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GENERATING FUNCTION FOR CLEBSCH-GORDAN COEFFICIENTS OF THE SUq(2) QUANTUM ALGEBRA

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Generating Function for Clebsch - Gordan Coefficients of the $su_q(2)$ Quantum Algebra

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Abstract

Some methods have been developed to calculate the $su_q(2)$ Clebsch-Gordan coefficients (CGC). Here we develop a method based on the calculation of Clebsch-Gordan generating function through the use of "quantum algebraic" coherent states. Calculating the $su_q(2)$ CGC by means of this generating function is an easy and straightforward task.

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I. INTRODUCTION

Quantum algebras have been extensively used in the literature for different purposes, in different areas of interest [1]. They are also known as quantum universal enveloping algebras and are mathematically not less than Hopf algebras.

The study of coherent states associated with quantum algebras for the q- harmonic oscillator has been established some time ago [2] and a resolution of unity for the q-analog oscillator coherent states has already been found through the use of the definitions of q-differentiation and q-integration, as seen in ref. [3] and [4]. For the $su_q(2)$ algebra, q-analogs of Perelomov [5] coherent states have been obtained and they are shown to have a resolution of unity related to the q-differential calculus [6].

The aim of this work is to calculate a generating function for the $su_q(2)$ Clebsch - Gordan coefficients. The method we use here is based on the idea developed in refs. [7] for the usual $su(2)$ Clebsch - Gordan coefficients. For calculating the $su_q(2)$ Clebsch - Gordan coefficients, other methods have also been developed , namely, some algebraic methods [8],[9] and a method based on the basic hypergeometric functions [10]. The underlying idea for the following derivation of the Clebsch - Gordan generating function is extremely simple and yields very easily handed expressions. Once the generating functions are obtained it becomes trivial to write the *wanted* $su_q(2)$ Clebsch - Gordan coefficients down, as will be shown latter on in this paper.

II. $SU_Q(2)$ COHERENT STATES

We start this section by giving some definitions and expressions which will be vital to the development of our method. For the sake of completeness and simplicity in the main text, some formulae are given in appendix 1.

The generators of the $su_q(2)$ algebra obey the following commutation relations [2]

$$[J_0, J_\pm] = \pm J_\pm$$
$$[J_+, J_-] = [2J_0] \qquad (1)$$

where the q-number $[x]$ is defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{2}$$

The above operators, when applied to a basis $|j\ m>$ of the carrier space V^j of the representation T^j of $su_q(2)$ yield

$$J_0|j\ m> = m|j\ m>$$
$$J_\pm|j\ m> = ([j \mp m][j \pm m + 1])^{1/2}|j\ m \pm 1>. \tag{3}$$

with $m = -j, -j+1, .., j$ and $j = 0,\ 1/2,\ 1, \ldots$.

The q-analogs of the $su(2)$ Perelomov coherent states are usually written as [6]

$$|z> = e_q^{zJ_+}|j\ -j> = \sum_{m=-j}^{j} C_{jm} \bar{z}^{j+m}|j\ m>, \tag{4}$$

where

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[x]!} \tag{5}$$

and

$$C_{jm} = \begin{bmatrix} 2j \\ j+m \end{bmatrix}^{1/2} \tag{6}$$

is the q-binomial defined in appendix 1.

For this representation, it can be shown that the resolution of unity is [6]

$$I = \int d\mu(z)|z><z|, \tag{7}$$

the measure being

$$d\mu(z) = \frac{[2j+1]}{\pi} \frac{1}{[1(+)|z|^2]^{2j+2}} d^2z, \tag{8}$$

where $d^2z = \frac{1}{2}d\theta d_q|z|^2$, the integration over θ running from 0 to 2π and the q-integration over $|z|^2$ from 0 to infinity.

Given an arbitrary state $|\phi>$ in V^j and due to the existence of the resolution of unity, one may find a holomorphic(or q-analog Bargmann [11]) representation that reads

$$\phi(z) = < z|\phi > \tag{9}$$

and the standard $su_q(2)$ basis is given by

$$\phi_{jm}(z) = < z|j\ m >= C_{jm}\ z^{j+m}. \tag{10}$$

The $su_q(2)$ operators in the holomorphic representation, which obey the commutation relations given by eqs. (1) are [6]

$$\begin{aligned} J_0 &= z\frac{\partial}{\partial z} - j \\ J_+ &= z[2j - z\frac{\partial}{\partial z}] \\ J_- &= D_z \end{aligned} \tag{11}$$

where D_z is the q-derivative [3] such that

$$D_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}. \tag{12}$$

Observing eq. (4), it follows that the scalar product between two coherent states $|z>$ and $|\chi>$ is

$$< \chi|z >= [1\ (+)\ \chi\bar{z}]^{2j}, \tag{13}$$

where $[x(\pm)y]^m$ is defined in appendix 1. Given an arbitrary state $|\phi>$ in V^j, such that $\phi(\chi) =< \chi|\phi>$ and utilizing eqs. (7) and (13), the reproducing kernel is shown to be

$$k(\chi, z) = [1\ (+)\ \chi\bar{z}]^{2j} \tag{14}$$

and therefore

$$\phi(\chi) = \int d\mu(z) k(\chi, z)\phi(z). \tag{15}$$

From now on, all calculations are performed in the q-deformed space.

