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M.H. Sarmadi

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

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# THE RING STRUCTURE OF CHIRAL OPERATORS FOR MINIMAL MODELS COUPLED TO 2D GRAVITY *

M.H. Sarmadi
International Centre for Theoretical Physics, Trieste, Italy
and
INFN, Sezione di Trieste, Trieste, Italy.

ABSTRACT

The BRST cohomology ring for $(p,q)$ models coupled to gravity is discussed. In addition to the generators of the ghost number zero ring, the existence of a generator of ghost number $-1$ and its inverse is proved and used to construct the entire ring. Some comments are made regarding the algebra of the vector fields on the ring and the supersymmetric extension.

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It was shown by Witten [1] and further elaborated by Witten and Zwiebach [2] that the physical operators of $c = 1$ model (compactified at the self-dual radius) coupled to 2D gravity have an interesting structure. The chiral operators of ghost number zero form a ring which is generated by two elements. Furthermore, the operators of ghost number $+1$ give rise to vector fields on the ring that satisfy the algebra of area preserving diffeomorphism. One would like to see if there is an analogous structure for the minimal models coupled to gravity. There is a major difference between these models and $c = 1$ model. For these models one has physical states for any ghost number $n_{gh} \in Z$ whereas for $c = 1$ case physical states exist only for $-1 \leq n_{gh} \leq 2$ (for the chiral states).

In this talk I will show that the chiral operators of the minimal models coupled to gravity can be generated by a few operators. The ones of ghost number zero can be generated [3] by two elements $x$ and $y$ which are related to the two generators of $c = 1$ ring mentioned above by an $SO(2,C)$ rotation on the two fields of $c = 1$, matter and Liouville $(X, \phi)$. Moreover, there are two operators of ghost numbers $-1$ and $+1$ denoted by $w$ and $v$, which we will show satisfy $v \cdot w \sim 1$ and therefore $v \sim w^{-1}$, which implies that $w^n \neq 0$ and $v^n \neq 0$ for all $n \in Z_+$. This means that all the chiral operators (of the relative cohomology) are of the form $w^n x^i y^j \quad n \in Z, \ 1 \leq i \leq p-2, \ 1 \leq j \leq q-2$.

To establish the notation, let us first recall how the chiral operators for these models are constructed. The most efficient way of obtaining these operators is by considering the cohomology classes of the BRST operator,

$$Q_B = \oint \frac{dz}{2\pi i} : (T^M(z) + T^L(z) + \frac{1}{2}T^G(z))c(z) :, \tag{1}$$

where $T^M$ and $T^L$ are the energy-momentum tensors of the matter and Liouville sectors, with central charges $c^M$ and $c^L = 26 - c^M$ respectively, and $T^G$ is the energy-momentum tensor of the ghost system $(b, c)$. For $(p,q)$ models, in which we will be interested, the matter sector has the central charge

$$c_{p,q} = 1 - \frac{6(q-p)^2}{pq}, \qquad (p < q, \ coprime) \tag{2}$$

and highest weight

$$\Delta_{r,s} = \frac{1}{4pq}\left[(qr - ps)^2 - (q-p)^2\right], \tag{3}$$

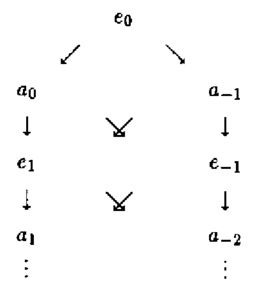
$$(1 \leq r \leq p-1, \ 1 \leq s \leq q-1).$$

An efficient way of formulating these $(p,q)$ CFT's is through the use of Coulomb gas.

They are described by a single boson $\phi^M$ which has a background charge $-\alpha_0$, where $\alpha_0 = (q-p)\sqrt{\frac{2}{pq}}$. The space corresponding to the irreducible verma module $\mathcal{H}_{r,s}$ is then obtained by using the screening operators $Q_+ = \int dz e^{i\alpha_+\phi^M}$ or $Q_- = \int dz e^{i\alpha_-\phi^M}$ where $\alpha_+ = \sqrt{\frac{2q}{p}}$ and $\alpha_- = -\sqrt{\frac{2p}{q}}$. It was shown by Felder [4] that $\mathcal{H}_{r,s}$ is isomorphic to the cohomology of $Q_F$ where $Q_F$ is an appropriate power of $Q_+$ or $Q_-$

After coupling these models to Liouville, and describing the Liouville sector by a free boson with the background charge $-\beta_0 = -(\beta_+ + \beta_-)$ where $\beta_+ = \alpha_+$ and $\beta_- = -\alpha_-$, then one obtains the physical states as the cohomology of $Q_B$ on the complex $\mathcal{H}_{r,s} \otimes \mathcal{F}^L_\beta \otimes \Lambda^{b,c}$ where $\mathcal{F}^L_\beta$ is the Fock module whose Liouville charge is $\beta$ and whose highest weight is $\Delta_\beta = -\frac{1}{2}\beta(\beta - \beta_0)$. It is more convenient to restrict oneself first to the relative cohomology which is the subspace annihilated by $b_0$ and then afterwards obtain the full space, i.e., the absolute cohomology, which has twice as many states as the relative one.

