_n(w) + \cdots + A_0(w) + O((z-w)) \,, \tag{17}$$

defines a ring structure of $H^*_{LZ} = \oplus_{n\in Z} H^n_{LZ}$. The expansion coefficient $A_n(w)$ commutes with the BRST charge $Q_B$ and has conformal weight $(-n)$, since $\mathcal{O}_1$ and $\mathcal{O}_2$ are BRST cohomologies with weight 0. But due to the relation $\big[Q_B, b_0\big] = L_0^{tot}$, only $\mathcal{O}_3 \equiv A_0$ may give a BRST non-trivial operator. Thus we get the ring structure

$$H^n_{LZ} \times H^m_{LZ} \longrightarrow H^{n+m}_{LZ} \,, \quad (n,m \in Z) \tag{18}$$

defined by the OPE modulo BRST exact terms

$$\mathcal{O}_1(z)\ \mathcal{O}_2(w) \sim \mathcal{O}_3(w) + \big[Q_B, *\big] \,, \tag{19}$$

which will be denoted simply by

$$\mathcal{O}_1 \cdot \mathcal{O}_2 = \mathcal{O}_3 \,. \tag{20}$$

The fundamental OPE of the vertex operators of the free scalar field:

$$:e^{\alpha\phi(z)}::e^{\beta\phi(w)}: = (z-w)^{-\alpha\beta} :e^{\alpha\phi(z)+\beta\phi(w)}: \,, \tag{21}$$

is valid by assumption of a free field realization of the Liouville field. The OPE (21) implies the conservation of the Liouville charge under the ring multiplication. The product of two cohomologies $\mathcal{O}_1[\beta_1], \mathcal{O}_2[\beta_2]$ with charges $\beta_1$ and $\beta_2$ can generate another cohomology $\mathcal{O}_3$ with charge $\beta_1 + \beta_2$:

$$\mathcal{O}_1[\beta_1] \cdot \mathcal{O}_2[\beta_2] = \mathcal{O}_3[\beta_1 + \beta_2] \,. \tag{22}$$

Hence, the whole 'spectrum' of the Liouville charge of observables in $H^*_{LZ}$ gives some insight into a possible choice of generators of the cohomology ring.

Let us introduce the standard notation for the Liouville charges;

$$\beta_+ := \sqrt{\frac{2p}{q}}\,, \quad \beta_- := \sqrt{\frac{2q}{p}}\,, \quad (\beta_+\beta_- \;=\; 2) \tag{23}$$

$$\begin{aligned} \beta_0 &:= \beta_+ \;+\; \beta_- \;=\; 2Q^L\,, \\ \beta_{n',n} &:= \frac{1}{2}\big[(1-n)\beta_+ \;+\; (1-n')\beta_-\big]\,, \end{aligned} \tag{24}$$

$\beta_{n',n}$ satisfies the following relation,

$$\begin{aligned} \beta_{n'\pm p, n\mp q} &= \beta_{n',n}\,, \\ \beta_{n',n} + \beta_{k',k} &= \beta_{n'+k'-1,n+k-1}\,. \end{aligned} \tag{25}$$

Solving the equation

$$\begin{aligned} 1+\frac{1}{2}\beta(\Delta)\big(\beta(\Delta)-\beta_0\big) &= \Delta\,, \\ \Delta = a_t,\; b_t &\in E_{m',m}(q,p)\,, \end{aligned} \tag{26}$$

we obtain the following 'spectrum' of the Liouville charge:

$$\begin{aligned} \beta^{\pm}(a_t) &= \frac{1}{2}\{\beta_0 \pm |(tq+m)\beta_+ + (tp+m')\beta_-|\}\,, \\ \beta^{\pm}(b_t) &= \frac{1}{2}\{\beta_0 \pm |(tq-m)\beta_+ + (tp+m')\beta_-|\}\,, \end{aligned} \tag{27}$$

where the two branches of the solution are given by the sign in front of the absolute value. Let $k$ be a positive integer. The ghost number $n_{gh}$ for each Liouville charge in (27) is given by (cf. Eq.(12)):

$$\begin{aligned} \beta^+(a_{k-1}),\quad \beta^+(a_{-k}) &\longrightarrow n_{gh} = 2k\,, \\ \beta^+(b_k),\quad \beta^+(b_{-k}) &\longrightarrow n_{gh} = 2k+1\,, \\ \beta^-(a_{k-1}),\quad \beta^-(a_{-k}) &\longrightarrow n_{gh} = -2k+2\,, \\ \beta^-(b_k),\quad \beta^-(b_{-k}) &\longrightarrow n_{gh} = -2k+1\,, \\ \beta^+(b_0),\quad \beta^-(b_0) &\longrightarrow n_{gh} = 1\,. \end{aligned} \tag{28}$$

Removing the absolute value by the condition $1 \le m \le q-1,\ 1 \le m' \le p-1$, we see the Liouville charges are $\beta_{m'-np,m-nq}$ and $\beta_{(p-m')-np,(q-m)-nq}$ for even ghost number ($n_{gh} = 2n$). On the other hand, for odd ghost number ($n_{gh} = 2n+1$), the charges are $\beta_{m'-np,-m-nq}$ and $\beta_{-m'-np,m-nq}$. Note that the $n_{gh} = 1$ case reproduces the usual gravitational dressing $\beta_{m',-m}$ and $\beta_{-m',m}$ for the primary field. Making use of the relation (25) for $\beta_{n',n}$, we can parametrize the $(q-1)(p-1)$ Liouville charges of observables with ghost number $n_{gh}$ as follows:

$$\begin{aligned} \beta(s,s') &= -\,n_{gh}\,\beta_w \;+\; s'\beta_x \;+\; s\beta_y\,, \\ &(0 \le s \le q-2\,,\quad 0 \le s' \le p-2)\,, \end{aligned} \tag{29}$$

where we have defined

$$\begin{aligned} \beta_w &:= \beta_{p+1,1} \;=\; \beta_{1,q+1}\,, \\ \beta_x &:= \beta_{2,1}\,,\quad \beta_y \;:=\; \beta_{1,2}\,. \end{aligned} \tag{30}$$

We will restrict ourselves to matter blocks whose labels are inside the Kac-table. Note that the labels of some Liouville charges, for instance $\beta_w$, can sit outside the table.