III. VECTOR ADDITION OF ANGULAR MOMENTA

The total angular momentum of an $su_q(2)$ system consisting of two sub-systems $\vec{j} = \vec{j}_1 + \vec{j}_2$, where $\vec{j}_1$ and $\vec{j}_2$ are the angular momenta of sub-systems 1 and 2 respectively, such that $|j_1 - j_2| \leq j \leq j_1 + j_2$ is given by [8]- [9]

$$J_0(12) = J_0(1) \otimes I(2) + I(1) \otimes J_0(2)$$
$$J_\pm(12) = J_\pm(1) \otimes q^{J_0(2)} + q^{-J_0(1)} \otimes J_\pm(2) \tag{16}$$

where $J_0(12)$ and $J_\pm(12)$ obey the following commutation relations:

$$[J_0(12), J_\pm(12)] = \pm J_\pm(12)$$
$$[J_+(12), J_-(12)] = [2J_0(12)]. \tag{17}$$

Notice that $J_0(i)$ and $J_\pm(i)$ are the operators defined in eqs.(11), where for $i = 1$, z and j become z_1 and j_1 and for $i = 2$, they become z_2 and j_2. The same modifications hold in eqs. (9) and (10).

IV. COHERENT STATES IN THE SPACE $V^{J_1} \otimes V^{J_2}$

The uncoupled basis in the space $V^{j_1} \otimes V^{j_2}$ has the form $|j_1 m_1 > |j_2 m_2 >$ and its representation is $\phi_{j_1 m_1}(z_1)\phi_{j_2 m_2}(z_2)$. It is well known that the direct product of representations $T^{j_1} \otimes T^{j_2}$ can be decomposed as a direct sum $T^{j_1} \otimes T^{j_2} = \sum_j \oplus T^j$ [12] where $|j_1 - j_2| \leq j \leq j_1 + j_2$. In the carrier space V^j of T^j, the coherent state is defined to be

$$|z >= e_q^{zJ_+(12)} |j \ -j > \tag{18}$$

in analogy with (4), where $|j \ -j >$ stands for the lowest weight state in the V^j space. From the above considerations, it is straightforwardly shown that the state obeying the conditions $J_-(12)\phi_{j,-j}(z_1, z_2) = 0$ and $J_0(12)\phi_{j,-j}(z_1, z_2) = -j\phi_{j,-j}(z_1, z_2)$ is

$$\phi_{j,-j}(z_1, z_2) = C_j \, [z_1 \, (-) \, q^{-(j+1)} z_2]^{j_1 + j_2 - j} \tag{19}$$

where

$$C_j = \left(\frac{[2j_1]![2j_2]![2j+1]!}{[j_1+j_2-j]![j_1-j_2+j]![j-j_1+j_2]![j_1+j_2+j+1]!} \right)^{1/2} \times$$
$$q^{\frac{1}{2}(j_1+j_2-j)(j+1+j_1-j_2)} \qquad (20)$$

is the normalization constant. In the notation used for this calculation, $\phi_{j,-j}(z_1,z_2)$ is defined as

$$\phi_{j,-j}(z_1,z_2) = < z_1\ z_2|j-j >$$

where $|z_1\ z_2> = |z_1>|z_2>$ with $|z_1>$ being the coherent state in the space V^{j_1} and $|z_2>$ the coherent state in V^{j_2}.

In the above calculations, the q-derivatives written in appendix 1 have been used.

V. GENERATING FUNCTIONS

From the definitions for the coherent states in the V^j space, vide eq. (18) and in the $V^{j_1} \otimes V^{j_2}$ space, we obtain

$$< z_1\ z_2|\chi > = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m=-j}^{j} \begin{bmatrix} 2j_1 \\ j_1+m_1 \end{bmatrix}^{1/2} \begin{bmatrix} 2j_2 \\ j_2+m_2 \end{bmatrix}^{1/2} \begin{bmatrix} 2j \\ j+m \end{bmatrix}^{1/2}$$
$$z_1^{j_1+m_1}\ z_2^{j_2+m_2}\ \bar{\chi}^{j+m} < j_1\ m_1\ j_2\ m_2\ |j\ m>_q \qquad (21)$$