It was proven by Lian and Zuckerman [5,6] and also by Bouwknegt, McCarthy and Pilch [7] that the relative cohomology $H^{(n)}(\mathcal{H}_{r,s} \otimes \mathcal{F}^L_\beta \otimes \Lambda^{b,c}, Q_B)$ is non-zero for any integer $n$. As they showed, there is a correspondence between these states and the singular vectors of Verma module. To see what these states are, recall the embedding diagram of singular vectors of the Verma module in the matter sector given below:




In this diagram $a_t$ and $e_t$ are the weights of the singular vectors and are given by the following expressions:

$$a_t = \frac{1}{4pq}\left[(2pqt + qr + ps)^2 - (q-p)^2\right] \,,$$
$$e_t = \frac{1}{4pq}\left[(2qpt + qr - ps)^2 - (q-p)^2\right] \,. \tag{4}$$

If $\beta$ is chosen such that $1 - \Delta_\beta$ is equal to one of the weights in the above diagram then one can find a physical state whose Liouville charge is $\beta$ and and whose ghost number is



$n_{gh} = d_\beta sign(\beta - \frac{1}{2}\beta_0)$ where $d_\beta$ is the number of steps from that particular node to top node ($e_0 = \Delta_{r,s}$). Solving the equation $1 - \Delta_\beta \in \{a_t, e_t\}$ for $\beta$ one finds that for weights of type $a_t$,

$$\beta = \frac{1}{\sqrt{2pq}}(p + q \pm (2pqt + qr + ps)), \tag{5}$$

and for those of type $e_t$,

$$\beta = \frac{1}{\sqrt{2pq}}(p + q \pm (2pqt + qr - ps)). \tag{6}$$

Using the notation $\beta_{r,s} = \frac{1-r}{2}\beta_+ + \frac{1-s}{2}\beta_-$ where $\beta_+ = \sqrt{2q/p}$ and $\beta_- = \sqrt{2p/q}$, from the above expressions for the Liouville charges we see that in the $(r,s)$ sector the states of ghost number $-1$ (which give rise to operators of ghost number zero) have the charges $\beta_{r,s}$ and $\beta_{p-r,q-s}$. In the sectors $(1,2)$ and $(2,1)$ one can easily write two such ghost number zero operators:

$$x = \left(bc + (i\alpha_{1,2}\partial\phi^M + \beta_{1,2}\partial\phi^L)\right)e^{i\alpha_{2,1}\phi^M + \beta_{2,1}\phi^L} \,, \tag{7}$$

and

$$y = \left(bc + (i\alpha_{2,1}\partial\phi^M + \beta_{2,1}\partial\phi^L)\right)e^{i\alpha_{1,2}\phi^M + \beta_{1,2}\phi^L} \,. \tag{8}$$

In fact these two operators are just the two generators of $c = 1$ chiral ring after an $SO(2,C)$ rotation on the (matter, Liouville) field space. Then the two ghost number zero operators in the $(r,s)$ sector whose Liouville charges are $\beta_{r,s}$ and $\beta_{p-r,q-s}$ are respectively $x^{r-1}y^{s-1}$ and $x^{p-r-1}y^{q-s-1}$.

Further inspection of eqs. (5,6) for ghost numbers $-2$ and $+1$ states (i.e. ghost numbers $-1$ and $+1$ operators) shows that in the sector $(p-1,1) = (1,q-1)$ there are two operators, which we denoted as $w$ and $v$, which have Liouville charges $\beta_w = \beta_{1,q+1}$ and $\beta_v = -\beta_w$. This suggests that if $v \cdot w$ is non-zero then $v \cdot w \sim 1$. Moreover, from eqs. (5,6) one sees that all the Liouville charges are of the form $n\beta_w + \beta_{m,m'}$ for $n \in Z$, $1 \le m \le p-1$ and $1 \le m' \le q-1$. Therefore, if $v \cdot w$ does not vanish then any power of $w$ and $v = w^{-1}$ would be non-zero and one should be able to obtain ghost number $-n$ operators by taking the product of the $n$'th power of the operator $w$ and the ghost number zero operators, i.e., all the operators are of the form $w^n x^{m-1} y^{m'-1}$. Before proving the non-vanishing of $v \cdot w$ for the general $(p,q)$ models, let us look at two examples, namely the models $c^M = 0$ and $c^M = \frac{1}{2}$.