Let $w, x$ and $y$ denote the observables with the charges $\beta_w, \beta_x$ and $\beta_y$, respectively. $x, y \in H^0_{LZ}$ and $w \in H^{-1}_{LZ}$. Then the linear spectrum (29) means a possible identification

$$\mathcal{O}\,\big[\beta(s,s')\big] \;=\; w^{-n_{gh}} x^{s'} y^{s}\,, \tag{31}$$

for the observables with ghost number $n_{gh}$, *if the right hand side does not vanish modulo BRST exact terms.* The generators $x$ and $y$ belonging to the ghost number zero sector can be obtained by an $SO(2,C)$ rotation from the generators of the chiral ground ring for $c = 1$ matter [17,20]. In the Coulomb gas description of the matter sector, they are given by the following expressions:

$$\begin{aligned} x &= \big(bc + (i\alpha_{1,2}\partial X + \beta_{1,2}\partial\phi)\big)e^{i\alpha_{2,1}X+\beta_{2,1}\phi}\,, \\ y &= \big(bc + (i\alpha_{2,1}\partial X + \beta_{2,1}\partial\phi)\big)e^{i\alpha_{1,2}X+\beta_{1,2}\phi}\,, \end{aligned}$$

where $\alpha_{n',n} = \frac{1-n}{2}\alpha_+ \;+\; \frac{1-n'}{2}\alpha_-$, $\alpha_+ = \beta_+$ and $\alpha_- = -\beta_-$. (The background matter charge is equal to $-\alpha_0$ where $\alpha_0 = \alpha_+ \;+\; \alpha_-$.) It is easy to see that $x^{p-1}$ and $y^{q-1}$ have matter charges which are not inside the Kac-table and if one is restricting oneself to only those operators whose charges $\alpha_{m',m}$ are inside the table, i.e. $1 \le m \le q-1$ and $1 \le m' \le p-1$, then one should set $x^{p-1} = y^{q-1} = 0$. Therefore $\mathcal{R}_0$ has a simple structure with some connection with the chiral ring for $c = 1$ matter. The novel feature of the minimal models arises from the generator $w^{\pm 1}$ with $n_{gh} = \mp 1$. To establish the identification (31) we have to prove that the product of generators indeed give BRST non-trivial cohomologies, which is the problem we will be concerned with in the following sections.

## 4. EXAMPLES

In this section we will discuss the above ring structure in some examples.

*(2,3) Model = Pure Liouville Gravity*

Let us take the $(2,3)$ model ($c_{2,3}=0$) as a simple example. We will see by explicit calculations that the ring structure of $H^*_{LZ}(Vir,(2,3))$ is exactly the one proposed in the last section. In this case the only matter primary is the identity $(1,1)=(1,2)$ sector. The matter Verma module $M(c=0,\Delta=0)$ has its first two singular states at level one and two ($a_0=1, a_{-1}=2$). Hence, we can take a representative of BRST cohomologies in the irreducible module such that there is no matter oscillator. (Recall that $L_{-n}$ $(n>0)$ is generated by $L_{-1}$ and $L_{-2}$.) In this sense the matter part is trivial as expected and we can think of it as pure Liouville gravity. The basic parameters for the Liouville charges are

$$\begin{aligned} \beta_+ &= \beta_{1,-1} = \sqrt{3}\,, \quad \beta_- = \beta_{-1,1} = \frac{2}{\sqrt{3}}\,, \\ \beta_x &= \beta_{2,1} = -\frac{1}{\sqrt{3}}\,, \quad \beta_w = \beta_{1,3} = -\sqrt{3}\,, \end{aligned} \tag{32}$$

The Lian-Zuckerman states with $n_{gh}=-1$ exist at level one and two. The one with charge $\beta_{1,1}=0$ gives the identity operator $\mathbf{1}$. By an explicit construction of the other state with charge $\beta_{2,1}$, we get a ghost number zero observable

$$x = (bc - \frac{\sqrt{3}}{2}\partial\phi)\, e^{-\frac{1}{\sqrt{3}}\phi}\,. \tag{33}$$

It is the existence of a vanishing null vector at level two

$$(L_{-2} + \frac{3}{2}L_{-1}{}^2)\,|p^L = -\frac{1}{\sqrt{3}}\rangle\,, \tag{34}$$

which makes $x$ be in the kernel of $Q_B$. The observable $w$ with charge $\beta_{1,3}$ comes from the Lian-Zuckerman state with $n_{gh}=-2$ at level $b_1=5$. Explicitly,

$$w = (b\partial bc - \frac{1}{\sqrt{3}}b\partial^2\phi + \frac{1}{2\sqrt{3}}\partial b\partial\phi + \frac{1}{6}\partial^2 b)\, e^{-\sqrt{3}\phi}\,. \tag{35}$$

The vanishing vectors at level 3 and 4,

$$\begin{aligned} &(L_{-3} + L_{-1}L_{-2} + \frac{1}{6}L_{-1}{}^3)\,|p^L = -\sqrt{3}\rangle\,, \\ &(L_{-4} + L_{-2}{}^2 + \frac{2}{3}L_{-1}{}^2L_{-2} + \frac{1}{12}L_{-1}{}^4)\,|p^L = -\sqrt{3}\rangle\,, \end{aligned} \tag{36}$$

are responsible for the BRST invariance of $w$. On the other hand, the observables with



