IC/92/266

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

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INTERNATIONAL ATOMIC ENERGY AGENCY

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ABSTRACT

In this paper, we treat a general method of quantization of Hamiltonian systems whose flow is a subgroup (not necessarily closed) of a torus acting freely and symplectically on the phase space. The quantization of some classes of completely integrable systems as well as the Borel–Weil–Bott version of representation theory are special cases.

MIRAMARE – TRIESTE
September 1992

* Research partially supported by Contract MM1/91 with the Bulgarian Ministry of Science and Education.

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Recently there has been great interest in the quantization of compact phase spaces [1-4]. These spaces do not have in general globally defined potentials of their symplectic forms. This is a serious obstacle for the application of the Schrödinger quantization scheme and the issue is settled by the geometric quantization of Kostant [5] and Souriau [6]. Quantization of classical phase spaces is also a general method for obtaining representations of semisimple Lie groups [5]. Essentially this is an interpretation of the Borel - Weil - Bott theorem. The procedure is straightforward, but the explicit description of a given representation requires explicit coordinates on the phase space. Obviously, the compact phase spaces cannot be parametrized globally. The way to surround the computations with local coordinates is proposed by Gates et al [2]. These authors show how to treat phase spaces of the form G/T where G is a compact Lie group and T is a maximal torus subgroup. Spaces of this type are known as flag manifolds.The key idea in [2] is that instead of direct quantization of relevant coset spaces one can consider larger flat spaces with Poisson bracket structures, and then following Dirac to quantize the constrained systems. The results obtained in [2] look so nice and the technique that has been used is so simple to deserve as such some comments. Especially one can ask the natural question about the place and role of geometric quantization in this situation. One of the purposes of this paper is by tracing back the principal stages of the mentioned construction to compare it with geometric quantization. We hope that the present paper will convince the reader that this is the proper scheme to be used for quantization of compact phase spaces. Besides we indicate how a slightly more general class then flag manifolds can be treated.

Now we turn to listing below deficiencies of the scheme given in [2].

1. The phase spaces of the type $\mathbb{G}/\mathbb{T}$ appear there in an artificial way.
2. The same holds for the Poisson brackets introduced in the ambient flat spaces.
3. The quantization conditions determining the admissible symplectic structures are rather unexplaned.
4. No argument is presented why quantizations of the extended phase spaces are in one to one correspondence with quantizations of the reduced phase spaces.
5. There is no sound reason why only half of the second class constraints are used for the appropriate physical states to be defined.
6. No method is even mentioned for counting dimensionalities of the unitary irreducible representations.

In what follows all these points will be discussed in some extent.

Let $X \xrightarrow{p} M$ be a principal fibre bundle, where the global space X is a compact smooth manifold, the structure group $\mathbb{T} = U(1)^{\times k}$ is a k-dimensional torus , and the base M is a symplectic manifold. The case when k = 1 is known and studied under the name of "regular contact manifold".

Let $\mathfrak{t}$ be the Lie algebra of $\mathbb{T}$. Obviously $\mathfrak{t} \cong \mathbb{R}^k$ with the abelian Lie algebra structure.

Let ω be a symplectic form on M.

Theorem 1. There exists a symplectic manifold (P,Ω) with a free symplectic action

$$\Phi: \mathbb{T}\times P \longrightarrow P,$$

with momentum map

$$J: P \longrightarrow t^*,$$

such that the Marsden-Weinstein reduction of the action Φ gives for each $\mu \in \mathfrak{t}^*$

$$X \cong J^{-1}(\mu) \;;\; P_\mu \cong M$$

Moreover, there exists a value $\nu \in \mathfrak{t}^*$, such that

$$\Omega_\nu = \omega$$

We start with a compact, connected, semisimple Lie group $\mathbb{G}$ which acts on itself by left translations

$$L_g \;:\; \mathbb{G} \longrightarrow \mathbb{G} \,,\quad h \longrightarrow L_g h = gh \tag{1}$$

