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STOCHASTIC MAGNETIC FIELD.

by

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Abstract.

The diffusive motion of charged particles in a stochastic magnetic field is investigated systematically in a model in which the statistics of both the collisions and the magnetic field are described by coloured noises characterized, respectively, by a finite correlation time and finite correlation lengths. An analytic solution is obtained for the basic nonlinear differential equation of the model. It describes asymptotically a pure diffusion process, in which the mean square displacement in the perpendicular direction, $\Gamma(t)$, grows proportionally to time (after a sufficiently long time). The corresponding diffusion coefficient scales like the fourth power of the magnetic fluctuation intensity. The values obtained are in very good agreement with experimental data in reverse-field pinch experiments. The present result contradicts earlier results predicting subdiffusive behaviour: $\Gamma(t) \sim t^{1/2}$ or $\Gamma(t) \sim t^{1/4}$. The relation of these results to ours is discussed in detail.

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I. INTRODUCTION.

It is well-known that fluctuations of the magnetic field can produce dramatic effects in magnetically confined plasmas[1-3]. In a first stage the topology of nested tori can be broken by the formation of magnetic islands near rational surfaces. As the intensity of the fluctuations grows, neighbouring islands end up overlapping and lead to the formation of regions in which the magnetic field lines are completely stochastic ("braided magnetic field"). The structure of such regions is vividly illustrated in the work of White[3] on tokamaks. An even more extreme case is provided by the reversed-field pinch experiments, in which the magnetic field is stochastic throughout most of the plasma [4]. The description of such a stochastic field requires a statistical treatment, which was first given in the work of Rechester & Rosenbluth[2].

An important problem arising here is the study of the behaviour of charged particles (hence, of the plasma) in such an environment. If collisions were the only mechanism at work, the particles would diffuse "normally" [5]. Their diffusion tensor would be anisotropic, with a diffusion coefficient parallel to the average magnetic field B_0, $\chi_{\parallel}$, much larger than the perpendicular diffusion coefficient $\chi_{\perp}$. It is also known that this conclusion is not borne out by experiments: anomalous diffusion processes lead to much stronger perpendicular diffusion than predicted by classical transport theory.

Turbulence produced by magnetic fluctuations is a possible candidate for explaining anomalous diffusion. We are in presence of a *doubly stochastic* process: the velocity of the particles undergoes random fluctuations due, on one hand to their mutual collisions and, on the other hand, to the random fluctuations of the magnetic field. Several authors have already considered this problem. In their pioneering work, Rechester & Rosenbluth[2] estimated semi--qualitatively a "diffusion coefficient" and found a typically "*subdiffusive behaviour*": the mean-square displacement of the particles in the perpendicular direction, $\Gamma(t) \equiv \langle \delta x^2(t) \rangle$ varies like $\sqrt{t}$ in their model. A much more detailed study of the problem was done by Krommes *et al.*[6], who confirmed the $\sqrt{t}$ - law. Finally, in two recent works, Rax & White[7,8] once more obtained this subdiffusive law. A common characteristic feature of all these works is that the magnetic field fluctuation is assumed to depend solely on the co-ordinate z, parallel to the unperturbed magnetic field.

In their papers, RAX & WHITE (especially in ref. 7) very briefly mention the generalized problem in which the fluctuations depend on all three spatial co-ordinates. They obtain in this case a much more strongly subdiffusive behaviour: $\Gamma(t) \sim t^{1/4}$. It must be stressed that the general case presents a truly non-trivial difference compared to the z-dependent fluctuations. Whereas in the latter case the mean square displacement can be calculated explicitly by quadratures, in the general case $\Gamma(t)$ is obtained as the solution of a nonlinear differential equation with time-dependent coefficients.

We decided to deal systematically with this equation, for which a well-defined initial value problem is established. The direct numerical integration of this equation does *not* support the asymptotic subdiffusive behaviour. In order to gain insight, we constructed an analytic solution which describes the behaviour of the mean square displacement $\Gamma(t)$ over the whole range of times.