$n_{gh}=1$ are nothing but the Liouville 'screening' operators multiplied by the $c$-ghost:

$$s_+ = ce^{\beta_+\phi}\,, \quad s_- = ce^{\beta_-\phi}\,. \tag{37}$$

By computing the OPE, we see

$$s_+(z)w(0) \sim -\frac{1}{6}\cdot\mathbf{1}\,, \quad s_-(z)w(0) \sim -\frac{1}{3}x(0)\,. \tag{38}$$

This proves the identification $s_+ = -\frac{1}{6}w^{-1}$ and $s_- = -\frac{1}{3}xw^{-1}$. For each fixed ghost number we have exactly two observables in the relative cohomology. The very existence of the inverse $w^{-1}$ in the cohomology ring $H^*_{LZ}$ is enough to prove that for any $n\in Z$, neither $w^n$ nor $w^n x$ vanishes modulo BRST exact terms. Therefore, $x\in H^0_{LZ}$ and $w\in H^{-1}_{LZ}$ are two of the generators of the ring. For example, we can obtain the other ghost number $-1$ observable by taking the OPE coefficient:

$$\begin{aligned} w(z)x(0) \sim \Big[&\frac{1}{48}\partial^4 b - \frac{2}{9}\partial^3 b(bc) + \frac{3}{4}\partial^2 b(\partial bc) + \frac{1}{4}\partial b(b\partial^2 c) + \frac{1}{2\sqrt{3}}\partial^2 b(bc)\partial\phi \\ &- \frac{\sqrt{3}}{4}\partial^2 b\partial^2\phi + \partial b(bc)(\frac{5}{2\sqrt{3}}\partial^2\phi - \frac{2}{3}(\partial\phi)^2) + \frac{1}{8\sqrt{3}}\partial b(\partial^3\phi + 2(\partial\phi)^3) \\ &+ b(\frac{-1}{12\sqrt{3}}\partial^4\phi + \frac{1}{2}(\partial^2\phi)^2 + \frac{1}{12}\partial^3\phi\partial\phi - \frac{1}{2\sqrt{3}}\partial^2\phi(\partial\phi)^2)\Big]\exp(\frac{-4}{\sqrt{3}}\phi)\,, \end{aligned} \tag{39}$$

which corresponds to the Lian-Zuckerman states at level $b_{-1}=7$.

So far we considered only the relative cohomology. It contained the operators $w^n$ and $xw^n$ $(n\in Z)$. The absolute cohomology contains twice as many physical operators. As in the $c=1$ case of ref. [19], the other half of the operators are conveniently obtained by multiplying the above operators by the physical operator

$$a = c\partial\phi + \frac{1}{2}\beta_0\partial c\,. \tag{40}$$

Therefore the full set of chiral operators are of the form $w^n, xw^n, aw^n$ and $axw^n$. Note that $aw=-wa$ which is due to $a$ and $w$ both carrying nonzero ghost numbers, and $xw=-wx$ which, even though $x$ has ghost number zero, is due to the anticommutation of the exponentials. However $ax=xa$. By introducing appropriate cocycle factors, say the Pauli matrices $\sigma^i$, one can instead have two anti-commuting generators, $\tilde{x},\tilde{a}$ and one commuting one $\tilde{w}$. As we will see shortly it is more natural to use these latter generators.

Using the above operators we can construct currents and the corresponding charges which act on the cohomology ring as derivations. If $\psi^{(n)}$ is a physical operator of ghost number $n$ then $j^{(n-1)}(z) = \oint dz' b(z')\psi(z)$ is an operator of ghost number $(n-1)$ and conformal weight one whose BRST variation is a total derivative. Therefore for some other physical operator $\tilde{\psi}$ the BRST variation of of $\oint dz j^{(n)}\tilde{\psi}^{(m)}$ vanishes and if it is not exact then it corresponds to a physical operator of ghost number $(m+n)$. For example, the operator $s_- = ce^{\beta-\phi}$ gives rise to the current $e^{\beta-\phi}$ and by considering the action of $\oint dz e^{\beta-\phi}$ on physical operators one sees that it should be identified with the vector field $xw^{-1}\partial_a$. Similarly from the operator $w$ one obtains the current $b\partial b e^{-\sqrt{3}\phi}$ whose integral should be identified with the vector $w\partial_a$. By considering the other physical operators one can show that one has the following four sets of vector fields:

$$
\begin{aligned}
G_n &= w^n \partial_a, \\
H_n &= x w^n \partial_a, \\
L_n &= -w^n(w\partial_w + 1/3x\partial_x - na\partial_a), \\
K_n &= -xw^n(w\partial_w - (n+1/3)a\partial_a).
\end{aligned}
\tag{41}
$$

The above vector fields are obtained respectively from the operators $w^n, xw^n, aw^n$ and $axw^n$. The generators $L_n$ satisfy a Virasoro algebra $[L_n, L_m] = (n-m)L_{n+m}$. In order to write the other commutators in a more recognizable form, we first write the vector fields in terms of $\tilde{w}$, $\tilde{x}$ and $\tilde{a}$ which we denote by $\tilde{L}, \tilde{K}, \tilde{G}$ and $\tilde{H}$. These set of generators satisfy the following algebra:

$$
\begin{aligned}
[\tilde{L}_n, \tilde{L}_m] &= (n-m)\tilde{L}_{n+m}, \\
[\tilde{L}_n, \tilde{K}_m] &= (n-m+1/3)\tilde{K}_{n+m}, \\
[\tilde{L}_n, \tilde{G}_m] &= -(n+m)\tilde{G}_{n+m}, \\
[\tilde{L}_n, \tilde{H}_m] &= (-n-m-1/3)\tilde{H}_{n+m}, \\
[\tilde{K}_n, \tilde{K}_m] &= 0, \\
[\tilde{K}_n, \tilde{G}_m] &= -(n+m+1/3)\tilde{H}_{n+m}, \\
[\tilde{K}_n, \tilde{H}_m] &= 0.
\end{aligned}
\tag{42}
$$