This action can be lifted to the symplectic action on $\mathbb{T}^*\mathbb{G}$ which is Hamiltonian and has Ad^* - equivariant momentum map

$$J_L \;:\; \mathbb{T}^*\mathbb{G} \longrightarrow g^* \;;\; \alpha_g \longrightarrow J_L(\alpha_g) = (T_eR_g)^* \alpha_g \tag{2}$$

It is easy to see that each $\mu \in g^*$ is a regular value of J_L. Also, because each fibre $T_g^*\mathbb{G} \in T^*\mathbb{G}$ can be translated onto g^* by J_L there exists only one point $\alpha_g \in T_g^*\mathbb{G}$ that maps to a given $\mu \in g^*$. Thus

$$J_L^{-1}(\mu) = \{ \alpha \in T^*\mathbb{G} \ ; \ J_L(\alpha_g) = (T_e R_g)^* \alpha_g = \mu \} \tag{3}$$

In fact, the momentum map J in our situation coincides with the standard map from $T^*\mathbb{G}$ to g obtained from identifying left invariant vector fields with their values at $T_e\mathbb{G}$. Next, let us define the right-invariant form α_μ on $\mathbb{G}$ according to the rule

$$\alpha_\mu(g) = (T_g R_{g^{-1}})^* \mu \quad \text{and} \quad \alpha_\mu(e) = \mu \tag{4}$$

The graph of the right-invariant one-form α_μ is

$$J_L^{-1}(\mu) = \{ (g, \alpha_\mu(g)); \ g \in \mathbb{G} \} \tag{5}$$

It is easy to see that the isotropy subgroup $\mathbb{G}_\mu$ at μ of the co-adjoint action can be equivalently defined as

$$\mathbb{G}_\mu = \{ g \in \mathbb{G}; \quad L_g^* \alpha_\mu = \alpha_\mu \} \tag{6}$$

while its Lie algebra

$$g_\mu = \{ \xi \in g; \quad ad_\xi^* \mu = 0 \} \tag{7}$$

is defined respectively by

$$g_\mu = \{ \xi \in g; \quad L(\xi_{\mathbb{G}})\alpha_\mu = 0 \} \tag{8}$$

Here $L(\xi_{\mathbb{G}})$ is the Lie-derivative with respect to the right-invariant vector field $\xi_{\mathbb{G}}$ generated by $\xi \in g$. Obviously, $\mathbb{G}_\mu$ acts on $J_L^{-1}(\mu)$ by left translations of the base points and this gives us the identification

$$\varphi_\mu : \quad (g, \alpha_\mu(g)) \longrightarrow g^{-1} \tag{9}$$

whence

$$J_L^{-1}(\mu) \cong \mathbb{G} \tag{10}$$

Further, denoting by O_μ the orbit of μ under co-adjoint action of $\mathbb{G}$ on g^* one can identify O_μ with $\mathbb{G}/\mathbb{G}_\mu$ via canonical projection $p_\mu : \mathbb{G} \longrightarrow O_\mu$ given by

$$p_\mu(g) = Ad^*_{g^{-1}}(\mu) \tag{11}$$

Hence, in the language of Marsden-Weinstein's reduction theorem [7]

$$\mathbb{P}_\mu \cong O_\mu \tag{12}$$

which follows from the formula

$$\pi_\mu(\alpha_\mu(g)) = Ad_g^*(\mu) \qquad \forall \ g \in \mathbb{G} \tag{13}$$

The diagram below summarizes the above remarks

$$\begin{array}{ccc} J_L^{-1}(\mu) & \overset{\varphi_\mu}{\longleftrightarrow} & \mathbb{G} \\ \pi_\mu \downarrow & & \downarrow p_\mu \\ \mathbb{P}_\mu & \longleftrightarrow & O_\mu \end{array} \tag{14}$$

Before proceeding with the second point we note that here the phase spaces in question appear in the context of reduction of classical mechanical systems with symmetries. This should be compared with motivation presented in [2] coming from Witten's ideas about topological quantum field theories [8]. There is no classical physics associated with these theories, and there the phase spaces appear as dequantized objects. Requantizing them we do not obtain more information than that we have at the start.