Our main conclusion is that the solution $\Gamma(t)$ of the physical initial value problem behaves asymptotically in a truly diffusive way, *i.e.*, $\Gamma(t) \sim \hat{D}t$, with a constant, positive diffusion coefficient $\hat{D}$. This coefficient approximately scales with the magnetic fluctuation intensity $\beta \equiv \left\langle (\delta B_x / B_0)^2 \right\rangle^{1/2}$ like $\hat{D} \sim \beta^4$, a definitely non-quasilinear law. Finally, when numbers are put into our expression, we find values of the diffusion coefficient in excellent agreement with the experimental results of Antoni & Ortolani[4,9] obtained in the ETA-BETA II reversed-field pinch experiments.

Our paper is organized as follows. The basic model is established in Sec. II. A specific novel feature here is the adoption of a coloured noise, with a finite correlation time τ_c, for modelling the collisions. This turns out to make the problem mathematically well-defined. The exact nonlinear differential equation for $\Gamma(t)$, together with its initial conditions are derived in this section. Various limiting cases are discussed in Sec. III. It appears, in particular, that the subdiffusive behaviour $\Gamma(t) \sim t^{1/2}$ mentioned above is obtained as an exact result in the limit of an infinitely large perpendicular correlation length. The solution of the general differential equation is constructed in Sec. IV by a non-trivial singular perturbation method, in the physically realistic case of small magnetic fluctuation intensity. The solution is completed by a long-time asymptotic solution technique. In Sec. V the status of the subdiffusive asymptotic solution $\Gamma(t) \sim t^{1/4}$ is considered. Although it is an exact solution of the (asymptotic) differential equation, it does not satisfy the required initial condition. Moreover, it appears (when expressed in appropriate variables) as an exceptional, unstable steady-state solution (saddle point). This analysis allows us to complete the analytical solution of the problem in Sec. VI and to determine the final form of the asymptotic diffusion coefficient. The comparison with the experimental data is also discussed at the end of this section. The final conclusions are discussed in Sec. VII.

II. THE BASIC DIFFERENTIAL EQUATION.

The model used here for the description of the diffusion of charged particles in a stochastic magnetic field is very similar to the one used by RAX & WHITE [7,8] [hereafter denoted by RW1, RW2, respectively]. The magnetic field consists of an unperturbed (average) homogeneous field, defining the z-direction (or parallel direction) of a cartesian reference frame, and a *fluctuating* component, perpendicular to the unit vector $\underline{e}_z$:

$$\underline{B} = B_0 \underline{e}_z + \delta B_x(\underline{x})\, \underline{e}_x + \delta B_y(\underline{x})\, \underline{e}_y \ . \tag{1}$$

The non-trivial difference with the RW2 model is that the fluctuating field is supposed to depend on all three position variables $\underline{x} \equiv (x,y,z)$, and not on z alone as in their model. The magnetic fluctuations are purely spatial: $\delta\underline{B}$ is assumed, for simplicity, not to depend *explicitly* on time. The statistics of the magnetic field fluctuations is defined by a *Gaussian process*, introducing two correlation lengths: $\lambda_\parallel$, $\lambda_\perp$ and a dimensionless measure of the magnetic fluctuation intensity, β^2 :

$$B_0^{-2}\langle \delta B_\alpha(\underline{x})\ \delta B_\alpha(\underline{x}')\rangle = \beta^2 \exp\left\{-\frac{(z-z')^2}{2\lambda_\parallel^2} - \frac{|\underline{x}_\perp - \underline{x}'_\perp|^2}{2\lambda_\perp^2}\right\}, \qquad \alpha = x,y. \tag{2}$$

where $\underline{x}_\perp \equiv (x,y,0)$.

We now consider a set of charged particles moving in this stochastic field; if the average magnetic field is sufficiently strong, they are assimilated to their guiding centres. In the RW model it is assumed that the motion is determined by the superposition of two factors:

a) The (dominant part of the) *guiding centre motion parallel to the* perturbed *magnetic field;*

b) The effect of the *collisions* is modelled by a *stochastic velocity*, with components $\eta_\parallel$ in the z-direction and $\eta_\perp$ in the perpendicular direction.