Therefore, the vectors $\tilde{K}, \tilde{G}$ and $\tilde{H}$ are primaries of the Virasoro generators $\tilde{L}_n$ with weights 2,0 and 0 respectively. Moreover $\tilde{G}$ has integer modes but $\tilde{K}$ and $\tilde{H}$ have $n+1/3$ and $n-1/3$ modings. The vanishing of the commutator of $\tilde{K}$ with itself and with $\tilde{H}$ is simply because $x^2 = 0$.



*(2,5) Model*

Our next example is the (2,5) model ($c_{2,5} = -\frac{22}{5}$), the Lee-Yang edge singularity model coupled to gravity. This is complicated enough, since the matter field enters in non-trivial way. Our strategy to prove the ring structure is the same as before. The generator $w$ is constructed from the Lian-Zuckerman state at level 7 in the $(1,1) = (1,4)$ sector. In the course of finding the Lian-Zuckerman state we must use the existence of vanishing null vectors at levels 3 and 6 in the Liouville Fock module. We also use the fact that there are singular vectors at level 1 and 4 in the identity sector of the matter Verma module. The BRST variation of the Lian-Zuckerman state is a linear combination of descendents of these singular vectors. (Of course, the Lian-Zuckerman state is defined up to descendents of singular vectors in addition to the freedom of BRST exact terms.) Expanded in the ghosts, $w$ takes the form

$$
\begin{aligned}
w = &\Big[\frac{a_1}{4!}\partial^4 b + \frac{a_2}{3!}\partial^3 b(bc) + \frac{a_3}{2!}\partial^2 b(\partial bc) + \frac{1}{3!}\mathcal{L}_{-1}\partial^3 b + \frac{1}{2!}\widetilde{\mathcal{L}_{-1}}\partial^2 b(bc) \\
&+ \frac{1}{2!}\mathcal{L}_{-2}\partial^2 b + \widetilde{\mathcal{L}_{-2}}\partial b(bc) + \mathcal{L}_{-3}\partial b + \mathcal{L}_{-4} b\Big] \exp(-\sqrt{5}\phi) ,
\end{aligned}
\tag{43}
$$

where $5a_1 + 3a_2 + a_3 = 0$ and $\mathcal{L}_{-n}$ ($\widetilde{\mathcal{L}_{-n}}$) is a differential polynomial in both the matter and the Liouville fields with conformal weight $n$. The explicit form is presented in the appendix. $\mathcal{L}_{-n}$ depends on the parameters $a_1$ and $a_2$, one of which corresponds to an overall factor. We have a one-parameter family of BRST cohomologies. The point $3a_1 + 2a_2 = 0$ gives a BRST trivial case. (to be precise, we have already fixed one freedom of adding an exact term by requiring that there is no $\partial c(b\partial b)$-term.) Hence the BRST cohomology class is unique in accord with the general theorem of Lian and Zuckerman.

Among the gravitationally dressed primaries in $H^{+1}_{LZ}$, we take

$$
\mathcal{O}_1 = ce^{\sqrt{5}\phi} , \quad \mathcal{O}_\sigma = ce^{\frac{4}{\sqrt{5}}\phi + \frac{1}{\sqrt{5}}X} ,
\tag{44}
$$

which are in the identity and the spin operator sectors, respectively. A little tedious calculation of OPE's gives the following results:

$$
\begin{aligned}
\mathcal{O}_1 \cdot w &= -\frac{3}{16}(3a_1 + 2a_2) \cdot 1 , \\
\mathcal{O}_\sigma \cdot w &= -\frac{3}{16}(3a_1 + 2a_2) \cdot \Big(bc - \frac{\sqrt{5}}{2}(\partial\phi + i\partial X)\Big) e^{\frac{1}{\sqrt{5}}(iX - \phi)} .
\end{aligned}
\tag{45}
$$

The OPE *coefficients* vanish *if* and *only if* $w$ is BRST trivial. For BRST *non*-trivial $w$, the OPE (45) again proves the existence of $w^{-1} = \mathcal{O}_1$ in the cohomology ring. Futhermore, in the second OPE coefficient we recognize the generator $x \in H^0_{LZ}$.

In order to write the operator $w$ or to check the non vanishing of the product $\mathcal{O}_1 \cdot w$ , we could have alternatively tried to write down the vector feild $\oint dz j^{(-2)} = w\partial_a$ and then consider its action on $a$ or $aw^{-1}$. In fact this is computationally simpler, and for the next two models for which the expressions become more involved we will do this. The current $j^{(-2)}$ can be obtained by demanding that its BRST variation be a total derivative up to null states. For the present model it has the expression:

$$j^{(-2)}(z) = \left[ [(\frac{-3}{2\sqrt{5}}\partial^2\phi - \frac{1}{2}(\partial\phi)^2) - 2(\frac{3i}{2\sqrt{5}}\partial^2 X - \frac{1}{2}(\partial X)^2)]b\partial b + \frac{7}{30}b\partial^3 b - \frac{3}{10}\partial b\partial^2 b \right] e^{-\sqrt{5}\phi}. \tag{46}$$

Note that $j$ does not contain the freedom in the choice of the cohomology class. This is due to the fact that in this case for the exact states $Q\chi$ where $\chi = L_{-1}b_{-1}b_{-2}b_{-3}c_1| - \sqrt{5} >$ or $\chi = b_{-1}b_{-2}b_{-4}c_1| - \sqrt{5} >$, we have $b_{-1}Q\chi = L_{-1}^{total}\chi$ because $b_{-1}\chi = 0$.