The invariant symplectic form ω_μ appears naturally in the setting of the reduction theorem. The two-form $d\alpha_\mu$ is $\mathbb{G}_\mu$ invariant and ω_μ on O_μ is defined by the formula

$$p_\mu^* \omega_\mu = d\alpha_\mu \tag{15}$$

We call μ <u>regular</u> if the Lie algebra

$$\eta_\mu = \{ \xi \in g_\mu ; \ i(\xi_{\mathbb{G}})\alpha_\mu = 0 \} \tag{16}$$

has one dimension less than g_μ, and if the connected component $\mathbb{N}_\mu^o$ of the Lie group $\mathbb{N}_\mu$ generated by η_μ is closed. It is easy to prove that η_μ is an ideal in g_μ. Really, if $\xi \in g_\mu$ and $\eta \in \eta_\mu$ one has

$$i([\xi_{\mathbb{G}},\eta_{\mathbb{G}}])\alpha_\mu = L(\xi_{\mathbb{G}})\ i(\eta_{\mathbb{G}})\alpha_\mu - i(\eta_{\mathbb{G}})L(\xi_{\mathbb{G}})\alpha_\mu = 0 \tag{17}$$

which means that $[\xi,\eta] \in \eta_\mu$ for any $\xi \in g_\mu$ and $\eta \in \eta_\mu$.

This situation suggests the following generalization:

Definition 1. A fibered dynamical system is a triple (M,Ω,D), where M is a smooth manifold, Ω is a closed two-form of constant rank and D is its kernel distribution

$$D \equiv \mathrm{Ker}\,\Omega = \{ X \in X(M);\ i(X)\Omega = 0 \} \tag{18}$$

which is a reductible foliation, i.e. the space of leaves M/D is a Hausdorff manifold.

Definition 2. When $\Omega = d\alpha$, $\alpha \in \Lambda^1(M)$ and $i(X_i)\alpha = \mu_i =$ const, where $\{X_i\}$ span globally D, $(M, d\alpha, D)$ is called an **exact fibered dynamical system.**

Definition 3. A foliation D will be called called **proper** if it

generates a global group of transformations of M.

Definition 4. The quotient space $O(D)$ of M with respect to the action of this group is called the **orbit manifold.**

Definition 5. A foliation D is called **regular** if the factor-topology of $O(D)$ is Hausdorff.

Theorem 2. (see Libermann & Marle [9]). If D is proper and regular then the orbit manifold $O(D)$ is a smooth symplectic manifold.

Example 1. (see Godbillon [10]). Let (M,α) be a contact manifold and $D \equiv Y$ be its contact vector field: $\alpha(Y) = 1$, $i(Y)d\alpha = 0$, then $(M, d\alpha, Y)$ is a fibered dynamical system.

Remark 1. It is easy to see that the definitions given above generalize the contact manifold structure and such manifolds can be referred to as hypercontact.

Going back to the phase spaces of the type $\mathbb{G}/\mathbb{T} = \mathbb{F}$ we can consider the short exact sequence

$$0 \longrightarrow \mathbb{T} \longrightarrow \mathbb{G} \longrightarrow \mathbb{F} \longrightarrow 0 \tag{19}$$

from which the long exact cohomology sequence follows

$$H^p(\mathbb{T},\mathbb{Z}) \longrightarrow H^p(\mathbb{G},\mathbb{Z}) \longrightarrow H^p(\mathbb{F},\mathbb{Z}) \longrightarrow H^{p-1}(\mathbb{T},\mathbb{Z}) \longrightarrow \tag{20}$$