The instantaneous guiding centre coordinate is decomposed, as usual, into an average and a fluctuating part: $\underline{x}(t) = \langle \underline{x}(t) \rangle + \delta\underline{x}(t)$. In order to concentrate on the diffusion process, it is assumed that the average velocity of the particle vanishes; hence: $\langle \underline{x}(t) \rangle = \langle \underline{x}(0) \rangle$. The equation of motion for the fluctuating position is obtained by using the equation of the field lines: $dx/\delta B_x = dy/\delta B_y = dz/B_0$:

$$\frac{d\,\delta x(t)}{dt} = \frac{\delta B_x[\underline{x}(t)]}{B_0}\,\frac{d\,\delta z(t)}{dt} + \eta_\perp(t)\ ,$$

$$\frac{d\,\delta y(t)}{dt} = \frac{\delta B_y[\underline{x}(t)]}{B_0}\,\frac{d\,\delta z(t)}{dt} + \eta_\perp(t)\ ,$$

$$\frac{d\,\delta z(t)}{dt} = \eta_\parallel(t)\,. \tag{3}$$

These equations must be completed with a definition of the statistical properties of the collisional noise. We assume here that $\eta_\parallel(t)$ and $\eta_\perp(t)$ have zero average and are modeled by a *Gaussian coloured noise*, introducing a finite correlation time τ_c:

$$\langle \eta_\parallel(t)\ \eta_\parallel(t') \rangle = \frac{1}{2\tau_c}\,\chi_\parallel\ \exp\left\{-\ \frac{|t - t'|}{\tau_c}\right\}. \tag{4}$$

A similar relation holds for $\eta_\perp(t)$; however, it will henceforth be assumed that $\chi_\perp \ll \chi_\parallel$, thus neglecting the term $\eta_\perp(t)$ in eqs. (3) [this is not an essential approximation, see refs. 7,8]. The introduction of a finite correlation time has the advantage of making the short time-behaviour well defined, thus avoiding ambiguities in the long-time solution (see below, Sec. IV).

We have now established a well-defined set of stochastic equations of motion, describing the behaviour of a charged particle under the influence of *two* distinct random processes: the *collisions* and the *magnetic field fluctuations*. The present model differs from the RW model in two respects: the dependence of the magnetic fluctuations on all three coordinates (as noted above) and the finite correlation time τ_c. In ref. 7, the case $\delta\underline{B} = \delta\underline{B}(\underline{x}(t))$ is very briefly considered (with a white noise for the collisions): their result is discussed in Sec. V below.

Considering that the x and y directions are equivalent, we may limit ourselves to the following simplified set of equations:

$$\frac{d\,\delta x(t)}{dt} = \frac{\delta B_x[\underline{x}(t)]}{B_0}\,\frac{d\,\delta z(t)}{dt},$$

$$\frac{d\,\delta z(t)}{dt} = \eta_{\|}(t)\ . \tag{5}$$

which is to be solved with the initial condition: $\delta x(0) = \delta z(0) = 0$. Solving formally for $\delta x(t)$, we can express the mean-square displacement in the x-direction as follows:

$$\Gamma(t) \equiv \langle\ \delta x^2(t)\ \rangle$$

$$= \frac{2}{B_0^2}\left\langle \int_0^t dt_1 \int_0^{t_1} dt_2\ \delta B_x[\underline{x}(t_1)]\ \delta B_x[\underline{x}(t_2)]\ \eta_{\|}(t_1)\ \eta_{\|}(t_2) \right\rangle . \tag{6}$$

Introducing the Fourier transform of $\delta B_x[\underline{x}(t)]$, this is written as:

$$\Gamma(t) = \frac{2}{B_0^2}\int_0^t dt_1 \int_0^{t_1} dt_2 \int d\underline{k}_1 \int d\underline{k}_2$$

$$\left\langle \delta B_x(\underline{k}_1)\ \delta B_x(\underline{k}_2)\ e^{i\underline{k}_1\cdot\underline{x}(t_1)+i\underline{k}_2\cdot\underline{x}(t_2)}\ \eta_{\|}(t_1)\ \eta_{\|}(t_2) \right\rangle . \tag{7}$$

Assuming that the average over the magnetic fluctuations can be averaged out of the global average, we note that eq. (2) implies:

$$\frac{1}{B_0^2}\langle\ \delta B_x(\underline{k})\ \delta B_x(\underline{k}')\ \rangle$$

$$= (2\pi)^{-3/2}\ \lambda_{\|}\lambda_{\perp}^2\ \beta^2\ \exp\left(-\frac{1}{2}\,k_{\|}^2\lambda_{\|}^2 - \frac{1}{2}\,k_{\perp}^2\lambda_{\perp}^2\right)\ \delta(\underline{k} + \underline{k}')\ . \tag{8}$$