*(2,7) model*

In this case, $c_{2,7} = -\frac{68}{7}$, the generator $w$ is constructed from the states at level 9 in the (1,1)=(1,6) sector. Since the expression for $w$ contains many terms, it is more convenient to write the current for $w\partial_a$ and then show that its action on $a(z)c(0)e^{\sqrt{7}\phi} = -\frac{5}{2\sqrt{7}}\partial cce^{\sqrt{7}\phi}$ is nonzero. As we mentioned above, this current is obtained by requiring that its BRST variation to be a total derivative up to matter and Liouville null states. These null states are at level 6 for the matter sector and level 3 for the Liouville sector. For the current one obtains the following expression:

$$\begin{aligned} j^{-2}(z) = [&G_{-4}b\partial b + \frac{1}{2!}G_{-3}b\partial^2 b + \frac{1}{3!}G_{-2}b\partial^3 b + \frac{1}{5!}(-\frac{17}{7}\eta + \frac{297}{49}\epsilon)b\partial^5 b \\ &+ \frac{1}{2!}\tilde{G}_{-2}\partial b\partial^2 b + \frac{1}{4!}(\frac{8}{7}\eta - \frac{90}{49}\epsilon)\partial b\partial^4 b + \frac{1}{2!3!}(-\frac{9}{7}\eta + \frac{105}{49}\epsilon)\partial^2 b\partial^3 b]e^{-\sqrt{7}\phi} \end{aligned} \tag{47}$$

The explicit expressions for $G_{-n}$ and $\tilde{G}_{-n}$ are given in the appendix. In this expression, setting $\epsilon = 0$ one obtains a BRST-trivial current. The action of $\oint j^{(-2)}dz$ on $aw^{-1}$ gives $-\frac{18}{49}\epsilon$ which again verifies that $w^{-1} = -\frac{49\sqrt{7}}{45\epsilon}ce^{\sqrt{7}\phi}$.

All the above examples were models of type $(2,p)$. For these models the ring is generated by $w, x$ and $a$ where $x^{p-1} = 0$, $wa = -aw$ and $wx = -xw$. The vector fields in eq.(41) which we obtained for the (2,3) model are easily generalized to $(2,p)$ models as follows. The ones obtained from the operators $x^i w^n$ have the form

$$K_n^{(i)} = -x^i w^n (w\partial_w + \frac{1}{p}x\partial_x - (n + \frac{i}{p})a\partial_a) , \tag{48}$$



and the ones obtained from the operators $ax^iw^n$ are

$$G_n^{(i)} = x^i w^n \partial_a . \tag{49}$$

After introducing the appropriate cocycle factors as in the (2,3) model, we find the following algebra:

$$\begin{aligned} [\tilde{K}_n^{(i)}, \tilde{K}_m^{(j)}] &= (n - m + \frac{i-j}{p})\tilde{K}_{n+m}^{(i+j)} , \\ [\tilde{K}_n^{(i)}, \tilde{G}_m^{(j)}] &= -(n + m + \frac{i+j}{p})\tilde{G}_{n+m}^{(i+j)} , \\ [\tilde{G}_n^{(i)}, \tilde{G}_m^{(j)}] &= 0 . \end{aligned}$$

Therefore $\tilde{K}_n^{(0)}$ generate a Virasoro algebra under which $\tilde{K}_n^{(i)}$ and $\tilde{G}_m^{(i)}$ are primaries of weights two and zero respectively.

*(3,4) model*

As a final example we consider the Ising model coupled to gravity. Again we will write the current $w\partial_a$ where in this case $w$ is made out of states at level 7 and it is in the energy sector (2,1)=(1,3). The null states are at levels 2 and 3 for the matter sector and at levels 4 and 5 for the Liouville sector. We obtain the following expression for the current:

$$\begin{aligned} j^{(-2)}(z) = \Big[ &[(-\frac{5}{2\sqrt{6}}\partial^2\phi - \frac{1}{2}(\partial\phi)^2) - \frac{11}{9}(-\frac{3}{2\sqrt{6}}\, i\partial^2 X - \frac{1}{2}(\partial X)^2)]b\partial b \\ &- \frac{4}{3\sqrt{6}}\, i\partial X b\partial^2 b + \frac{1}{3}b\partial^3 b - \frac{11}{12}\partial b\partial^2 b \Big] e^{-2/\sqrt{6}\, iX - \sqrt{6}\phi}. \end{aligned} \tag{50}$$

The action of $\oint j^{(-2)}dz$ on the state $a(z)c(0)e^{3i/\sqrt{6}X+\sqrt{6}\phi} = -\frac{5}{2\sqrt{6}}\ \partial cce^{3i/\sqrt{6}X+\sqrt{6}\phi}$ gives $\frac{1}{2}e^{i\alpha_0 X}$, which again verifies that $w^{-1} = -4\sqrt{6}/5\ ce^{3i/\sqrt{6}X+\sqrt{6}\phi}$. Therefore, for this model the states are of the type $x^iy^jw^n$ and $ax^iy^jw^n$ $(n \in Z)$ where $x^3 = 0$ and $y^2 = 0$.

## 5. Discussion

For the proof of the ring structure

$$\mathcal{R} \;=\; \mathcal{R}_0 \otimes \mathbf{C}[w, w^{-1}] , \tag{51}$$

the existence of the 'inverse' $w^{-1}$ is a crucial point. The spectrum of the Liouville momentum discussed in Sect.3 tells us that there is an observable $v$ in $H_{LZ}^{+1}$ with the desired momentum $-\beta_w$. The product $w \cdot v$ has vanishing momentum and, hence, is proportional to the identity.