and we are left with the calculation of $< z_1\ z_2|\chi >$, i.e., the generating function from which the Clebsh-Gordan coefficients appearing in the right-hand side of the above expression can be easily obtained by q- differentiation, as is discussed later. One has to bear in mind that $|z_1>|z_2> = |z_1\ z_2>$ is the coherent state in the $V^{j_1} \otimes V^{j_2}$ space and $|\chi>$ is the coherent state coming from the V^j space. Using the definition for the resolution of unity in space $V^{j_1} \otimes V^{j_2}$ (which follows in a straightforward way from eqs. (7) - (8)), we can write

$$< z_1\ z_2|\chi > = \int d\mu(\xi_1)d\mu(\xi_2) < z_1|\xi_1 >< z_2|\xi_2 >< \xi_1\ \xi_2|e_q^{\bar{\chi}J_+(12)}|j\ -j > \qquad (22)$$

where $d\mu(\xi_1)$ and $d\mu(\xi_2)$ are the measures in spaces V^{j_1} and V^{j_2} respectively (see eq. (8)). With the help of eqs. (14)- (15) we obtain

$$< z_1\ z_2|\chi >= e_q^{\bar{\chi} J_+(12)} \phi_{j_1 - j}(z_1, z_2). \tag{23}$$

From the fact that [13]

$$e_q^{\bar{\chi} J_+(12)} |j_1\ m_1 > |j_2\ m_2 >= e_q^{\bar{\chi} q^{-m_2} J_+(1)} |j_1\ m_1 >\ e_q^{\bar{\chi} q^{-m_1} J_+(2)} |j_2\ m_2 > \tag{24}$$

and

$$e_q^{\lambda J_+}\ z^n = [1\ (+)\ \lambda z]^{2j-n}, \tag{25}$$

$< z_1\ z_2|\chi >$ becomes

$$< z_1\ z_2|\chi >= C_j \sum_{m=0}^{j_1+j_2-j} \begin{bmatrix} j_1 + j_2 - j \\ m \end{bmatrix} (-q^{-(j+1)})^m z_1^{(j_1+j_2-j-m)} z_2^m \times$$
$$[1\ (+)\ \bar{\chi} q^{(m-j_2)} z_1]^{(j_1-j_2+j+m)} [1\ (+)\ \bar{\chi} q^{(j-j_2+m)} z_2]^{(2j_2-m)}. \tag{26}$$

With the use of the identity [13]

$$[1\ (+)\ q^a z]^r [1\ (+)\ q^{r+s+a} z]^s = [1\ (+)\ q^{s+a} z]^{r+s}, \tag{27}$$

eq.(26) reads

$$< z_1\ z_2|\chi >= C_j\ [1\ (+)\ \bar{\chi} q^{-j_2} z_1]^{(j_1-j_2+j)}\ [1\ (+)\ \bar{\chi} q^{j_1} z_2]^{(j_2-j_1+j)} \times$$
$$\sum_{m=0}^{j_1+j_2-j} \begin{bmatrix} j_1 + j_2 - j \\ m \end{bmatrix} (-q^{-(j+1)})^m z_1^{(j_1+j_2-j-m)} z_2^m \times$$
$$[1\ (+)\ \bar{\chi} q^{(j_1-2j_2+j+m)} z_1]^m\ [1\ (+)\ \bar{\chi} q^{(j_1-2j_2+m)} z_2]^{(j_1+j_2-j-m)}. \tag{28}$$

Finally, another simplification can be performed with the help of the expression

$$\sum_{m=0}^{J} \begin{bmatrix} J \\ m \end{bmatrix} (-1)^m [x\ (+)\ y q^{m-1}]^m [z\ (+)\ y q^m]^{J-m} = [z\ (-)\ x]^J \tag{29}$$

yielding

$$< z_1\ z_2|\chi >= C_j\ [1\ (+)\ \bar{\chi} q^{-j_2} z_1]^{(j_1-j_2+j)} [1\ (+)\ \bar{\chi} q^{j_1} z_2]^{(j_2-j_1+j)} \times$$
$$[z_1\ (-)\ q^{-(j+1)} z_2]^{(j_1+j_2-j)}, \tag{30}$$

which is the Clebsch-Gordan generating function we are looking for. This generating function is the left-hand side of formula (21) and a simple example of how to obtain the Clebsch - Gordan coefficient itself by means of q-differentiation is worked out in appendix 2.

VI. CONCLUSION

In this paper we have developed a way of calculating $su_q(2)$ Clebsch - Gordan coefficients generating function through the use of coherent states in a holomorphic (or q- analog Bargmann) representation. The advantage of this method is that the generating function is obtained in a straightforward way. With the help of q-differentiation, the $su_q(2)$ Clebsch - Gordan coefficients are easily worked out from the generating function, as shown in appendix 2.