Assuming also that the averages over z and $\underline{x}_\perp$ can be factored, we write eq. (7) in the form:

$$\Gamma(t) = 2(2\pi)^{-3/2}\ \lambda_\parallel\lambda_\perp^2\beta^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \int dk_\parallel \int d\underline{k}_\perp$$

$$\times \exp\left(-\frac{1}{2}k_\parallel^2\lambda_\parallel^2 - \frac{1}{2}k_\perp^2\lambda_\perp^2\right) \langle \exp[\ i\underline{k}_\perp\cdot(\delta\underline{x}_\perp(t_1) - \delta\underline{x}_\perp(t_2))\ ]\ \rangle$$

$$\times \langle \exp[\ ik_\parallel\ (\delta z(t_1) - \delta z(t_2))\ ]\ \ \eta_\parallel(t_1)\ \eta_\parallel(t_2)\ \rangle\ . \qquad (9)$$

The first average can be evaluated in the second cumulant approximation, assuming that the turbulence is stationary and isotropic in the x-y plane:

$$\langle \exp[\ i\underline{k}_\perp\cdot(\delta\underline{x}_\perp(t_1) - \delta\underline{x}_\perp(t_2))\ ]\ \rangle \approx \exp\left\{-\frac{1}{2}\left\langle [\ \underline{k}_\perp\cdot(\delta\underline{x}_\perp(t_1) - \delta\underline{x}_\perp(t_2))\ ]^2 \right\rangle\right\}$$

$$= \exp\left\{-\frac{1}{2}k_\perp^2\ \Gamma(t_1 - t_2)\right\}\ . \qquad (10)$$

This result exhibits the main difference between the RW2 model and the case when δB_x depends on all three coordinates. In the latter case eq. (9) is a *nonlinear integral equation* in $\Gamma(t)$, whereas in the former eq. (9) gives an explicit expression for $\Gamma(t)$, which can be calculated by quadratures (see Sec. III). Indeed, the RW2 model is obtained by letting $\lambda_\perp \to \infty$, in which case eq. (8) becomes:

$$\lim_{\lambda_\perp\to\infty} \frac{1}{B_0^2} \langle\ \delta B_x(\underline{k})\ \delta B_x(\underline{k}')\ \rangle$$

$$= (2\pi)^{-1/2}\ \lambda_\parallel\ \beta^2\ \exp\left(-\frac{1}{2}k_\parallel^2\lambda_\parallel^2\right)\ \delta(\underline{k}_\perp)\ \delta(\underline{k} + \underline{k}')\ . \qquad (11)$$

Hence, the average involving $\delta\underline{x}_\perp$ in eq. (9) reduces to 1.

The last average in eq. (9) can be written in the form:

$$\mathfrak{J}(k_\parallel;t_1,t_2) \equiv \left\langle \exp\left[\ ik_\parallel \int_{t_2}^{t_1} dt'\ \eta_\parallel(t')\ \right] \eta_\parallel(t_1)\ \eta_\parallel(t_2)\ \right\rangle\ . \qquad (12)$$

This expresion is evaluated exactly by expanding the exponential, calculating the resulting averages by using eq. (4) and the gaussian character of $\eta_\parallel(t)$ and resumming the resulting (standard) series. The result of this somewhat lengthy, but elementary calculation is:

$$\mathfrak{Z}(k_\parallel;t_1,t_2) = \left\{ \frac{\chi_\parallel}{2\tau_c}\exp\left(-\frac{t_1-t_2}{\tau_c}\right) - \frac{k_\parallel^2\chi_\parallel^2}{4}\varphi^2\left(\frac{t_1-t_2}{\tau_c}\right)\right\}$$

$$\times \exp\left\{-\frac{k_\parallel^2\chi_\parallel}{2}\tau_c\,\psi\left(\frac{t_1-t_2}{\tau_c}\right)\right\}, \qquad (t_1 > t_2) . \tag{13}$$

where:

$$\varphi(x) = 1 - e^{-x} , \qquad \psi(x) = x - \varphi(x) . \tag{14}$$

In obtaining eq. (13) we made use of the following integrals:

$$\int_{t_2}^{t_1} dt_3 \langle \eta_\parallel(t_1)\,\eta_\parallel(t_3) \rangle = \int_{t_2}^{t_1} dt_3 \langle \eta_\parallel(t_2)\,\eta_\parallel(t_3) \rangle = \frac{1}{2}\chi_\parallel\,\varphi\left(\frac{t_1-t_2}{\tau_c}\right)$$

$$\int_{t_2}^{t_1} dt_3 \int_{t_2}^{t_1} dt_4 \langle \eta_\parallel(t_3)\,\eta_\parallel(t_4) \rangle = \chi_\parallel\,\tau_c\,\psi\left(\frac{t_1-t_2}{\tau_c}\right) . \tag{15}$$

The results (10) and (13) are now substituted into eq. (9) and the latter expression is transformed into a differential equation by differentiating both sides twice with respect to t and performing the elementary integrations over $k_\parallel$ and $\underline{k}_\perp$:

$$\frac{d^2\Gamma(t)}{dt^2} = F(t/\tau_c)\,\frac{\lambda_\perp^2}{\lambda_\perp^2 + \Gamma(t)} , \tag{16}$$

with:

$$F(x) = \frac{1}{2}\,\frac{\beta^2\lambda_\parallel\chi_\parallel}{\left(\lambda_\parallel^2 + \chi_\parallel\tau_c\,\psi(x)\right)^{1/2}}\left\{2\,\frac{e^{-x}}{\tau_c} - \frac{\chi_\parallel\,\varphi^2(x)}{\lambda_\parallel^2 + \chi_\parallel\tau_c\,\psi(x)}\right\} . \tag{17}$$

This is the basic differential equation determining the mean-square displacement $\Gamma(t) = \langle \delta x^2(t) \rangle$ in our model of collisional diffusion in a stochastic magnetic field. This equation is to be solved with the initial condition [which is obvious from eq. (9)]:

$$\Gamma(0) = C, \qquad \left.\frac{d\Gamma(t)}{dt}\right|_{t=0} = 0 \,. \tag{18}$$

III. OISCUSSION OF THE BASIC EQUATION.

It is interesting to consider several limiting cases. The limit $\tau_c \to 0$ corresponds to a *white collisional noise*. In this case $\varphi(t/\tau_c) \to 1$, and $\tau_c \psi(t/\tau_c) \to t$, thus:

$$\lim_{\tau_c \to 0} F(t/\tau_c) = \frac{1}{2} \frac{\beta^2 \lambda_\parallel \chi_\parallel}{(\lambda_\parallel^2 + \chi_\parallel t)^{1/2}} \left\{ 4\, \delta(t) - \frac{\chi_\parallel}{\lambda_\parallel^2 + \chi_\parallel t} \right\} . \qquad (19)$$

This form is inconvenient, because of the occurrence of a δ-function in the differential equation. Moreover, the choice of an initial condition for $d\Gamma/dt$ is not well defined (see the discussion below, Sec. IV). It is because of these two difficulties that we have chosen a *coloured noise* model for an unambiguous treatment of the problem.

Another interesting case is the limit of an *infinitely large perpendicular correlation length*, $\lambda_\perp \to \infty$ [a problem alluded to in Sec. II, eq. (11)]. In this case, the differential equation (16) reduces to:

$$\frac{d^2\Gamma(t)}{dt} = F(t/\tau_c) , \qquad \lambda_\perp \to \infty . \qquad (20)$$

Not only has the equation become *linear*, but its integration is straightforward through a double quadrature. The *exact* solution of eq. (20) [with (18)] is:

$$\Gamma(t) = 2\, \beta^2 \lambda_\parallel \left[\left(\lambda_\parallel^2 + \chi_\parallel \tau_c\, \psi(t/\tau_c) \right)^{1/2} - \lambda_\parallel \right] , \qquad \lambda_\perp \to \infty . \qquad (21)$$

For long times, $\tau_c \psi(t/\tau_c) \to t$, and eq. (21) reduces to:

$$\Gamma(t) = 2\, \beta^2 \lambda_\parallel \left[\left(\lambda_\parallel^2 + \chi_\parallel t \right)^{1/2} - \lambda_\parallel \right] , \qquad \lambda_\perp \to \infty, \quad t \gg \tau_c \, . \qquad (22)$$

We now take advantage of this shape in order to choose the point at

This is precisely the solution given in refs. 7,8 (for $\chi_\perp = 0$). In this limit we find the *subdiffusive asymptotic behaviour* $\Gamma(t) \sim \sqrt{t}$, previously obtained in refs. 2, 6-8.

