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The Weyl-van der Waerden spinor formalism is applied to the evaluation of helicity invariant amplitudes in the framework of linearized gravitation. The graviton couplings to spin-0, $\frac{1}{2}$, 1, and $\frac{3}{2}$ particles are given, and, to exhibit the reach of this method, the helicity amplitudes for the process electron + positron $\rightarrow$ photon + graviton are obtained.

I. INTRODUCTION

The calculation of invariant amplitudes involving a large number of particles is greatly simplified with the use of the helicity method [1]. Matrix elements are evaluated taking into account all possible helicity configurations for the external legs, and the squared amplitudes are obtained by adding them up incoherently. The method avoids much of the the algebra involving traces of several Dirac matrices, intrinsic to the usual procedure. For instance, the helicity method leads to very compact expressions for processes involving multiple bremsstrahlung in gauge theories [2].

An important improvement in the helicity method was achieved with the use of Weyl–van der Waerden SL(2,C) spinors together with spinor calculus [3]. This method [4, 5] unifies the descriptions of Dirac spinors and Minkowski four-vectors. The couplings and helicity wave functions of the particles can be written in terms of momenta spinors, allowing the use of gauge invariance to eliminate some Feynman diagrams.

The method of Weyl-van der Waerden spinors for spin-$\frac{1}{2}$ and spin-1 particles was derived in Ref [5]. The technique was used to construct helicity wave functions for massless and massive spin-$\frac{3}{2}$ fermions [6] and spin-2 bosons [7, 8]. In Ref. [6] we have shown how useful this method could be in the calculation of processes with high spin particles.

In this paper we apply the Weyl–van der Waerden spinor technique to processes involving the weak field approximation of matter–coupled gravity. In Section II we present the Lagrangians involving the couplings of matter fields of spin-0, $\frac{1}{2}$, 1, and $\frac{3}{2}$ to the graviton, as well as the graviton self-coupling interactions. In Section IV we present, as an illustrative example of the spinor method, the evaluation of helicity amplitudes for the process electron + positron $\rightarrow$ photon + graviton. Finally, in the Appendices, we define our notations and summarize some useful results for helicity states of particles of different spins.

II. MATTER-COUPLED GRAVITY

The coupling of the matter fields with the gravity can be obtained using the weak field approximation. The metric is assumed to be close to the Minkowski one $(\eta_{\mu\nu})$ *i.e.*,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \,, \tag{1}$$

where $h_{\mu\nu}$ is the graviton field and $\kappa \equiv \sqrt{32\pi G} = 8.211 \times 10^{-19}$ GeV^{-1} in natural unities. In order to satisfy $g_{\mu\nu}g^{\nu\alpha} = \delta_{\mu}{}^{\alpha}$, the tensor $g^{\mu\nu}$ will have the expansion

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}{}_{\alpha} h^{\alpha\nu} + \cdots \,. \tag{2}$$

We can also write the expansion for $\sqrt{g}$, where $g \equiv -\det(g_{\mu\nu})$, as a series in κ,

$$\sqrt{g} = 1 + \frac{\kappa}{2}h - \frac{\kappa^2}{4}h^{\nu}{}_{\mu}h^{\mu}{}_{\nu} + \frac{\kappa^2}{8}h^2 + \cdots \tag{3}$$

We are, as usual, taking cartesian coordinates on Minkowski space. Indices are lowered and raised with $\eta_{\mu\nu}$ and $\eta^{\mu\nu} = (\eta^{-1})_{\mu\nu}$. In our notation, $h \equiv h^{\mu}{}_{\mu}$.

As a general procedure to introduce the coupling between the graviton and the usual bosonic matter we write the action as

$$S_{\chi} = \int \mathrm{d}^4x \, \sqrt{g} \, \mathcal{L}[\chi(x), g^{\mu\nu}(x)] \,,$$

where $\mathcal{L}[\chi(x), g^{\mu\nu}(x)]$ is the Lagrangian of the scalar or vector field χ, in terms of $g^{\mu\nu}$.

Since the diffeomorphism group of general relativity cannot account for spinor representations, fermionic matter requires the introduction of vierbein fields $V^{a}{}_{\mu}(x)$, which are related to the metric by

$$\eta_{ab} \, V^{a}{}_{\mu}(x) \, V^{b}{}_{\nu}(x) = g_{\mu\nu}(x) \tag{4}$$

where a, b refer to the local flat space and μ, ν to the curved space. From Eq.(4) we learn that $\sqrt{g} = \det[\mathcal{V}^a{}_\mu]$. In matrix notation, Eq. (1) and (4) are simply $g = \mathcal{V}^T\eta\mathcal{V} = \eta + \kappa h$. Vierbeine are peculiar amongst fields, as they are always present on space-time [9]. They embody the soldering of flat tangent Minkowski space to each point of curved space [10] and translate the Lorentz metric of one into the Riemannian metric of the other, which is the meaning of Eq. (4). Thus, each vierbein field leads to a Riemann metric. At each point of space-time, vierbeine are obtained from each other by Lorentz transformations $\mathcal{V}' = \Lambda\mathcal{V}$ which, with the Lorentz group condition $\Lambda^T\eta\Lambda = \eta$, leave g invariant. To choose a vierbein field means to choose a gauge in the flat sector, to fix a Lorentz transformation at each point. The 16 fields $\mathcal{V}^a{}_\mu$ are in this way refunded into 6 local Lorentz transformations plus the 10 metric components. A most convenient choice is $\mathcal{V} = \sqrt{\eta^{-1}g}$, corresponding to symmetric vierbeine $\mathcal{V}^a{}_\mu = \mathcal{V}_\mu{}^a$ [11], which can be obtained from any tetrad $\mathcal{U}$ by a Lorentz transformation $\Lambda = \sqrt{\eta^{-1}g}\,\mathcal{U}^{-1}$. In this gauge $\mathcal{V} = (1 + \kappa\eta^{-1}h)^{1/2}$, that is,

$$\mathcal{V}^a{}_\mu = \delta^a{}_\mu + \frac{\kappa}{2}h^a{}_\mu - \frac{\kappa^2}{8}h^a{}_\alpha\delta^\alpha{}_b h^b{}_\mu + \cdots .$$

The theory becomes a purely metric theory, as for bosonic matter. By imposing

$$\mathcal{V}^a{}_\mu \mathcal{V}_b{}^\mu = \delta^a{}_b\,, \tag{5}$$

we obtain the expansion of the inverse vierbein fields,

$$\mathcal{V}_a{}^\mu = \delta_a{}^\mu - \frac{\kappa}{2}h_a{}^\mu + \frac{3\kappa^2}{8}h_a{}^\alpha\delta_\alpha{}^b h_b{}^\mu + \cdots \tag{6}$$

This expression will be used to generate the couplings of the graviton with the fermionic matter.

A. Bosonic Fields

In this section we write the Lagrangian describing the couplings of a spin-0 particle and the photon with gravitons. These results have been presented before in the literature [12] and are given here for the sake of completeness.

The action of a scalar field is

$$S_\phi = \int d^4x \sqrt{g} \left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right] .$$

Using Eqs. (2) and (3), and neglecting terms $\mathcal{O}(\kappa^3)$ we obtain

$$S_\phi = \int d^4x \left\{\mathcal{L}_\phi + \frac{\kappa}{4}\left[h\left(\partial^\alpha\phi\partial_\alpha\phi - m^2\phi^2\right) - 2h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] + \frac{\kappa^2}{16}\left[\left(h^2 - 2h^\mu_\nu h^\nu_\mu\right)\left(\partial^\alpha\phi\partial_\alpha\phi - m^2\phi^2\right) - 4\left(hh^{\mu\nu} - 2h^\mu_\alpha h^{\alpha\nu}\right)\partial_\mu\phi\partial_\nu\phi\right]\right\} , \quad (7)$$

where $\mathcal{L}_\phi$ is the free scalar Lagrangian.