The problem is whether the proportionality constant is non-vanishing or not. In the general $(q,p)$ model, by looking at the charge $\beta_w$, we see that the observable $w$ belongs to the $(1,p-1)=(q-1,1)$ sector of the matter conformal block. (Note that $w$ is not necessarily in the identity sector.) However, an explicit construction of the observable $w$ gets involved in general, since we have to manage singular vectors at higher levels. We cannot make an OPE calculation without knowing the form of $w$ explicitly. Instead of OPE coefficients, however, one may consider an appropriate three point function on the sphere. If there exists a non-vanishing three point function containing $w$ and $v$, the proportionality constant cannot vanish and we can identify $v$ with the inverse of $w$. By ghost number counting the three point function $\langle wv\mathcal{O}\rangle$ can be non-zero only for observables $\mathcal{O}$ with ghost number three. For such a correlation function with matching ghost number, we can reduce it to a correlation function of Dotsenko-Kitazawa type by making use of the descent equation trick discussed in [21]. This descent equation comes from the double complex consisting of two coboundary operators, the BRST operator $Q_B$ and the operator $Q_F$ of Felder's resolution of the Virasoro irreducible module. Noting that the Liouville charge does not change in 'descending' the descent equation, we can identify relevant Dotsenko-Kitazawa type operators which are products of matter and Liouville vertex operators. (The matter momentum is not restricted to the inside of the Kac table.) If one uses only the operators in relatve cohomology then after using the descent equations only the operators of Dotsenko-Kitazawa type will appear and then one can use the results of ref. [5] on the calculation of these types of correlators to conclude that $\langle wv\mathcal{O}\rangle$ is non-vanishing.

In ref. [5], in calculating the correlators, a continuation to a negative number of screening operators had to be performed. Since this issue is not well understood, it is preferable to modify the above arguement such that no insertions of screening operators are required. In ref. [21], in relating the correlators of Lian-Zuckerman operators to those of Dotsenko-Kitazawa operators, only the states in relative cohomology were used. However, if one does not restrict oneself to using the states which are only in relative cohomology then it is possible to write correlators which require no screening operators. Let us consider the general $(q,p)$ minimal model. The operator

$$\mathcal{O}^{(3)}=av^2x^{p-2}y^{q-2},$$

which has ghost number three and matter and Liouville charges equal to $\alpha_0$ and $\beta_0$, has a non-vanishing one-point function. In order to show that the three point function $\langle w\ v\ \mathcal{O}^{(3)}\rangle$



is non-zero, we first use the descent equation

$$\partial^2 cce^{\beta_0\phi}=[Q_B\ ,\ c\partial Xe^{\beta_0\phi}], \tag{52}$$

then the action of $Q_B$ on $w$ gives:

$$[Q_B\ ,\ w]=[Q_F\ ,\ \Phi^{(0)}], \tag{53}$$

where $\Phi^{(0)}$ is an operator of ghost number zero and whose matter charge is outside the Kac-table. In the above equation one has two choices for $Q_F$, namely $Q_+$ and$Q_-$ where $Q_\pm=\oint e^{i\alpha_\pm X}$ are the two screening operators in the matter sector. Choosing $Q_F=Q_-$ then, by a proper choice of the representative for $w$, one can take $\Phi^{(0)}=x^p$‡ ( The choice $Q_F=Q_+$ corresponds to taking $\Phi^{(0)}=y^q$.) Now the action of $Q_F$ on $c\partial Xe^{\beta_0\phi}$ gives the state $ce^{i\alpha_{-1,1}X+\beta_0\phi}$. Moreover, it is easy to show:

$$x^p\ v\ \sim\ ce^{i\alpha_{1,-1}X}. \tag{54}$$

Therefore, we have reduced the above three-point function to the two-point function

$$\langle\alpha_{1,-1},0|\ c_{-1}c_0c_1\ |\alpha_{-1,1},\beta_0\rangle,$$

which is obviously non-zero and thus implies that the product $v.w$ does not vanish.

Finally we note that for the general $(q,p)$ model, the vector fields $\tilde{K}_n^{i,j}$ and $\tilde{G}_n^{i,j}$ which are obtained from the operators $ax^iy^jw^n$ and $x^iy^jw^n$ should have the froms*:

$$K_n^{i,j}=-x^iy^jw^n\left(w\partial_w+\frac{1}{p}x\partial_x+\frac{1}{q}y\partial_y-(n+\frac{i}{p}+\frac{j}{q})a\partial_a\right)\ , \tag{55}$$

and

$$G_n^{i,j}=x^iy^jw^n\partial_a\ . \tag{56}$$

In addition to the $(2,p)$ models discussed in the previous section, we have checked these expressions for a few currents of the other models. These vector fields $\tilde{K}_n^{0,0}$ satisfy a Virasoro algebra.

‡ In the previous sections, when defining the ring we set $x^{p-1}=y^{q-1}=0$ since in the Coulomb gas description of the matter sector $x^{p-1}$ and $y^{q-1}$ have charges outside the Kac-table and we were restricting ourselves only to operators inside the table. However, when discussing Felder's resolution one needs to allow for operators like $x^p$ and $y^q$ whose charges are outside the table.

* Recall from the previous section that $\tilde{K}$ and $\tilde{G}$ have the same expressions as $K$ and $G$ but written in terms of $\tilde{w}$, $\tilde{x}$, $\tilde{y}$ and $\tilde{a}$. These latter generators include the cocycle factors. The original generators of the ring satisfy the (anti-)commutations $wx=-xw$, $wy=-yw$, $wa=-aw$ and $xy=yx$. After including the cocycle factors, $\tilde{w}$ is a commuting generator whereas $\tilde{x}$, $\tilde{y}$ and $\tilde{a}$ are anti-commuting ones.