This result also shows that the asymptotic power law (scaling) of $\Gamma(t)$ does not depend on τ_c, the coloured and white noise models are identical in this respect. But for short times ($t \ll \tau_c$), the behaviour of eq. (21) is quite different from (22), and this difference will show up in the global solution:

$$\Gamma(t) \sim \frac{1}{2}\beta^2\chi_\| \tau_c \left(\frac{t}{\tau_c}\right)^2 , \qquad t \ll \tau_c . \tag{23}$$

This form clearly satisfies the initial condition (18) and exhibits the well-known ballistic short-time behaviour: $\Gamma(t) \sim t^2$. On the contrary, the RW2 solution (22) starts linearly for short times $\left(t \ll \lambda_\|^2/\chi_\| \right)$:

$$\Gamma_{RW}(t) \sim \beta^2\chi_\| \, t . \tag{24}$$

Thus, the Rax-White solution starts at short times as a *diffusive* process with a non-zero slope, which is progressively slowed down and ends up in the *subdiffusive* $\sqrt{t}$-regime for long times. This means that the initial ballistic phase is wiped out in their (white noise) model.

We now return to the general case, with finite τ_c and finite $\lambda_\perp$. We note that in this problem there are two characteristic lengths, viz. the two *correlation lengths*: $\lambda_\|$, $\lambda_\perp$; moreover, there are two characteristic times: the *correlation time*, τ_c and the *diffusion time*, defined as follows:

$$\tau_d = \frac{\lambda_\|^2}{\chi_\|} . \tag{25}$$

With these quantities, together with the amplitude of the magnetic fluctuations, we may construct three independent dimensionles parameters:

$$\eta = \frac{\tau_c}{\tau_d}, \qquad \delta = \frac{\lambda_\parallel}{\lambda_\perp}, \qquad \alpha = \frac{1}{2}\beta^2\delta^2 . \tag{26}$$

The parameter α has a simple and important physical meaning. In the quasilinear approximation, the diffusion coefficient of the magnetic field lines is[10]:

$$D_F = \frac{1}{4}\beta^2\lambda_\parallel . \tag{27}$$

Hence, the square of the perpendicular displacement of the field lines over a length z is $4D_Fz$. The type of "diffusional behaviour" depends crucially on the ratio between this perpendicular displacement over the parallel correlation length $\lambda_\parallel$ and the perpendicular correlation length of the field $\lambda_\perp$:

$$\frac{4D_F\lambda_\parallel}{\lambda_\perp^2} = \beta^2\delta^2 = 2\alpha . \tag{28}$$

When $\alpha \ll 1$ (as will be assumed here), we deal with a "*quasilinear regime*"[6-8,10,11]. [Note that in some of these references[10,11] a parameter $R = \beta\delta = \sqrt{2\alpha}$ is used]. In the opposite case, $\alpha \gg 1$, called the "*percolation regime*", the behaviour of the plasma is quite different and must be treated by altogether different methods[12,13].

We now scale the mean-square displacement with $\lambda_\perp^2$ and the time with τ_d, introducing the dimensionless quantities:

$$Y = 1 + \frac{\Gamma}{\lambda_\perp^2}, \qquad \theta = \frac{t}{\tau_d} . \tag{29}$$

The diferential equation (16) reduces to the dimensionless form:

$$Y(\theta)\ \ddot{Y}(\theta) = \frac{\alpha}{\eta}\,\Phi\left(\frac{\theta}{\eta}\,;\,\eta\right) , \tag{30}$$

where $\dot{Y}(\theta) \equiv dY/d\theta$, and:

59. . This e uation then ields the followin value for the constant in the

$$\Phi(x;\eta) = \frac{1}{[\,1 + \eta\,(e^{-x} - 1 + x)\,]^{1/2}}$$

$$\times \left[2\,e^{-x} - \eta\,\frac{(1 - e^{-x})^2}{1 + \eta\,(e^{-x} - 1 + x)} \right] . \tag{31}$$