In a similar way we can write, for the photon,

$$S_A = \int d^4x \sqrt{g} \left[-\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\right]$$

which becomes, with the aid of the expansions Eqs. (2) and (3),

$$S_A = \int d^4x \Big[\mathcal{L}_A - \frac{\kappa}{8}\left(\eta^{\mu\alpha}\eta^{\nu\beta}h - 4h^{\mu\alpha}\eta^{\nu\beta}\right)F_{\mu\nu}F_{\alpha\beta} - \frac{\kappa^2}{32}\Big(\eta^{\mu\alpha}\eta^{\nu\beta}h^2 - 2\eta^{\mu\alpha}\eta^{\nu\beta}h^\lambda{}_\rho h^\rho{}_\lambda - 8\eta^{\nu\beta}hh^{\mu\alpha} + 16\eta^{\nu\beta}h^\alpha{}_\lambda h^{\lambda\mu} + 8h^{\mu\alpha}h^{\nu\beta}\Big)F_{\mu\nu}F_{\alpha\beta}\Big] \quad (8)$$

where $\mathcal{L}_A$ is the free photon Lagrangian and terms $\mathcal{O}(\kappa^3)$ have been neglected.

B. Fermionic Fields

The action in the case of a spinorial field can be obtained by replacing all the partial derivatives by the covariant derivative,

$$\partial_a \rightarrow \mathcal{D}_a = V_a{}^\mu(\partial_\mu + \Gamma_\mu) , \quad (9)$$

where

$$\Gamma_\mu = -\frac{i}{4}\mathcal{V}_a{}^\alpha \partial_\mu \mathcal{V}_{b\alpha}\Sigma^{ab} + \frac{i}{4}\mathcal{V}_a{}^\beta \Gamma^\alpha{}_{\beta\mu}\mathcal{V}_{b\alpha}\Sigma^{ab} ,$$

with

$$\Sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b] ,$$

and

$$\Gamma^\mu{}_{\alpha\beta} \equiv \frac{1}{2}g^{\mu\lambda}\left(\partial_\beta g_{\alpha\lambda} + \partial_\alpha g_{\lambda\beta} - \partial_\lambda g_{\beta\alpha}\right) . \tag{10}$$

Therefore, the action becomes

$$S_S = \int d^4x \, \sqrt{g}\left\{\frac{i}{2}\left[\bar\psi\gamma^a\mathcal{V}_a{}^\mu\partial_\mu\psi - \left(\partial_\mu\bar\psi\right)\mathcal{V}_a{}^\mu\gamma^a\psi\right] + \frac{i}{2}\bar\psi\left(\gamma^a\mathcal{V}_a{}^\mu\Gamma_\mu + \Gamma_\mu\mathcal{V}_a{}^\mu\gamma^a\right)\psi - m\bar\psi\psi\right\} . \tag{11}$$

Using the expansion Eq.(6), we obtain, to second order in κ,

$$\Gamma_\mu = \frac{i}{8}\kappa\Sigma^{ab}\left(\partial_a h_{b\mu} - \partial_b h_{a\mu}\right) - \frac{i}{16}\kappa^2\Sigma^{ab}\left[h_b{}^\lambda\left(\partial_\mu h_{\lambda a} + \partial_a h_{\lambda\mu} - \partial_\lambda h_{\mu a}\right) + h_a{}^\beta\left(\partial_\beta h_{b\mu} - \partial_b h_{\mu\beta}\right)\right] . \tag{12}$$

Therefore the action Eq.(11) can be written as

$$\begin{aligned} S_S = \int d^4x \, \Big\{ \mathcal{L}_\psi &+ \frac{\kappa}{2}\Big[\Big[-\frac{i}{4}\bar\psi(\gamma^a\partial_\mu + \gamma_\mu\partial^a)\psi + \frac{i}{4}\left(\partial_\mu\bar\psi\gamma^a + \partial^a\bar\psi\gamma_\mu\right)\psi\Big] h_a{}^\mu \\ &+ \left(\frac{i}{2}\bar\psi\gamma^a\partial_a\psi - \frac{i}{2}\partial_a\bar\psi\gamma^a\psi - m\bar\psi\psi\right) h\Big] \\ &+ \frac{\kappa^2}{16}\Big[-\frac{1}{2}\bar\psi(\gamma^c\Sigma^{ab} + \Sigma^{ab}\gamma^c)\psi\left[h_a{}^\lambda\partial_b h_{c\lambda} - h_b{}^\lambda(\partial_c h_{\lambda a} + \partial_a h_{\lambda c}) - h_c{}^\lambda(\partial_a h_{b\lambda} - \partial_b h_{a\lambda})\right] \\ &+ \left[i\bar\psi(\gamma^a\partial_\mu + \gamma_\mu\partial^a)\psi - i(\partial_\mu\bar\psi\gamma^a + \partial^a\bar\psi\gamma_\mu)\psi\right]\left(\frac{3}{2}h_a{}^d h_d{}^\mu - hh_a{}^\mu\right) \\ &+ 4\left(\frac{i}{2}\bar\psi\gamma^a\partial_a\psi - \frac{i}{2}\partial_a\bar\psi\gamma^a\psi - m\bar\psi\psi\right)\left(\frac{1}{2}h^2 - h^\nu{}_\mu h^\mu{}_\nu\right)\Big]\Big\} , \end{aligned} \tag{13}$$

where $\mathcal{L}_\psi$ is the free fermion Lagrangian. Now both sets of indices $a, b, \cdots$ and $\mu, \nu, \cdots$ may be treated on the same footing. When we compute specific processes the terms involving the free Lagrangian can be discarded since they give a vanishing contribution when the Dirac equation is used.

We can also introduce the coupling of a spin-$\frac{3}{2}$ fermion with the graviton. Starting with the free Lagrangian for a massless spin-$\frac{3}{2}$ fermion,

$$\begin{aligned}\mathcal{L}_\Psi &= \varepsilon^{abcd}\bar{\Psi}_a\gamma_5\gamma_b\partial_c\Psi_d \\ &= \frac{i}{2}\Big[\bar{\Psi}_a\left(\gamma^a\gamma^c\gamma^d + \eta^{ad}\gamma^c - \eta^{ac}\gamma^d - \eta^{cd}\gamma^a\right)\partial_c\Psi_d \\ &\quad - \left(\partial_c\bar{\Psi}_a\right)\left(\gamma^a\gamma^c\gamma^d + \eta^{ad}\gamma^c - \eta^{ac}\gamma^d - \eta^{cd}\gamma^a\right)\Psi_d\Big]\,, \end{aligned} \tag{14}$$

we introduce the vierbein fields to get

$$\begin{aligned}S_{SV} = \int \mathrm{d}^4x\,\sqrt{g}\,\frac{i}{2}\Big\{&\mathcal{V}_a{}^\alpha\mathcal{V}_b{}^\nu\mathcal{V}_c{}^\beta\left(\bar{\Psi}_\alpha\gamma^a\gamma^b\gamma^c\partial_\nu\Psi_\beta - \partial_\nu\bar{\Psi}_\alpha\gamma^a\gamma^b\gamma^c\Psi_\beta\right) \\ &+ \mathcal{V}_a{}^\alpha\mathcal{V}_b{}^\nu\mathcal{V}_c{}^\beta\bar{\Psi}_\alpha\left(\gamma^a\gamma^b\gamma^c\Gamma_\nu + \Gamma_\nu\gamma^a\gamma^b\gamma^c\right)\Psi_\beta \\ &+ \left(g^{\alpha\beta}\mathcal{V}_a{}^\nu - g^{\alpha\nu}\mathcal{V}_a{}^\beta - g^{\nu\beta}\mathcal{V}_a{}^\alpha\right)\Big[\bar{\Psi}_\alpha\gamma^a\partial_\nu\Psi_\beta - \partial_\nu\bar{\Psi}_\alpha\gamma^a\Psi_\beta \\ &+ \bar{\Psi}_\alpha\left(\gamma^a\Gamma_\nu + \Gamma_\nu\gamma^a\right)\Psi_\beta\Big]\Big\}\,. \end{aligned} \tag{15}$$