Imposing a mild restriction on $\mathbb{G}$, $H^1(\mathbb{G},\mathbb{Z}) = 0$ one derives

$$H^2(\mathbb{F},\mathbb{Z}) = H^1(\mathbb{T},\mathbb{Z}) = \mathbb{Z}^r \tag{21}$$

where $r = \mathrm{rank}\mathbb{G} = \dim\mathbb{T}$ and $\mathbb{Z}^r$ corresponds exactly to the lattice of characters of $\mathbb{T}$. This can be viewed also in the following way. The Lie algebra g of $\mathbb{G}$ is the real span of the set

$$\{H_i, E_{\pm\alpha}\} \qquad i = 1,2,\dots,r\ , \quad \pm\alpha \in \Delta \tag{22}$$

where H_i is a basis of the maximal subset of commuting elements which form the Cartan subalgebra h of $\mathbb{G}$, while the label α denotes an element of the root vector space Δ.

The Lie algebra relations are of the following form:

$$[H, E_\alpha] = \alpha(H)E_\alpha \qquad H \in h \text{ and } \alpha \in \Delta$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta \\ H_\alpha \in h & \text{if } \alpha + \beta = 0 \end{cases} \tag{23}$$

The simultaneous eigenvector $|\mu> := |\mu_1, \ldots, \mu_r>$ of the Cartan subalgebra h

$$H_i|\mu> = \mu_i|\mu> \qquad i = 1,2,\ldots,r \tag{24}$$

is called a weight vector. The remaining elements of g, $E_{+\alpha}$, $E_{-\alpha}$ act as raising, respectively lowering operators. Among weight vectors there exists highest $|m>$, respectively lowest $|-m>$ weight vectors which satisfy

$$E_\alpha|m> = E_{-\alpha}|-m> = 0 \qquad \forall \alpha \in \Delta \tag{25}$$

The infinitesimal version of the equivariance property of the momentum map (2) is

$$\hat{J}([\xi,\eta]) = \{\hat{J}(\xi), \hat{J}(\eta)\} \tag{26}$$

where $\hat{J}(\xi) : \mathbb{T}^*\mathbb{G} \longrightarrow \mathbb{R}$ is defined by $\hat{J}(\xi)(\alpha_g) = <J_L(\alpha_g),\xi>$ and the meaning of (26) is that $\hat{J}$ is the homomorphism from the Lie algebra (23) to the Lie subalgebra of functions under Poisson bracket.

These functions are the constraints discussed in [2]. Following Dirac they can be divided into two groups. The first group consists of all constraints whose bracket with all other constraints vanishes. Second-class constraints are those which have at least one non-trivial bracket. Transputing (23) into a Lie algebra of functions we recognize

$\hat{J}(H_i) = \mu_i \qquad i = 1,2,\ldots,r$ as first-class

and $\hat{J}(E_{\pm\alpha}) = 0$ as second-class constraints.

Together $\hat{J}(H_i)$, $\hat{J}(E_{\pm\alpha})$ define the constrained manifold $J_L^{-1}(\mu)$ but the first-class constraints generate the action of $\mathbb{G}_\mu$ on this manifold. Removing these gauge degrees of freedom one obtains the reduced phase spaces (O_μ, ω_μ). Choosing μ to be regular, which means that all μ_i are distinct, $O_\mu = \mathbb{F}$ and then the symplectic form ω_μ can be written in the form

$$\omega_\mu = \sum_{i=1}^{r} \mu_i \Omega_i \tag{27}$$

where $\{\Omega_i\}_{i=1}^r$ generate $H^2(\mathbb{F},\mathbb{Z})$ (see (21)). Applying the prequantization condition to the symplectic form ω_μ

$$\frac{1}{2\pi}\int_{\sigma_i}\omega_\mu \in \mathbb{Z} \quad \text{for all } \sigma_i \in H_2(\mathbb{F}, 2\pi\mathbb{Z}) \tag{28}$$

one gets immediately

$$\mu_i \in \mathbb{Z}, \quad i = 1,2,\ldots,r \tag{29}$$

It is instructive to compare these quantization conditions with the prescriptions used in [2] to select the admissible Poisson structures. The scheme of [2] gives:

$$\alpha = \frac{2}{2n+m}, \qquad n,m \in \mathbb{Z} .$$

The prequantization conditions (29) imply the existence of the prequantum line bundle L_μ over $\mathbb{F}$. The Hilbert space is spanned by the global polarized sections of the quantum line bundle $L_\mu \otimes N^{1/2}$ (see Guillemin & Sternberg [11] for details). The first Chern class of the manifold of complete flags $\mathbb{G}/\mathbb{T}$ is always even (this might not be the case for an arbitrary flag manifold e.g. $\mathbb{P}^2$).