It is worth pointing out that for $q = 1$, the generating function written in eq. (30) becomes exactly expression (14) given by Belissard and Holtz [7] for the $su(2)$ algebra.

In writing this paper we have tried to use a notation which makes evident the similarities between the q-deformed and the usual expressions.

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VII. APPENDIX 1

Some formulae mentioned in the main text are shown below. The expression

$$[a\,(\pm)\,b]^m = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} a^{m-k}(\pm b)^k \tag{31}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]!}{[m-n]![n]!} \tag{32}$$

is the q-binomial definition. Some q-derivatives are

$$D_z(e_q^{az}) = a\, e_q^{az} \tag{33}$$

$$D_z\, z^n = [n]\, z^{n-1} \tag{34}$$

$$D_z(f(z)g(z)) = (D_z f(z))g(qz) + f(q^{-1}z)D_z g(z) \tag{35}$$

$$D_{z_2}[az_1(\pm)bz_2]^m = \pm[m]b[az_1(\pm)bz_2]^{m-1} \tag{36}$$

where a and b are constants and $[az_1(\pm)bz_2]^m$ is defined in eq. (31) above.

VIII. APPENDIX 2

Here we show two simple examples of how to obtain the Clebsch - Gordan coefficients by q- differentiating the generating function. From eq. (21) one can easily see that

$$< j_1\ m_1\ j_2\ m_2|j\ m >_q = \left(\frac{[j_1-m_1]!}{[2j_1]![j_1+m_1]!}\right)^{1/2}\left(\frac{[j_2-m_2]!}{[2j_2]![j_2+m_2]!}\right)^{1/2}\left(\frac{[j-m]!}{[2j]![j+m]!}\right)^{1/2}$$
$$D_{z_1}^{j_1+m_1} D_{z_2}^{j_2+m_2} D_{\bar{\chi}}^{j+m} < z_1\ z_2|\chi > |_{z_1=z_2=\bar{\chi}=0}. \tag{37}$$

1^{st} *example*) Here we obtain $< j_1\ m\ 2\ 0|j_1+2\ m >_q$, a necessary coefficient when calculating quadrupole transition probabilities in a large number of atomic and nuclear models [14]. Substituting the correct values for j_1, m_1, j_2, m_2, j and m in eq. (30) we have

$$< z_1\ z_2|\chi > = C_j\ [1(+)\bar{\chi}q^{-2}z_1]^{2j_1}\ [1(+)\bar{\chi}q^{j_1}z_2]^4. \tag{38}$$

By q-differentiating the above expression, we obtain

$$D_{\bar{\chi}}^{j_1+2+m} D_{z_2}^{2} D_{z_1}^{j_1+m} < z_1\ z_2|\chi > \Big|_{z_1=z_2=\bar{\chi}=0} = C_j \frac{[4][3][2j_1]![j_1+m+2]!}{[j_1-m]!} q^{-2m}, \tag{39}$$

where $C_j = 1$. Finally, substituting the above expression in (37), we end up with

$$< j_1 m 2\ 0|j_1+2\ m >_q = q^{-2m}\sqrt{\frac{[4][3][j_1-m+2][j_1-m+1][j_1+m+2][j_1+m+1]}{[2][2j_1+4][2j_1+3][2j_1+2][2j_1+1]}}$$

which is exactly the expression found in table IV of ref. [8].

2^{nd} *example*) The $< j_1\ m_1\ 1/2\ 1/2|j_1-1/2\ m >_q$ coefficient, where $m = m_1+1/2$ is calculated below. Again, from eq. (30) we may write

$$< z_1\ z_2|\chi > = C_j\ [1(+)\bar{\chi}q^{-1/2}z_1]^{2j_1-1}[z_1(-)q^{-(j_1+1/2)}z_2] \tag{40}$$

from where we calculate

$$D_{\bar{\chi}}^{j_1+m_1} D_{z_2}^{1} D_{z_1}^{j_1+m_1} < z_1\ z_2|\chi > \Big|_{z_1=z_2=\chi=0} = -C_j \frac{[2j_1-1]![j_1+m_1]!}{[j_1-m_1-1]!} \times$$
$$q^{-1/2(j_1+m_1)-(j_1+1/2)}, \tag{41}$$

where

$$C_j = q^{j_1} \frac{[2j_1]!}{\sqrt{[2j_1-1]![2j_1+1]!}} \tag{42}$$

and hence

$$< j_1\ m_1\ 1/2\ 1/2|j_1 - 1/2\ m >_q = -q^{-1/2(j_1+m+1/2)} \sqrt{\frac{[j_1 - m + 1/2]}{[2j_1+1]}}, \tag{43}$$

which is the $su_q(2)$ Clebsch - Gordan coefficient also found in table I of ref. [8].

In the same way, all Clebsch - Gordan coefficients can be obtained.

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