First order expansion in κ gives

$$\begin{aligned}S_{SV} = \int \mathrm{d}^4x\,\frac{i}{2}\Big\{&\mathcal{L}_\Psi + \frac{\kappa}{2}h\Big[\bar{\Psi}_a\left(\gamma^a\gamma^b\gamma^c + \eta^{ac}\gamma^b - \eta^{ab}\gamma^c - \eta^{bc}\gamma^a\right)\partial_b\Psi_c \\ &- \partial_b\bar{\Psi}_a\left(\gamma^a\gamma^b\gamma^c + \eta^{ac}\gamma^b - \eta^{ab}\gamma^c - \eta^{bc}\gamma^a\right)\Psi_b\Big] \\ &- \frac{\kappa}{2}\Big[h_c{}^\beta\left(\bar{\Psi}_a\gamma^a\gamma^b\gamma^c\partial_b\Psi_\beta - \partial_b\bar{\Psi}_a\gamma^a\gamma^b\gamma^c\Psi_\beta\right) \\ &+ h_b{}^\nu\left(\bar{\Psi}_a\gamma^a\gamma^b\gamma^c\partial_\nu\Psi_c - \partial_\nu\bar{\Psi}_a\gamma^a\gamma^b\gamma^c\Psi_c\right) \\ &+ h_a{}^\alpha\left(\bar{\Psi}_\alpha\gamma^a\gamma^b\gamma^c\partial_b\Psi_c - \partial_b\bar{\Psi}_\alpha\gamma^a\gamma^b\gamma^c\Psi_c\right)\Big] \\ &+ \kappa\Big[-h^{\alpha\beta}\left(\bar{\Psi}_\alpha\gamma^a\partial_a\Psi_\beta - \partial_a\bar{\Psi}_\alpha\gamma^a\Psi_\beta\right) + h^{\alpha\nu}\left(\bar{\Psi}_\alpha\gamma^a\partial_\nu\Psi_a - \partial_\nu\bar{\Psi}_\alpha\gamma^a\Psi_a\right) \\ &+ h^{\nu\beta}\left(\bar{\Psi}_a\gamma^a\partial_\nu\Psi_\beta - \partial_\nu\bar{\Psi}_a\gamma^a\Psi_\beta\right) - \frac{1}{2}h_a{}^\nu\left(\bar{\Psi}_\alpha\gamma^a\partial_\nu\Psi^\alpha - \partial_\nu\bar{\Psi}_\alpha\gamma^a\Psi^\alpha\right) \\ &+ \frac{1}{2}h_a{}^\beta\left(\bar{\Psi}_\alpha\gamma^a\partial^\alpha\Psi_\beta - \partial^\alpha\bar{\Psi}_\alpha\gamma^a\Psi_\beta\right) + \frac{1}{2}h_a{}^\alpha\left(\bar{\Psi}_\alpha\gamma^a\partial^\beta\Psi_\beta - \partial^\beta\bar{\Psi}_\alpha\gamma^a\Psi_\beta\right)\Big] \end{aligned}$$

$$+i\frac{\kappa}{8}\Big[(\partial_d h_{cb}-\partial_c h_{db})\bar{\Psi}_a\left(\gamma^a\gamma^b\gamma^c\Sigma^{dc}+\Sigma^{dc}\gamma^a\gamma^b\gamma^c\right)\Psi_c$$

$$-(\partial_b h_c{}^a-\partial_c h_b{}^a)\bar{\Psi}_a\left(\gamma^a\Sigma^{bc}+\Sigma^{bc}\gamma^a\right)\Psi_a$$

$$-\left(\partial_b h_c{}^\beta-\partial_c h_b{}^\beta\right)\bar{\Psi}_a\left(\gamma^a\Sigma^{bc}+\Sigma^{bc}\gamma^a\right)\Psi_\beta\Big]\Big\}\,. \qquad (16)$$

C. Graviton Self-Interaction

In order to have the complete Feynman rules we should consider the graviton self-interaction in the present linearized case [13, 14]. This can be obtained from the action for pure gravity,

$$S_G=\frac{2}{\kappa^2}\int d^4x\,\sqrt{g}\,g^{\mu\nu}R_{\mu\nu}$$

where the Ricci tensor is

$$R_{\mu\nu}=\partial_\nu\Gamma^\alpha_{\mu\alpha}-\partial_\alpha\Gamma^\alpha_{\mu\nu}+\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}-\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}$$

with affine connection given by Eq.(10).

Using the expansion Eq.(2) we find

$$\Gamma^\mu{}_{\alpha\beta}\simeq\frac{\kappa}{2}\Big[\partial_\beta h^\mu_\alpha+\partial_\alpha h^\mu_\beta-\partial^\mu h_{\beta\alpha}-\kappa h^{\mu\lambda}(\partial_\beta h_{\lambda\alpha}+\partial_\alpha h_{\lambda\beta}-\partial_\lambda h_{\beta\alpha})$$

$$+\kappa^2 h^\mu_\rho h^{\rho\lambda}(\partial_\beta h_{\lambda\alpha}+\partial_\alpha h_{\lambda\beta}-\partial_\lambda h_{\beta\alpha})\Big]+\mathcal{O}(\kappa^4)\,, \qquad (17)$$

and the Lagrangian comes out as

$$\mathcal{L}_G=\partial_\nu h\partial^\alpha h^\nu_\alpha-\partial^\mu h^\alpha_\nu\partial_\alpha h^\nu_\mu+\frac{1}{2}\left(\partial_\alpha h^\mu_\nu\partial^\alpha h^\nu_\mu-\partial_\alpha h\partial^\alpha h\right)+\mathcal{L}_{int} \qquad (18)$$

where

$$\mathcal{L}_{int}=\kappa\mathcal{L}_3+\kappa^2\mathcal{L}_4+\cdots$$

The terms independent of κ give rise to the usual graviton propagator. An important consequence of having a purely metric theory is that we can use the same propagator when treating with fermionic and bosonic matter. The Lagrangians $\mathcal{L}_i$ describes the self-coupling of i gravitons. Since we are interested in process $2 \to 2$ involving matter-coupled gravity we need only the trilinear vertex between gravitons. The portion of the Lagrangian containing this coupling is given by

$$\begin{aligned}\mathcal{L}_{hhh} = \kappa \mathcal{L}_3 = \kappa \Big[& \frac{1}{2}\left(\partial_\alpha h^\mu_\nu \partial^\beta h^\nu_\mu h^\alpha_\beta - h \partial_\beta h^\mu_\nu \partial^\nu h^\beta_\mu - h \partial^\beta \partial_\nu h h^\nu_\beta\right) \\ & + h^\mu_\nu \partial^\beta h^\alpha_\mu \partial_\alpha h^\nu_\beta - h^\mu_\nu \partial^\alpha h^\nu_\beta \partial_\alpha h^\beta_\mu + \partial_\alpha h h^\mu_\nu \partial^\alpha h^\nu_\mu + \partial^\alpha \partial_\beta h h^\nu_\alpha h^\beta_\nu + h^\alpha_\beta \partial^\mu \partial_\nu h^\beta_\alpha h^\nu_\mu \\ & + 2\partial^\alpha h^\mu_\nu h^\beta_\mu \partial_\beta h^\nu_\alpha - \frac{1}{4}\left(h \partial^\nu h^\alpha_\beta \partial_\nu h^\beta_\alpha - h \partial^\mu h \partial_\mu h\right)\Big] . \end{aligned} \tag{19}$$

In the case of a external graviton line, since we describe the physical graviton states as a traceless and null four-divergence tensor field [8], we can drop all the terms proportional to h and to $\partial_\mu h^{\mu\nu}$. The same remains true, with the minimal gauge choice, when these fields correspond to a graviton propagation [5, 14]. Using the fact that $\partial_\nu h^\alpha_\beta \partial^\mu h^\beta_\alpha h^\nu_\mu = -h^\alpha_\beta \partial^\mu \partial_\nu h^\beta_\alpha h^\nu_\mu$ up to a total divergence and a divergence of h^ν_μ, $\mathcal{L}_{hhh}$ becomes finally

$$\mathcal{L}_{hhh} = 2\kappa \left[\partial^\alpha h^\mu_\nu h^\beta_\mu \partial_\beta h^\nu_\alpha + \frac{1}{2}\left(h^\mu_\nu \partial^\beta h^\alpha_\mu \partial_\alpha h^\nu_\beta - h^\mu_\nu \partial^\alpha h^\nu_\beta \partial_\alpha h^\beta_\mu\right) - \frac{1}{4}\left(\partial_\alpha h^\mu_\nu \partial^\beta h^\nu_\mu h^\alpha_\beta\right)\right] . \tag{20}$$

III. WEYL–VAN DER WAERDEN FORMALISM

The group $SL(2, C)$ associates every four-vector with an hermitian matrix. In the space of these matrices, the Lorentz transformation is given as a linear transformation induced by an unimodular matrix Λ of $SL(2, C)$. The two-component spinors transform as [15]

$$\xi'_a = \Lambda_a{}^b \xi_b \quad , \quad \bar{\xi}'_{\dot{a}} = (\Lambda^*)_{\dot{a}}{}^{\dot{b}} \, \bar{\xi}_{\dot{b}}$$

where $\bar{\xi}_{\dot{a}} = (\xi_a)^*$. The undotted spinor transforms under the $(\frac{1}{2}, 0)$ (left handed spinor) Lorentz representation, whereas the dotted one transforms under the $(0, \frac{1}{2})$

(left handed spinor), the conjugate representation. The linear combination $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ yields a Dirac spinor.