Thus the correction coming from the bundle of half-form amounts only to a shift of the spectrum in the most important case.

These have to be identified with the physical states over the extended phase space T^*G whenever the quantization scheme is coherent. It is known that geometric quantization of extended phase spaces is equivalent to that of reduced phase spaces within some categories of symplectic manifolds (see Gotay [12] and Guillemin & Sternberg [13]) but unfortunately the cases in which we are interested here do not fall among them.

Dirac's strategy for quantization in the presence of constraints is that they must be enforced quantum-mechanically if they have not been eliminated classically. Following this rule the authors of [2] define Fock states replacing canonical coordinates in constraints for creation and annihilation operators. Doing this they faced the operator ordering problem and choose to follow Weyl's prescription. Neverthless it is not possible to incorporate all second-class constraints and this difficulty is beset in [2] requiring only half of these constraints to annihilate physical states. At this point in [2] appealed to weaker condition that matrix elements between these states should vanish. In fact, the argument for this step is pretty clear - after quantization of the momentum map, one half of its components acts as raising and the second half as lowering operators.

Finally, the multiplicities of the representations associated with (O_μ, ω_μ) can be computed using Riemann-Roch-Hirzebruch theorem. For more details we refer to Guillemin & Sternberg [13] and Duistermaat [14].

ACKNOWLEDGMENTS

One of the authors (I.M.M.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

1. A. Alekseev, L. Faddeev and S. Shatashvili, "Quantization of Symplectic Orbits of Compact Lie Groups by Means of the Functional Integral", JGP, vol.5, pp.391-406, 1989.
2. E. Gates, R. Potting, C. Taylor and B. Velikson, "Quantizing Compact Phase Spaces:Irreducible Representations from Constrained Dynamics", Phys. Rev. Lett., vol.63, pp.2617-2620, 1989.
3. A. Bulgac and D. Kusnezov, "Classical Limit of Lie Algebras", Ann. Phys., vol.199, pp.187-224, 1990.
4. H. Kuratsuji,"Canonical Phases and Quantum Connection",to appear.
5. B. Kostant, "Quantization and Unitary Representations", Lect. Notes in Math., vol.170, pp.87-208, Springer, Berlin, 1970.
6. J.-M. Souriau, Structure des Systemes Dynamiques, Dunod, Paris, 1970.
7. J. Marsden and A. Weinstein, "Reduction of Symplectic Manifolds with Symmetry", Rep. Math. Phys., vol.5, pp.121-130, 1974.
8. E. Witten, "Quantum Field Theory and the Jones Polynomial", Comm. Math. Phys., vol.121, pp.351-399, 1989.
9. P. Libermann and C. Marle, Symplectic Geometry and Analytical Mechanics, Reidel, Dordrecht, 1987.
10. C. Godbillon, Geometrie Differentielle et Mecanique Analytique, Hermann, Paris, 1969.
11. V. Guillemin and S. Sternberg, Geometric Asymptotics, AMS Providence R.I., 1977.
12. M. Gotay, "Constraints, Reduction and Quantization", J. Math. Phys., vol.27, pp.2051-2066, 1986.
13. V. Guillemin and S. Sternberg, "Geometric Quantization and Multiplicities of Group Representations", Invent. Math., vol.67, pp.515-538, 1982.
14. J.Duistermaat, "Multiplicites de Representations de Groupes et l'Espace de Phase Reduit", Travaux en Cours, vol.27, pp.71-81, Hermann, Paris, 1988.