*(2,3) Model*

For this example there are two operators of ghost number zero, namely the identity operator and the operator $y$. One can also explicitly find the operator $w$ of ghost number $-1$ and Liouville charge $\beta_{3,1} = -\beta_+ = -\sqrt{3}$ by solving the equation

$$[Q_B, w] = 0 \text{ mod null operators.} \tag{9}$$

One finds the follwing expression for $w$:

$$w = (b\partial bc - \frac{1}{\sqrt{3}} b\partial^2\phi^L + \frac{1}{2\sqrt{3}}\partial b\partial\phi^L + \frac{1}{6}\partial^2 b)\ e^{-\sqrt{3}\phi^L}\,. \tag{10}$$

The operator $v$ of ghost number $+1$ and Liouville charge $\beta_+$ can be represented by $ce^{i\alpha_0\phi^M+\beta_+\phi^L}$ or by $ce^{\beta_+\phi^L}$ since both 1 and $e^{i\alpha_0\phi^M}$ represent the identity in the coulomb gas description of the matter sector. In any case after working out the operator product expansion one finds

$$v(z)w(0) = -\frac{1}{6}\mathbf{1},$$

and therefore the relative cohomology is generated by $w$ and $y$. In general, as in the case of $c = 1$ of ref. [2], to obtain the absolute cohomology which has twice as many states as the relative cohomology, one multiplies the operators in the relative cohomology by the physical operator

$$a = c\partial\phi + \frac{1}{2}\beta_0\partial c\,. \tag{11}$$

Thus all the operators are of the form $w^n y^i a^k$ where $n \in Z$ and $i, k = 0, 1$.

Two comments are in order. First, one can write a current whose charge acts on the ring as the vector field $w\partial_a$. This current should have ghost number $-2$ and the action of $Q_B$ on it should give a total derivative. For this example, the only such current one can write is $b\partial be^{-\sqrt{3}\phi^L}$ and it is in fact $(b_{-1}w) = \oint b(z)w$. and therefore

$$[Q_B, \oint b\partial be^{-\sqrt{3}\phi^L}] = 0.$$

One can explicitly check that it acts on the ring as $w\partial_a$. The second comment concerns eq. (9) which determines $w$. One has the freedom of adding null operators to $w$ and thereby change the representative for $w$. In fact by adding the term $\frac{1}{4}(\frac{i}{\sqrt{3}}\partial^2\phi^M - (\partial\phi^M)^2)e^{-\sqrt{3}\phi^L}$ to $w$



given in eq. (10) one obtains a representative which satisfies the equation $[Q_B, w] = [Q_+, x^2]$. In general one can choose a representative for $w$ which satisfies

$$[Q_B, w] = [Q_+, x^p].$$

*(3,4) model*

For this example, the ghost number zero operators are $x^i y^j$ $i = 0, 1,$ $j = 0, 1, 2$. To show that $v \cdot w$ is non-zero it is simpler to write first $w\partial_a$ and then show that its action on $av$ is not zero. To write $w\partial_a$ one needs to write a current of ghost number $-2$ on which the action of $Q_B$ gives a total derivative up to null operators. One finds that it is given by the following expression

$$\begin{aligned} j^{(-2)}(z) = \Big[ &[(-\frac{5}{2\sqrt{6}}\partial^2\phi^L - \frac{1}{2}(\partial\phi^L)^2) - \frac{11}{9}(-\frac{3}{2\sqrt{6}}\ i\partial^2\phi^M - \frac{1}{2}(\partial\phi^M)^2)]\,b\partial b \\ &- \frac{4}{3\sqrt{6}}\ i\partial\phi^M b\partial^2 b + \frac{1}{3}b\partial^3 b - \frac{11}{12}\partial b\partial^2 b\Big] e^{-2/\sqrt{6}\ i\phi^M - \sqrt{6}\phi^L}. \end{aligned} \tag{12}$$

Using this expression one finds that the action of $\oint j^{(-2)}$ on $av = a(z)c(0)e^{3i/\sqrt{6}\phi^M + \sqrt{6}\phi^L} = -\frac{5}{2\sqrt{6}}\ \partial cce^{3i/\sqrt{6}\phi^M + \sqrt{6}\phi^L}$ is non-zero.