In order to construct Lorentz invariants we introduce, in the space of $(2,2)$ matrices, the spinor metric

$$\epsilon^{ab}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{21}$$

satisfying $\epsilon^{ab}=-\epsilon_{ab}=\epsilon^{\dot a\dot b}=-\epsilon_{\dot a\dot b}$, and in terms of which

$$\xi^{a}=\epsilon^{ab}\xi_{b}\,,\qquad \bar\xi^{\dot a}=\epsilon^{\dot a\dot b}\bar\xi_{\dot b}\,. \tag{22}$$

The inner product between two spinors is defined as

$$<\xi\eta>\equiv\xi^{a}\eta_{a}\,,\qquad <\xi\eta>^{\dagger}\equiv\bar\xi^{\dot a}\bar\eta_{\dot a}\,. \tag{23}$$

We define the σ-matrices as:

$$\sigma^{\mu}_{a\dot b}=(\sigma^{0},\sigma^{i}) \tag{24}$$

$$\bar\sigma^{\mu\dot a b}=\epsilon^{\dot a\dot c}\epsilon^{bd}\bar\sigma^{\mu}_{\dot c d}=\epsilon^{\dot a\dot c}\epsilon^{bd}\sigma^{\mu}_{d\dot c}\,, \tag{25}$$

which form a basis for the $(2,2)$-matrices, and satisfy

$$\sigma^{\mu}_{a\dot b}=\bar\sigma^{\mu}_{\dot b a}=[\sigma^{\mu}_{b\dot a}]^{\dagger}=[\bar\sigma^{\mu}_{\dot a b}]^{\dagger}\,. \tag{26}$$

One can establish a relation between $(2,2)$-matrices and Lorentz four-vectors. For any four-vector $p^{\mu}=(p^{0},p^{i})$, we construct an hermitian matrix P according to

$$P_{a\dot b}=\sigma^{\mu}_{a\dot b}p_{\mu}\,,\ \text{or}\ \ \bar P_{\dot a b}=\bar\sigma^{\mu}_{\dot a b}p_{\mu}\,. \tag{27}$$

The relation between p^{μ} and $P_{a\dot b}$ is one to one, and the inverse formula is

$$p^{\mu}=\frac{1}{2}\bar\sigma^{\mu\dot a b}P_{b\dot a}=\frac{1}{2}\sigma^{\mu a\dot b}\bar P_{\dot b a}\,.$$

The scalar product between two four-vectors can now be written as a function of the associated matrices

$$p \cdot q = \frac{1}{2} P_{b\dot{a}} \bar{Q}^{\dot{a}b} \equiv \frac{1}{2}\{P, \bar{Q}\}$$

For a light-like four-vector p_μ, $\{P, \bar{P}\} = 0$, and the matrix P is given by $P_{a\dot{b}} = p_a \bar{p}_{\dot{b}}$. The spinor p_a can be written in terms of the four-momentum components as

$$p_a = \exp(i\omega) \begin{pmatrix} -\frac{(p_x - ip_y)}{\sqrt{p_0 + p_z}} \\ \sqrt{p_0 + p_z} \end{pmatrix} , \tag{28}$$

ω being an arbitrary phase.

The scalar product between two light-like four-vector becomes, consequently,

$$p \cdot q = \frac{1}{2} P_{b\dot{a}} \bar{Q}^{\dot{a}b} \equiv \frac{1}{2}\{P, \bar{Q}\} = \frac{1}{2} \bar{q}^{\dot{a}} \bar{p}_{\dot{a}} q^b p_b = \frac{1}{2} | < pq > |^2 .$$

We can also write the helicity wave functions of the different fields in terms of spinors. On the fermionic side, we write the Dirac spinor as

$$\psi^A = \begin{pmatrix} \xi_a \\ \bar{\eta}^{\dot{a}} \end{pmatrix} , \tag{29}$$

and the spinor–vector field of a spin-$\frac{3}{2}$ fermion as

$$\Psi^{A,\mu} = \begin{pmatrix} \xi_a{}^\mu \\ \bar{\eta}^{\dot{a},\mu} \end{pmatrix} = \frac{1}{2} \bar{\sigma}^{\mu\dot{c}b} \begin{pmatrix} \Xi_{ab\dot{c}} \\ N^{\dot{a}}{}_{b\dot{c}} \end{pmatrix} . \tag{30}$$

On the bosonic side, we write the photon field in the form

$$A^\mu = \frac{1}{2} \bar{\sigma}^{\mu\dot{a}b} A_{b\dot{a}} ,$$

and the graviton field as

$$h^{\mu\nu} = \frac{1}{4} \bar{\sigma}^{\mu\dot{a}b} \bar{\sigma}^{\nu\dot{c}d} H_{b\dot{a}d\dot{c}} .$$

In the Appendix we present the explicit form of these fields.

As an example of the use of Weyl–van der Waerden formalism, we give in the following some Feynman rules coupling a scalar field and the photon to the graviton. In all cases we consider all the momenta as incoming at the vertex.

From the action Eq. (7) we obtain the following vertex for the trilinear and quartic couplings of the scalar field

- $H_{b\dot{a}d\dot{c}}(k_1) - \phi(k_2) - \phi(k_3)$

$$V_{H\phi\phi} = i\frac{\kappa}{8}\left[\bar{K}_2^{\dot{a}b}\bar{K}_3^{\dot{c}d} + \bar{K}_3^{\dot{a}b}\bar{K}_2^{\dot{c}d} - 2\epsilon^{\dot{a}\dot{c}}\epsilon^{bd}\left(\frac{1}{2}\{K_2, \bar{K}_3\} + m^2\right)\right] . \tag{31}$$

- $H_{b\dot{a}d\dot{c}}(k_1) - H_{f\dot{e}h\dot{g}}(k_2) - \phi(k_3) - \phi(k_4)$

$$\begin{aligned} V_{HH\phi\phi} = -i\frac{\kappa^2}{32}\Big[& 2\left(\epsilon^{\dot{a}\dot{c}}\epsilon^{bd}\epsilon^{\dot{e}\dot{g}}\epsilon^{fh} - \epsilon^{\dot{a}\dot{g}}\epsilon^{bh}\epsilon^{\dot{c}\dot{e}}\epsilon^{df} - \epsilon^{\dot{c}\dot{g}}\epsilon^{dh}\epsilon^{\dot{a}\dot{e}}\epsilon^{bf}\right) \\ & \times\left(\frac{1}{2}\{K_3, \bar{K}_4\} + m^2\right) \\ & - \epsilon^{\dot{e}\dot{g}}\epsilon^{fh}\left(\bar{K}_3^{\dot{a}b}\bar{K}_4^{\dot{c}d} + \bar{K}_4^{\dot{a}b}\bar{K}_3^{\dot{c}d}\right) - \epsilon^{\dot{a}\dot{c}}\epsilon^{bd}\left(\bar{K}_3^{\dot{e}f}\bar{K}_4^{\dot{g}h} + \bar{K}_4^{\dot{e}f}\bar{K}_3^{\dot{g}h}\right) \\ & + \epsilon^{\dot{c}\dot{e}}\epsilon^{df}\left(\bar{K}_3^{\dot{a}b}\bar{K}_4^{\dot{g}h} + \bar{K}_4^{\dot{a}b}\bar{K}_3^{\dot{g}h}\right) + \epsilon^{\dot{a}\dot{g}}\epsilon^{bh}\left(\bar{K}_3^{\dot{e}f}\bar{K}_4^{\dot{c}d} + \bar{K}_4^{\dot{e}f}\bar{K}_3^{\dot{c}d}\right) \\ & + \epsilon^{\dot{a}\dot{e}}\epsilon^{bf}\left(\bar{K}_3^{\dot{c}d}\bar{K}_4^{\dot{g}h} + \bar{K}_4^{\dot{c}d}\bar{K}_3^{\dot{g}h}\right) + \epsilon^{\dot{c}\dot{g}}\epsilon^{dh}\left(\bar{K}_3^{\dot{e}f}\bar{K}_4^{\dot{a}b} + \bar{K}_4^{\dot{e}f}\bar{K}_3^{\dot{a}b}\right)\Big] . \end{aligned} \tag{32}$$