## Acknowledgments

The authors would like to thank V.K. Dobrev, E. Gava, S. Govindarajan, K.S. Narain and C. Vafa for discussions. They would also like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.



## Appendix

The generator of ghost number -1 in the (2,5) model has the form:

$$w = \Big[\frac{a_1}{4!}\partial^4 b + \frac{a_2}{3!}\partial^3 b(bc) + \frac{a_3}{2!}\partial^2 b(\partial bc) + \frac{1}{3!}\mathcal{L}_{-1}\partial^3 b + \frac{1}{2!}\widetilde{\mathcal{L}_{-1}}\partial^2 b(bc) + \frac{1}{2!}\mathcal{L}_{-2}\partial^2 b + \widetilde{\mathcal{L}_{-2}}\partial b(bc) + \mathcal{L}_{-3}\partial b + \mathcal{L}_{-4} b\Big] \exp(-\sqrt{5}\phi)\,.$$

where

$$5a_1 + 3a_2 + a_3 = 0\,.$$

Here we the expressions for $\mathcal{L}_{-n}$:

$$\mathcal{L}_{-1} = -\frac{\sqrt{5}}{12}(7a_1 + 3a_2)\partial\phi\,,$$

$$\widetilde{\mathcal{L}_{-1}} = \frac{\sqrt{5}}{6}(10a_1 + 3a_2)\partial\phi\,,$$

$$\begin{aligned}\mathcal{L}_{-2} = &-\frac{1}{16}(25a_1 + 22a_2)\Big(-\frac{3}{2\sqrt{5}}\partial^2\phi - \frac{1}{2}(\partial\phi)^2\Big) + \frac{1}{8}(25a_1 + 14a_2)\Big(\frac{3}{2\sqrt{5}}i\partial^2 X - \frac{1}{2}(\partial X)^2\Big)\\ &-\frac{3}{32}(3a_1 + 2a_2)\Big(-\sqrt{5}\partial^2\phi + 5(\partial\phi)^2\Big)\,,\end{aligned}$$

$$\begin{aligned}\widetilde{\mathcal{L}_{-2}} = &\frac{25}{16}(3a_1 + 2a_2)\Big(-\frac{3}{2\sqrt{5}}\partial^2\phi - \frac{1}{2}(\partial\phi)^2\Big) - \frac{25}{8}(3a_1 + 2a_2)\Big(\frac{3}{2\sqrt{5}}i\partial^2 X - \frac{1}{2}(\partial X)^2\Big)\\ &+\frac{5}{96}(5a_1 + 6a_2)\Big(-\sqrt{5}\partial^2\phi + 5(\partial\phi)^2\Big)\,,\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{-3} = &-\frac{5}{16}(3a_1 + 2a_2)\Big(\frac{1}{\sqrt{5}}\partial^3\phi - \partial^2\phi\partial\phi\Big) + \frac{5}{16}(3a_1 + 2a_2)\Big(\frac{3}{2\sqrt{5}}i\partial^3 X - \partial^2 X\partial X\Big)\\ &+\frac{5}{24}a_1\Big(-\frac{\sqrt{5}}{2}\partial^3\phi + \frac{3}{2}\partial^2\phi\partial\phi + \frac{\sqrt{5}}{2}(\partial\phi)^3\Big) + \frac{5}{48}(7a_1 + 6a_2)\partial\phi\Big(\frac{3}{2}i\partial^2 X - \frac{\sqrt{5}}{2}(\partial X)^2\Big)\,,\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{-4} = &-\frac{1}{8}(25a_1 + 14a_2)\Big(\frac{11}{12\sqrt{5}}\partial^4\phi - \frac{1}{2}\partial^3\phi\partial\phi - \frac{1}{2}(\partial^2\phi)^2\Big)\\ &-\frac{1}{8}(55a_1 + 34a_2)\Big(\frac{3}{4\sqrt{5}}i\partial^4 X - \frac{1}{2}\partial^3 X\partial X - \frac{1}{2}(\partial^2 X)^2\Big)\\ &+\frac{5}{48}(7a_1 + 6a_2)\Big(-\frac{\sqrt{5}}{6}\partial^4\phi - \partial^3\phi\partial\phi + \sqrt{5}\partial^2\phi(\partial\phi)^2\Big) + \frac{5}{24}a_1\sqrt{5}\partial\phi\Big(\frac{3}{2\sqrt{5}}i\partial^3 X - \partial^2 X\partial X\Big)\\ &-\frac{5}{32}(3a_1 + 2a_2)\Big(-\frac{4\sqrt{5}}{3}\partial^4\phi + 6(\partial^2\phi)^2 + 2\sqrt{5}\partial^2\phi(\partial\phi)^2\Big)\\ &-\frac{5}{8}(3a_1 + 2a_2)\partial^2\phi\Big(\frac{3}{2}i\partial^2 X - \frac{\sqrt{5}}{2}(\partial X)^2\Big)\\ &+\frac{25}{8}(3a_1 + 2a_2)\Big(\frac{1}{2\sqrt{5}}i\partial^4 X - \frac{1}{2}\partial^3 X\partial X + \frac{9}{20}(\partial^2 X)^2 - \frac{3}{2\sqrt{5}}i\partial^2 X(\partial X)^2 + \frac{1}{4}(\partial X)^4\Big)\,.\end{aligned}$$