Now we consider the general case of $(p, q)$ models. In order to prove that the product $v \cdot w$ does not vanish, we will show that a correlator of the type $< w\ v\ \mathcal{O}^{(3)} >$ on the sphere is non-zero. Here the operator $\mathcal{O}^{(3)}$ is a physical operator of ghost number 3 and Liouville charge $\beta_0$. It is just the product of the operator $a$ with the operator $\partial^2 cce^{\beta_0\phi}$. This latter operator satisfies the following equation [6,8]:

$$\partial^2 cce^{\beta_0\phi} = [Q_B\ ,\ c\partial\phi^M e^{\beta_0\phi^L}]. \tag{13}$$

Note that the operator $c\partial\phi^M e^{\beta_0\phi^L}$ is not a physical operator since $\partial\phi^M$ does not correspond to any of the states in the irreducible module. now inserting the right hand side of eq. (13) inside the correlator and pulling the contour of $Q_B$ to have it act on $w$ we can then use

$$[Q_B\ ,\ w] = [Q_+\ ,\ x^p]. \tag{14}$$

Now the contour of $Q_+$ can be pulled to have it act on $c\partial\phi^M e^{\beta_0\phi^L}$ to give $ce^{i\alpha_{-1,1}\phi^M + \beta_0\phi^L}$. One can also show that

$$x^p\ v\ \sim\ ce^{i\alpha_{1,-1}\phi^M}. \tag{15}$$

Therefore, the above correlator is reduced to the following two point function

$$\langle\alpha_{1,-1}, 0|\ c_{-1}c_0c_1\ |\alpha_{-1,1}, \beta_0\rangle,$$

which is clearly non-zero. This implies that the product $v \cdot w$ cannot vanish and we have therefore proven the proposed ring structure. Note that this ring is a non-commutative one, the operators $x$, $y$ and $a$ commute with each other but anti-commute with $w$.

For the above ring structure, since all integer powers of $w$ are present, it is natural to expect a Virasoro algebra for the vector fields. One can construct the vector fields as follows. Given a physical operator $\psi^{(n)}$ of ghost number $n$ one can construct $\oint(b_{-1}\psi^{(n)})$ which has ghost number $n-1$ and commutes with $Q$. It acts as a vector field on the ring. By this construction one finds that for the operators of type $w^n x^i y^j$ the vector fields are

$$G_n^{i,j} = w^n x^i y^j \partial_a \ ,$$

and for those of the type $aw^n x^i y^j$ the vector fields are

$$K_n^{i,j} = -x^i y^j w^n \left(w\partial_w + \frac{1}{p}x\partial_x + \frac{1}{q}y\partial_y - (n + \frac{i}{p} + \frac{j}{q})a\partial_a\right) \ . \tag{16}$$

After introducing the appropriate cocycle factors one finds that they satisfy the following algebra:

$$\begin{aligned}
[\tilde{K}_n^{i,j}, \tilde{K}_m^{k,l}] &= (n - m + \frac{i-k}{p} + \frac{j-l}{q})\tilde{K}_{n+m}^{i+k,j+l} \ , \\
[\tilde{K}_n^{i,j}, \tilde{G}_m^{k,l}] &= -(n + m + \frac{i+k}{p} + \frac{j+l}{q})\tilde{G}_{n+m}^{i+k,j+l} \ , \\
[\tilde{G}_n^{i,j}, \tilde{G}_m^{k,l}] &= 0 \ .
\end{aligned}$$

Therefore the vector fields $\tilde{K}_n^{0,0}$ satisfy a Virasoro algebra under which $\tilde{K}^{i,j}$ and $\tilde{G}^{i,j}$ are primaries of weights 2 and 0 respectively.

One can also consider the ring structure for $N = 1$ super-minimal models. Assuming that one can generalize Felder's discussion to these models, in refs. [9,10,6] it was pointed out that the results about the BRST chomology carry over to the supersymmetric case, that the cohomology is non-trivial for all ghost numbers. For $\hat{c} = 1$, it has been shown [11] that the ring of ghost number zero operators is also generated by two elements. After the appropriate $SO(2,C)$ rotation one obtains the ghost number zero ring for the super-minimal models. Moreover, one can write the Liouville charges for all the non-zero ghost number operators by solving the equation $\frac{1}{2} - \Delta_\beta \in \{a_t, e_t\}$, where $\{a_t, e_t\}$ are the set of weights of singular vectors for the super conformal case. One finds that again all these charges are of the form $n\beta_w + \beta_{m,m'}$ which suggests that the non-zero ghost number operators are obtained from integer powers of an element $w$.



The talk presented here was based on the work [12] done in collaboration with H. Kanno whom I would like to thank. A different ring structure for the minimal models has been proposed in ref. [13].

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