In the case of the photon, we can derive using Eq.(8) the following Feynman rules for the couplings:

- $H_{b\dot{a}d\dot{c}}(k_1) - A_{f\dot{e}}(k_2) - A_{h\dot{g}}(k_3)$

$$\begin{aligned} V_{HAA} = i\frac{\kappa}{16}\Big[& \epsilon^{\dot{a}\dot{c}}\epsilon^{bd}\left(\epsilon^{\dot{e}\dot{g}}\epsilon^{fh}\{K_2, \bar{K}_3\} - \bar{K}_2^{\dot{g}h}\bar{K}_3^{\dot{e}f}\right) \\ & - \epsilon^{\dot{e}\dot{g}}\epsilon^{fh}\left(\bar{K}_2^{\dot{a}b}\bar{K}_3^{\dot{c}d} + \bar{K}_3^{\dot{a}b}\bar{K}_2^{\dot{c}d}\right) \\ & + \left(\epsilon^{\dot{c}\dot{g}}\epsilon^{dh}\bar{K}_2^{\dot{a}b} + \epsilon^{\dot{a}\dot{g}}\epsilon^{bh}\bar{K}_2^{\dot{c}d}\right)\bar{K}_3^{\dot{e}f} + \left(\epsilon^{\dot{c}\dot{e}}\epsilon^{df}\bar{K}_3^{\dot{a}b} + \epsilon^{\dot{a}\dot{e}}\epsilon^{bf}\bar{K}_3^{\dot{c}d}\right)\bar{K}_2^{\dot{g}h} \\ & - \left(\epsilon^{\dot{a}\dot{e}}\epsilon^{bf}\epsilon^{\dot{c}\dot{g}}\epsilon^{dh} + \epsilon^{\dot{a}\dot{g}}\epsilon^{bh}\epsilon^{\dot{c}\dot{e}}\epsilon^{df}\right)\{K_2, \bar{K}_3\}\Big] ; \end{aligned} \tag{33}$$

- $H_{f\dot{e}h\dot{g}}(k_1) - H_{s\dot{r}u\dot{t}}(k_2) - A_{b\dot{a}}(k_3) - A_{d\dot{c}}(k_4)$

$$
\begin{aligned}
V_{HHAA} = i\frac{\kappa^2}{64}\Big[& \Big(\epsilon^{\dot r\dot\imath}\epsilon^{su}\epsilon^{\dot e\dot g}\epsilon^{fh} - \epsilon^{\dot g\dot\imath}\epsilon^{hu}\epsilon^{\dot e\dot r}\epsilon^{fs} - \epsilon^{\dot g\dot r}\epsilon^{hs}\epsilon^{\dot e\dot\imath}\epsilon^{fu}\Big) \\
& \cdot\Big(\epsilon^{\dot a\dot c}\epsilon^{bd}\{\bar K_3, \bar K_4\} - \bar K_3^{\dot c d}\bar K_4^{\dot a b}\Big) \\
& - \epsilon^{\dot e\dot g}\epsilon^{fh}\Big(\mathcal{K}^{\dot a b,\dot c d,\dot r s,\dot\imath u} + \mathcal{K}^{\dot a b,\dot c d,\dot\imath u,\dot r s}\Big) - \epsilon^{\dot r\dot\imath}\epsilon^{su}\Big(\mathcal{K}^{\dot a b,\dot c d,\dot e f,\dot g h} + \mathcal{K}^{\dot a b,\dot c d,\dot g h,\dot e f}\Big) \\
& + \epsilon^{\dot g\dot r}\epsilon^{hs}\Big(\mathcal{K}^{\dot a b,\dot c d,\dot e f,\dot\imath u} + \mathcal{K}^{\dot a b,\dot c d,\dot\imath u,\dot e f}\Big) + \epsilon^{\dot e\dot r}\epsilon^{fs}\Big(\mathcal{K}^{\dot a b,\dot c d,\dot g h,\dot\imath u} + \mathcal{K}^{\dot a b,\dot c d,\dot\imath u,\dot g h}\Big) \\
& + \epsilon^{\dot e\dot\imath}\epsilon^{fu}\Big(\mathcal{K}^{\dot a b,\dot c d,\dot r s,\dot g h} + \mathcal{K}^{\dot a b,\dot c d,\dot g h,\dot r s}\Big) + \epsilon^{\dot g\dot\imath}\epsilon^{hu}\Big(\mathcal{K}^{\dot a b,\dot c d,\dot r s,\dot e f} + \mathcal{K}^{\dot a b,\dot c d,\dot e f,\dot r s}\Big) \\
& + \Big(\epsilon^{\dot a\dot r}\epsilon^{bs}\epsilon^{\dot c\dot\imath}\epsilon^{du} + \epsilon^{\dot a\dot\imath}\epsilon^{bu}\epsilon^{\dot c\dot r}\epsilon^{ds}\Big)\Big(\bar K_3^{\dot e f}\bar K_4^{\dot g h} + \bar K_4^{\dot e f}\bar K_3^{\dot g h}\Big) \\
& + \Big(\epsilon^{\dot a\dot e}\epsilon^{bf}\epsilon^{\dot c\dot g}\epsilon^{dh} + \epsilon^{\dot a\dot g}\epsilon^{bh}\epsilon^{\dot c\dot e}\epsilon^{df}\Big)\Big(\bar K_3^{\dot r s}\bar K_4^{\dot\imath u} + \bar K_4^{\dot r s}\bar K_3^{\dot\imath u}\Big) \\
& - \epsilon^{\dot c\dot g}\epsilon^{dh}\Big(\epsilon^{\dot a\dot r}\epsilon^{bs}\bar K_3^{\dot e f}\bar K_4^{\dot\imath u} + \epsilon^{\dot a\dot\imath}\epsilon^{bu}\bar K_3^{\dot e f}\bar K_4^{\dot r s}\Big) \\
& - \epsilon^{\dot c\dot e}\epsilon^{df}\Big(\epsilon^{\dot a\dot\imath}\epsilon^{bu}\bar K_3^{\dot g h}\bar K_4^{\dot r s} + \epsilon^{\dot a\dot r}\epsilon^{bs}\bar K_3^{\dot g h}\bar K_4^{\dot\imath u}\Big) \\
& - \epsilon^{\dot c\dot\imath}\epsilon^{du}\Big(\epsilon^{\dot a\dot e}\epsilon^{bf}\bar K_3^{\dot r s}\bar K_4^{\dot g h} + \epsilon^{\dot a\dot g}\epsilon^{bh}\bar K_3^{\dot r s}\bar K_4^{\dot e f}\Big) \\
& - \epsilon^{\dot c\dot r}\epsilon^{ds}\Big(\epsilon^{\dot a\dot g}\epsilon^{bh}\bar K_3^{\dot\imath u}\bar K_4^{\dot e f} + \epsilon^{\dot a\dot e}\epsilon^{bf}\bar K_3^{\dot\imath u}\bar K_4^{\dot g h}\Big)\Big] ,
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
\mathcal{K}^{\dot a b,\dot c d,\dot e f,\dot g h} \equiv\ & \epsilon^{\dot a\dot c}\epsilon^{bd}\bar K_3^{\dot e f}\bar K_4^{\dot g h} - \epsilon^{\dot a\dot e}\epsilon^{bf}\bar K_3^{\dot c d}\bar K_4^{\dot g h} - \epsilon^{\dot c\dot g}\epsilon^{dh}\bar K_3^{\dot e f}\bar K_4^{\dot a b} \\
& + \epsilon^{\dot a\dot e}\epsilon^{bf}\epsilon^{\dot c\dot g}\epsilon^{dh}\{\bar K_3, \bar K_4\} .
\end{aligned}
\tag{35}
$$