The expressions for $G_{-n}$ and $\tilde{G}_{-n}$ of eq. (47) for the current $j^{(-2)}$ of the (2,7) model are the following:

$$G_{-2} =(-\eta + \frac{23}{7}\epsilon)( - \frac{5}{2\sqrt{7}}\partial^2\phi - \frac{1}{2}(\partial\phi)^2) - \frac{18}{7}\epsilon(\frac{5}{2\sqrt{7}}i\partial^2 X - \frac{1}{2}(\partial X)^2)$$

$$\tilde{G}_{-2} =(\frac{2}{3}\eta - \frac{10}{7}\epsilon)( - \frac{5}{2\sqrt{7}}\partial^2\phi - \frac{1}{2}(\partial\phi)^2) + (\eta + \frac{3}{7}\epsilon)(\frac{5}{2\sqrt{7}}i\partial^2 X - \frac{1}{2}(\partial X)^2)$$

$$G_{-3} =(-\eta + \frac{3}{7}\epsilon)(\frac{5}{2\sqrt{7}}i\partial^3 X \; - \; \partial^2 X\partial X)$$

$$\begin{aligned} G_{-4} =&\eta(\frac{13}{12\sqrt{7}}\partial^4\phi - \frac{1}{2}\partial^3\phi\partial\phi - \frac{1}{2}(\partial^2\phi)^2) \\ &+ (\eta - \frac{24}{7}\epsilon)(\frac{5}{4\sqrt{7}}i\partial^4 X - \frac{1}{2}\partial^3 X\partial X - \frac{1}{2}(\partial^2 X)^2) \\ &+ \epsilon[-\frac{5}{6\sqrt{7}}\partial^4\phi - \frac{1}{2}\partial^3\phi\partial\phi + ( - \frac{5}{2\sqrt{7}}\partial^2\phi - \frac{1}{2}(\partial\phi)^2)^2] \\ &+ 3\epsilon[\frac{5}{6\sqrt{7}}i\partial^4 X - \frac{1}{2}\partial^3 X\partial X + (\frac{5}{2\sqrt{7}}i\partial^2 X - \frac{1}{2}(\partial X)^2)^2] \\ &- 2\epsilon( - \frac{5}{2\sqrt{7}}\partial^2\phi - \frac{1}{2}(\partial\phi)^2)(\frac{5}{2\sqrt{7}}i\partial^2 X - \frac{1}{2}(\partial X)^2) \end{aligned}$$

## REFERENCES


1. E. Brézin and V. Kazakov, *Phys. Lett.* **B236** (1990) 144;
   M.R. Douglas and S. Shenker, *Nucl. Phys.* **B335** (1990) 635;
   D.J. Gross and A.A. Migdal, *Phys. Rev. Lett.* **64** (1990) 127.
2. J. Distler and H. Kawai, *Nucl. Phys.* **B321** (1989) 509;
   F. David *Mod. Phys. Lett.* **A3** (1989) 1651.
3. N. Seiberg, *Prog.Theor. Phys. Suppl.* **102** (1991) 319;
   J. Polchinski, *Nucl. Phys.* **B346** (1990) 253, *Nucl. Phys.* **B357** (1991) 241.
4. M. Goulian and M. Li, *Phys. Rev. Lett.* **66** (1991) 2051;
   P. Di Francesco and D. Kutasov, *Phys. Lett.* **B261** (1991) 385.
5. Y. Kitazawa, *Phys. Lett.* **B265** (1991) 262;
   Vl.S. Dotsenko, *Mod. Phys. Lett.* **A6** (1991) 3601, preprints, PAR-LPTHE 92-4 (January 1992), CERN-TH.6502/92,PAR-LPTHE 92-17 (May 1992).
6. N. Sakai and Y. Tanii, *Prog. Theor. Phys* **86** (1991) 547, *Int. J. Mod. Phys.* **A6** (1991) 2743, *Phys. Lett.* **B276** (1992) 41.





7. B.H. Lian and G.J. Zuckerman, *Phys. Lett.* **B254** (1991) 417.
8. D.J. Gross and I. Klebanov, *Nucl. Phys.* **B359** (1990) 3;
   D.J. Gross, I. Klebanov and M. Newman *Nucl. Phys.* **B350** (1990) 333;
   U.H. Danielsson and D.J. Gross, *Nucl. Phys.* **B366** (1991) 3.
9. A.M. Sengupta and S.R. Wadia, *Int. J. Mod. Phys.* **A6** (1991) 1961;
   G. Mandal, A.M. Sengupta and S.R. Wadia, *Mod. Phys. Lett.* **A6** (1991) 1465.
10. M. Bershadsky and I. Klebanov, *Nucl. Phys.* **B360** (1991) 559.
11. A.M. Polyakov, *Mod. Phys. Lett.* **A6** (1991) 635;
    I. Klebanov and A.M. Polyakov, *Mod. Phys. Lett.* **A6** (1991) 3273.
12. B.H. Lian and G.J. Zuckerman, *Phys. Lett.* **B266** (1991) 21
13. P. Bouwknegt, J.M. McCarthy and K. Pilch, *Comm. Math. Phys.* **145** (1992) 541.
14. S. Mukherji, S. Mukhi and A. Sen, *Phys. Lett.* **B266** (1991) 337;
    C. Imbimbo, S. Mahapatra and S. Mukhi, *Nucl. Phys.* **B375** (1992) 399.
15. K. Itoh and N. Ohta, preprint FERMILAB-PUB-91/228-T.
16. E. Witten, *Nucl. Phys.* **B373** (1992) 187.
17. D. Kutasov, E. Martinec and N. Seiberg, *Phys. Lett.* **B276** (1992) 187.
18. I.R. Klebanov *Mod. Phys. Lett.* **A7** (1992) 723.
19. E. Witten and B. Zwiebach, preprint IASSNS-HEP-92/4 (January 1992).
20. N. Chair, V.K. Dobrev and H. Kanno, *Phys. Lett.* **B283** (1992) 194.
21. S. Govindarajan, T. Jayaraman, V. John and P. Majumdar, preprint IMSc-91/40, (December 1991).