IV. CALCULATION OF INVARIANT AMPLITUDES — AN EXAMPLE

In order to evaluate the process $e^+ + e^- \rightarrow \gamma + g$ it is necessary to obtain the fermion-photon-graviton coupling. Therefore, we include the photon field in the covariant derivative Eq.(9), *i.e.*

$$
\partial_a \rightarrow \mathcal{D}_a = V_a{}^\mu\left(\partial_\mu + \Gamma_\mu + ieA_\mu\right) .
$$

The action containing this coupling is

$$
S_{QED} = \int \mathrm{d}^4x\, \sqrt{g}\, V_a{}^\mu\left[-e\bar\psi\gamma^a\psi A_\mu\right]
$$

and, expanding the determinant and the vierbein [Eqs.(3), and (6)] we obtain the Lagrangian

$$\mathcal{L}_{QED} = -\frac{e\kappa}{4}\Big[\delta_a{}^\mu h - h_a{}^\mu + \frac{\kappa}{2}(-hh_a{}^\mu - \delta_a{}^\mu h^\alpha{}_\beta h^\beta{}_\alpha + \frac{3}{2}h^a{}_\alpha \delta^\alpha{}_b h^b{}_\mu + \frac{1}{2}\delta_a{}^\mu h^2)\Big]\bar{\psi}(A_\mu\gamma^a + A^a\gamma_\mu)\psi \,, \qquad (36)$$

where we have left unwritten the usual QED fermion-photon coupling.

Let us denote the particle momenta as: $e^+(p) + e^-(q) \rightarrow \gamma(k) + g(l)$. In order to apply the Weyl van der Waerden spinor technique to the massive fermion it is useful to write the fermion momenta as the sum of two null four-vectors,

$$p = x + \alpha y \;, \quad q = \alpha x + y \,,$$

with $x^2 = y^2 = 0$. Since $p^2 = q^2 = m^2 = 2\alpha(x \cdot y)$, the spinor inner product of momenta spinors becomes

$$< xy >= \frac{m}{\sqrt{\alpha}} e^{i\theta}$$

where m is the fermion masses, and

$$\alpha = \alpha_\pm \equiv \frac{1 \pm \beta}{1 \mp \beta} \,,$$

where β is the fermion velocity in the center of mass system. Therefore, we can write the scalar products

$$(x \cdot l) = (y \cdot k) = \frac{1}{2(\alpha^2 - 1)}\left[u - m^2 - \alpha(t - m^2)\right]$$
$$(x \cdot k) = (y \cdot l) = \frac{1}{2((\alpha^2 - 1)}\left[t - m^2 - \alpha(u - m^2)\right] \,. \qquad (37)$$

We will denote the helicity amplitudes by $\mathcal{M}(\lambda_{e^+}, \lambda_{e^-}; \lambda_\gamma, \lambda_g)$ where $\lambda_{e^+(e^-)} = \pm$ is the electron (positron) helicity $(\pm\frac{1}{2})$, $\lambda_{\gamma(g)} = \pm$ is the photon (graviton) helicity $(\pm 1(2))$. There are four Feynman diagrams contributing to this process: two

"Compton-like" diagrams involving the exchange of the fermion in the t and u channels (denoted by A, and B respectively), the exchange of the photon in the s channel (C), and one involving the quartic coupling (D).

We are free to choose different gauge spinors in the photon and graviton helicity wave function for each set of helicity configuration $\{\lambda_{e^+}, \lambda_{e^-}; \lambda_\gamma, \lambda_g\}$. We make the so called "minimal gauge choice", in such a way that the largest number of contributions is zero. With this choice, all the helicity configurations of D contribution is null, and for the other contributions we have:

$$
\begin{aligned}
\mathcal{M}_A(+,+;-,+) &= \frac{ie\kappa\sqrt{2}\alpha^2 e^{-i\theta}}{2(t-m^2)} \frac{<xy>(<lx>^\dagger)^3<yk>}{<kx>^\dagger<xy>} \\
\mathcal{M}_A(+,+;+,-) &= \frac{ie\kappa\sqrt{2}e^{-i\theta}}{2(t-m^2)} \frac{<ly>^3<yx>^\dagger<xk>^\dagger}{<ky><lx>^\dagger} \\
\mathcal{M}_A(+,-;+,+) &= \frac{-ie\kappa\sqrt{2\alpha}\alpha^2}{2(t-m^2)} \frac{<xy>^3(<xl>^\dagger)^2<xk>^\dagger}{<ky><ly>^2} \\
\mathcal{M}_A(+,-;-,-) &= \frac{ie\kappa\sqrt{2\alpha}}{2(t-m^2)} \frac{<yl>^2(<yx>^\dagger)^3<yk>}{<kx>^\dagger(<lx>^\dagger)^2} \\
\mathcal{M}_A(+,-;+,-) &= \frac{-ie\kappa\sqrt{2\alpha}(\alpha-1)^2}{(t-m^2)} \frac{<xl>^3(<xk>^\dagger)^3}{<kl>(<lk>^\dagger)^2} \\
\mathcal{M}_A(+,-;-,+) &= \frac{-ie\kappa\sqrt{2\alpha}(\alpha-1)^2}{(t-m^2)} \frac{(<xl>^\dagger)^3<xk>^3}{<kl>^\dagger<kl>^2} \\
\mathcal{M}_A(-,+;+,+) &= \frac{ie\kappa\sqrt{2\alpha}\alpha}{2(t-m^2)} \frac{<xy>^3(<xl>^\dagger)^2<xk>^\dagger}{<ky><ly>^2} \\
\mathcal{M}_A(-,+;-,-) &= \frac{-ie\kappa\sqrt{2\alpha}\alpha}{2(t-m^2)} \frac{<yl>^2(<yx>^\dagger)^3<yk>}{<kx>^\dagger(<lx>^\dagger)^2} \\
\mathcal{M}_A(-,+;-,+) &= \frac{-ie\kappa\sqrt{2\alpha}(\alpha-1)^2}{(t-m^2)} \frac{<xk>^3(<xl>^\dagger)^3}{<kl>^\dagger<lk>^2} \\
\mathcal{M}_A(-,+;+,-) &= \frac{-ie\kappa\sqrt{2\alpha}(\alpha-1)^2}{(t-m^2)} \frac{<xl>^3(<xk>^\dagger)^3}{<kl>(<lk>^\dagger)^2} \\
\mathcal{M}_A(-,-;-,+) &= \frac{-ie\kappa\sqrt{2}e^{i\theta}}{2(t-m^2)} \frac{<yx>(<ly>^\dagger)^3<xk>}{<ky>^\dagger<lx>} \\
\mathcal{M}_A(-,-;+,-) &= \frac{-ie\kappa\sqrt{2}\alpha^2 e^{i\theta}}{2(t-m^2)} \frac{<lx>^3<xy>^\dagger<yk>^\dagger}{<kx><ly>^\dagger} \,.
\end{aligned}
\qquad (38)
$$

For the contribution B,

$$
\begin{aligned}
\mathcal{M}_B(+,+;-,+) &= \frac{ie\kappa\sqrt{2}e^{-i\theta}}{2(u-m^2)} \frac{<xy>(<lx>^\dagger)^3<yk>}{<kx>^\dagger<ly>} \\
\mathcal{M}_B(+,+;+,-) &= \frac{ie\kappa\sqrt{2}\alpha^2e^{-i\theta}}{2(u-m^2)} \frac{<ly>^3<yx>^\dagger<xk>^\dagger}{<ky><lx>^\dagger} \\
\mathcal{M}_B(+,-;+,+) &= \frac{ie\kappa\sqrt{2\alpha}\alpha}{2(u-m^2)} \frac{<xy>^3(<xl>^\dagger)^2<xk>^\dagger}{<ky><ly>^2} \\
\mathcal{M}_B(+,-;-,-) &= \frac{-ie\kappa\sqrt{2\alpha}\alpha}{2(u-m^2)} \frac{<yl>^2(<yx>^\dagger)^3<yk>}{<kx>^\dagger(<lx>^\dagger)^2} \\
\mathcal{M}_B(+,-;+,-) &= \frac{ie\kappa\sqrt{2\alpha}(\alpha-1)^2}{(u-m^2)} \frac{<yl>^3(<yk>^\dagger)^3}{<kl>(<lk>^\dagger)^2} \\
\mathcal{M}_B(+,-;-,+) &= \frac{ie\kappa\sqrt{2\alpha}(\alpha-1)^2}{(u-m^2)} \frac{<yk>^3(<yl>^\dagger)^3}{<kl>^\dagger<lk>^2} \\
\mathcal{M}_B(-,+;+,+) &= \frac{-ie\kappa\sqrt{2\alpha}}{2(u-m^2)} \frac{<xy>^3(<xl>^\dagger)^2<xk>^\dagger}{<ky><ly>^2} \\
\mathcal{M}_B(-,+;-,-) &= \frac{ie\kappa\sqrt{2\alpha}\alpha^2}{2(u-m^2)} \frac{<yl>^2(<yx>^\dagger)^3<yk>}{<kx>^\dagger(<lx>^\dagger)^2} \\
\mathcal{M}_B(-,+;-,+) &= \frac{-ie\kappa\sqrt{2\alpha}(\alpha-1)^2}{(u-m^2)} \frac{<xk>^3(<xl>^\dagger)^3}{<kl>^\dagger<lk>^2} \\
\mathcal{M}_B(-,+;+,-) &= \frac{ie\kappa\sqrt{2\alpha}(\alpha-1)^2}{(u-m^2)} \frac{<yl>^3(<yk>^\dagger)^3}{<kl>(<lk>^\dagger)^2} \\
\mathcal{M}_B(-,-;-,+) &= \frac{-ie\kappa\sqrt{2}\alpha^2e^{i\theta}}{2(u-m^2)} \frac{<yx>(<ly>^\dagger)^3<xk>}{<ky>^\dagger<lx>} \\
\mathcal{M}_B(-,-;+,-) &= \frac{-ie\kappa\sqrt{2}e^{i\theta}}{2(u-m^2)} \frac{<lx>^3<xy>^\dagger<yk>^\dagger}{<kx><ly>^\dagger} \, .
\end{aligned}
\tag{39}
$$

The other helicity contributions are zero, i.e.,

$$
\mathcal{M}_{A,B}(+,+;\pm,\pm) = \mathcal{M}_{A,B}(-,-;\pm,\pm) = 0
$$

For the contribution C we have

$$
\begin{aligned}
\mathcal{M}_C(+,+;-,+) &= \frac{ie\kappa\sqrt{2}(1+\alpha)e^{-i\theta}}{2s} \frac{<ky>^3<kl>^\dagger<xl>^\dagger}{<ly>^2} \\
\mathcal{M}_C(+,+;+,-) &= \frac{ie\kappa\sqrt{2}(1+\alpha)e^{-i\theta}}{2s} \frac{(<kx>^\dagger)^3<kl><yl>}{(<xl>^\dagger)^2}
\end{aligned}
$$

$$\mathcal{M}_C(-,-;-,+)=\frac{-ie\kappa\sqrt{2}(1+\alpha)e^{i\theta}}{2s}\frac{<kx>^3<kl>^\dagger<yl>^\dagger}{<lx>^2}$$
$$\mathcal{M}_C(-,-;+,-)=\frac{-ie\kappa\sqrt{2}(1+\alpha)e^{i\theta}}{2s}\frac{(<ky>^\dagger)^3<kl><xl>}{(<ly>^\dagger)^2}\,, \qquad (40)$$

and the other contributions are zero,

$$\mathcal{M}_C(+,+;\pm,\pm)=\mathcal{M}_C(-,-;\pm,\pm)=\mathcal{M}_C(\pm,\mp;\pm,\pm)=\mathcal{M}_C(\pm,\mp;\pm,\mp)=0\,.$$

By adding up all the contributions using that $<kx><ky>^\dagger=-<lx><ly>^\dagger$, we obtain, for the invariant amplitude squared,

$$|\mathcal{M}(+,+;-,+)|^2=|\mathcal{M}(-,-;+,-)|^2=\frac{2e^2\kappa^2m^2}{\alpha}\left(\frac{\alpha^2}{t-m^2}+\frac{1}{u-m^2}+\frac{\alpha}{m^2}\right)^2\frac{(x\cdot l)^4}{(x\cdot k)^2}$$
$$|\mathcal{M}(+,+;+,-)|^2=|\mathcal{M}(-,-;-,+)|^2=\frac{2e^2\kappa^2m^2}{\alpha}\left(\frac{1}{t-m^2}+\frac{\alpha^2}{u-m^2}+\frac{\alpha}{m^2}\right)^2\frac{(x\cdot k)^4}{(x\cdot l)^2}$$
$$|\mathcal{M}(+,-;+,-)|^2=|\mathcal{M}(-,+;-,+)|^2=|\mathcal{M}(+,-;-,+)|^2=|\mathcal{M}(-,+;+,-)|^2$$
$$=128e^2\kappa^2\alpha(\alpha-1)^4\left(\frac{1}{t-m^2}+\frac{1}{u-m^2}\right)^2\frac{(x\cdot k)^3(x\cdot l)^3}{s^3}$$
$$|\mathcal{M}(+,-;+,+)|^2=\alpha^2|\mathcal{M}(-,+;+,+)|^2=\frac{e^2\kappa^2m^6}{2}\left(\frac{\alpha}{t-m^2}-\frac{1}{u-m^2}\right)^2\frac{(x\cdot l)}{(x\cdot k)}$$
$$|\mathcal{M}(-,+;-,-)|^2=\alpha^2|\mathcal{M}(+,-;-,-)|^2=\frac{e^2\kappa^2m^6}{2}\left(\frac{1}{t-m^2}-\frac{\alpha}{u-m^2}\right)^2\frac{(x\cdot k)}{(x\cdot l)}\,. \qquad (41)$$

It is important to note that the above result remains the same whether we choose $\alpha=\alpha_+$ or α_-. Finally,

$$\left|\mathcal{M}(e^++e^-\rightarrow\gamma+g)\right|^2=4e^2\kappa^2\frac{(m^4-tu)}{s}\left[\left(\frac{m^2}{t-m^2}+\frac{m^2}{u-m^2}\right)^2\right.$$
$$\left.+\frac{m^2}{t-m^2}+\frac{m^2}{u-m^2}-\frac{1}{4}\left(\frac{t-m^2}{u-m^2}+\frac{u-m^2}{t-m^2}\right)\right]\,. \qquad (42)$$

APPENDIX: HELICITY WAVE FUNCTIONS

We present here the explicit form of the helicity wave functions for the different fields involved.

1. Spin-$\frac{1}{2}$ Fermions

Using the Weyl representation of the γ-matrices

$$[\gamma^\mu]^A{}_B = \begin{pmatrix} 0 & \sigma^\mu_{a\dot b} \\ \bar\sigma^{\mu\dot a b} & 0 \end{pmatrix} , \tag{A1}$$

the Dirac spinor is written

$$\psi^A = \begin{pmatrix} \xi_a \\ \bar\eta^{\dot a} \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} . \tag{A2}$$

For massless fermions, the helicity spinors are [5]

$$u(p,+) = v(p,-) = \begin{pmatrix} 0 \\ \bar p^{\dot a} \end{pmatrix} ; \quad u(p,-) = v(p,+) = \begin{pmatrix} p_a \\ 0 \end{pmatrix} ; \tag{A3}$$

where + (−) signs denote the right (left) handed spinors. The spinor p_a is given by Eq.(28).

In the massive case the momentum of the fermion (k) is expressed as a sum of two light-like momenta

$$k^\mu = p^\mu + q^\mu, \qquad (p^2 = q^2 = 0)$$

and the spinors K, P and Q are built up in the same way of Eq.(27)

$$K_{a\dot b} = P_{a\dot b} + Q_{a\dot b} \equiv p_a \bar p_{\dot b} + q_a \bar q_{\dot b} . \tag{A4}$$

Since $k^\mu k_\mu = m^2$, we should have $< pq >= m \exp(i\theta)$, and we can write the solutions of Dirac equation as

$$u(k,+) = \begin{pmatrix} q_a \\ e^{i\theta}\, \bar p^{\dot a} \end{pmatrix}$$

$$u(k,-) = \begin{pmatrix} p_a \\ -e^{i\theta}\, \bar q^{\dot a} \end{pmatrix} . \tag{A5}$$

The phase factor can be written in terms of the four-momentum components as

$$e^{i\theta} = \frac{1}{m} < pq >= \frac{1}{m}\left[(p_x - ip_y)\left(\frac{q_0 + q_z}{p_0 + p_z}\right)^{1/2} - (q_x - iq_y)\left(\frac{p_0 + p_z}{q_0 + q_z}\right)^{1/2}\right] .$$

The negative energy solution can be obtained through the phase convention $v(k,\lambda) = C\bar{u}^T(k,\lambda)$.

2. Photon

The spinors associated with the polarization vectors of a massless vector particle with momentum p, corresponding to states of definite helicity ($\lambda = \pm$), can be written as [5]

$$A^\mu = \frac{1}{2}\bar{\sigma}^{\mu\dot{a}b}A_{b\dot{a}} \, ,$$

with

$$\begin{aligned} A_{a\dot{b}}(p,+1) &= \sqrt{2}\frac{g_a\bar{p}_{\dot{b}}}{< pg >} \\ A_{a\dot{b}}(p,-1) &= \sqrt{2}\frac{p_a\bar{g}_{\dot{b}}}{< pg >^\dagger} \, , \end{aligned} \tag{A6}$$

where g is a gauge spinor: it can be arbitrarily chosen except for the spinor momentum p itself.

3. Spin-$\frac{3}{2}$ Fermions

Assuming the convention $\mathcal{V}_\mu(p,\lambda) = C\bar{\mathcal{U}}_\mu^T(p,\lambda)$, with C the charge conjugation matrix, the positive and negative energy spinor-vectors with helicity $\pm\frac{3}{2}$ ($\pm\pm$) for massless particles can be put into the form [6]

$$\begin{aligned} \mathcal{U}^{A,\mu}(p,++) = \mathcal{V}^{A,\mu}(p,--) &= \frac{1}{2}\bar{\sigma}^{\mu\dot{c}b}\begin{pmatrix} 0 \\ \sqrt{2}e^{i\theta}\frac{g^a p_b p_c}{<gp>} \end{pmatrix} \\ \mathcal{U}^{A,\mu}(p,--) = \mathcal{V}^{A,\mu}(p,++) &= \frac{1}{2}\bar{\sigma}^{\mu\dot{c}b}\begin{pmatrix} \sqrt{2}e^{-i\theta}\frac{p_a p_b \bar{g}_c}{<gp>^\dagger} \\ 0 \end{pmatrix} \, , \end{aligned} \tag{A7}$$

the double $\pm\pm$ signs denoting the helicities $\pm\frac{3}{2}$, and g being a gauge spinor.

4. Graviton

The helicity states of a massless spin-2 particle are [8]

$$h^{\mu\nu} = \frac{1}{4}\bar{\sigma}^{\mu\dot{a}b}\bar{\sigma}^{\nu\dot{c}d}H_{b\dot{a}d\dot{c}} ,$$

with

$$\begin{aligned} H_{b\dot{a}d\dot{c}}(p,+2) &= 2\frac{\bar{p}_{\dot{a}}r_b}{<pr>}\frac{\bar{p}_{\dot{c}}r_d}{<pr>} \\ H_{b\dot{a}d\dot{c}}(p,-2) &= 2\frac{\bar{r}_{\dot{a}}p_b}{<pr>^\dagger}\frac{\bar{r}_{\dot{c}}p_d}{<pr>^\dagger} . \end{aligned} \tag{A8}$$

Here r is also a gauge spinor. For instance, two different choices of the spinor r (r and r') would lead to

$$H_{b\dot{a}d\dot{c}}(r') = H_{b\dot{a}d\dot{c}}(r) - i\left(P_{d\dot{c}}\Lambda_{b\dot{a}} + P_{b\dot{a}}\Lambda_{d\dot{c}}\right) ,$$

with

$$\Lambda_{b\dot{a}} = \frac{2i<r'r>}{<pr'>^2<pr>^2}\left(<pr'>r_b\bar{p}_{\dot{a}} + \frac{1}{2}<r'r>p_b\bar{p}_{\dot{a}}\right) .$$

The sum over helicities is written as

$$\sum_{\lambda=\pm 2} H_{a\dot{b}c\dot{d}}(p,\lambda)\left[H_{e\dot{f}g\dot{h}}(p,\lambda)\right]^\dagger = \frac{16}{|<pr>|^4}\left[r_a\bar{p}_{\dot{b}}r_c\bar{p}_{\dot{d}}\bar{r}_{\dot{e}}p_f\bar{r}_{\dot{g}}p_h + (p\leftrightarrow r)\right] ,$$

which is equivalent to the vectorial form

$$\sum_{\lambda=\pm 2} h^{\mu\nu}(p,\lambda)\left[h^{\alpha\beta}(p,\lambda)\right]^\dagger = \frac{1}{2}\left(\mathcal{P}^{\mu\alpha}\mathcal{P}^{\nu\beta} + \mathcal{P}^{\mu\beta}\mathcal{P}^{\nu\alpha} - \mathcal{P}^{\mu\nu}\mathcal{P}^{\alpha\beta}\right) , \tag{A9}$$

where

$$\mathcal{P}^{\mu\nu} = -g^{\mu\nu} + \frac{(p^\mu r^\nu + p^\nu r^\mu)}{(p\cdot r)} ,$$

which shows that only the physical states contribute to Eq.(A9).

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REFERENCES

[1] P. de Causmaecker, R. Gastmans, W. Troost and T. T. Wu, Phys. Lett. **105B** (1981) 215;

J. F. Gunion and Z. Kunszt, Phys. Lett. **161B** (1985) 333;

R. Kleiss and W. J. Stirling, Nucl. Phys. **B262** (1985) 235;

Z. Xu, D.-H. Zhang and L. Chang, Nucl. Phys. **B291** (1987) 392

[2] R. Gastmans and T. T. Wu, The Ubiquitous Photon: Helicity Method for QED and QCD (Clarendon Press, 1990)

[3] H. Weyl, The Theory of Groups and Quantum Mechanics (Methuen, London, 1931); Zeit. Phys. **56** (1929) 330;

B.L. van der Waerden, Gött. Nachr. **K1** (1929) 100;

E.M. Corson, Introduction to Tensors, Spinors, and Relativistic Wave-Equations (Hafner Pub. Co., New York, 1953)

[4] G.R. Farrar and F. Neri, Phys. Lett. **130B** (1983) 109;

A. Kersch and F. Scheck, Nucl. Phys. **B263** (1986) 475;

K. Hagiwara and D. Zeppenfeld, Nucl. Phys. **B274** (1986) 1

[5] F. A. Berends and W. T. Giele, Nucl. Phys. **B294** (1987) 700

[6] S. F. Novaes and D. Spehler, Nucl. Phys. **B371** (1992) 618

[7] F. A. Berends, W. T. Giele and H. Kuijf, Phys. Lett. **211B** (1988) 91

[8] D. Spehler and S. F. Novaes, Phys. Rev. **D44** (1991) 3990

[9] D. Ivanenko and G. Sardanashvily, Phys. Rep. **94** (1983) 1

[10] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. 1 (Interscience Pub., New York, 1963)

[11] B. S. DeWitt and C. M. DeWitt, Phys. Rev. **87** (1952) 116; V. I. Ogievetskii and I. V. Polubarinov, Sov. Phys. JETP **21** (1965) 1093

[12] F. A. Berends and R. Gastmans, Nucl. Phys. **B88** (1975) 99

[13] B. S. DeWitt, Phys. Rev. **162** (1967) 1239

[14] J. G. M. Kuijf, Multiparton Production at Hadron Colliders, Ph.D. Thesis, Leiden Univ. (1990)

[15] We adopt the conventions of J. Wess and J. Bagger, Supersymmetry and Supergravity (Princeton Univ. Press, Princeton, 1983) with a different signature for the metric. We adopt the metric tensor $\eta_{\mu\nu}$ with $diag(\eta_{\mu\nu}) = (1, -1, -1, -1)$