The initial condition is:

$$Y(0) = 1\,, \qquad \dot{Y}(0) = 0\,. \tag{32}$$

On the basis of realistic orders of magnitude, it appears that α is usually a small parameter. On the other hand, the correlation time is the shortest time scale in the problem. We therefore assume the following ordering:

$$\eta \ll \alpha \ll 1\,. \tag{33}$$

IV. SOLUTION OF THE INITIAL VALUE PROBLEM.

The nonlinear differential equation (30) can be solved numerically by standard methods. Much more insight is gained, however, from an analytical solution. This can be achieved in a systematic way if we reduce the number of dimensionless parameters to one. We therefore assume, from here on:

$$\eta = \alpha^2 \ . \tag{34}$$

This assumption is consistent with the ordering (33). Although it is rather arbitrary (it actually amounts to a choice of the correlation time τ_c), and could be replaced by a different one, it has the advantage of giving a simple and internally consistent solution. With this new assumption, eq. (30) becomes:

$$\alpha \, Y \, \ddot{Y} = \Phi(\alpha^{-2}\theta \ ; \ \alpha^2) \ . \tag{35}$$

This is a typical problem of *singular perturbation*, with the small parameter multiplying the highest derivative of the unknown function. It is, moreover, very pathological in the limit $\alpha \to 0$ [this is due, of course, to the appearance of the $\delta(t)$ function in the limit of a white noise, see eq. (19)].

Eq. (35) can be solved by the method of the *matched asymptotic expansion* (Ref. 14, Chap. 12.3). We consider a boundary layer near $\theta = 0$, in the region $0 \leqslant \theta \leqslant \theta_0$, (where θ_0 will be defined afterwards), and seek different forms of the solution inside and outside the boundary layer. In order to look "through a microscope" into the boundary layer, we introduce the scaled time variable:

$$\xi = \alpha^{-2}\theta \ . \tag{36}$$

The function Φ is expanded in powers of α, keeping ξ fixed:

$$\Phi(\xi;\alpha^2) = \Phi_0(\xi) + \alpha^2\,\Phi_2(\xi) + \alpha^4\,\Phi_4(\xi) + \dots\ , \tag{37}$$

with:

$$\Phi_0(\xi) = 2\,e^{-\xi},$$

$$\Phi_2(\xi) = -\,[2\,e^{-2\xi} + (\xi - 3)\,e^{-\xi} + 1]\ ,$$

$$\Phi_4(\xi) = \frac{3}{4}\,[3\,e^{-3\xi} + (4\xi - 8)\,e^{-2\xi} + (\xi^2 - 6\xi + 7)\,e^{-\xi} + 2\xi - 2]\ . \tag{38}$$

The differential equation for the inner solution becomes:

$$Y_i(\xi)\,\frac{d^2Y_i(\xi)}{d\xi^2} = 2\alpha^3\,\left[\Phi_0(\xi) + \alpha^2\Phi_2(\xi) + \dots\right]\ . \tag{39}$$

We seek a solution in the form:

$$Y_i(\xi) = \sum_n \alpha^n\,Y_i^{(n)}(\xi)\ , \tag{40}$$

with:

$$Y_i^{(n)}(0) = \delta_{n,0}\ , \qquad \left.\frac{dY_i^{(n)}(\xi)}{d\xi}\right|_{\xi=0} = 0\ . \tag{41}$$

Substitution of (40) and (37) into eq. (39) yields a set of equations which are very easily solved, with the result:

$$Y_i(\xi) = 1 + 2\alpha^3\,[\xi - 1 + e^{-\xi}]$$

$$+\,\frac{\alpha^5}{2}\,[-\,\xi^2 + 2\xi - 1 - 2(\xi - 1)\,e^{-\xi} - e^{-2\xi}]$$

$$+ \alpha^6 [- 2\xi + 5 - 4(\xi + 1) e^{-\xi} - e^{-2\xi}] + O(\alpha^7) . \tag{42}$$

(we do not write down the lengthier $O(\alpha^7)$ term, which was however retained in our calculation). The inner solution is plotted in Fig. 1, together with its first derivative. It clearly exhibits the initial ballistic behaviour, $Y(\theta) \sim 1 + \alpha^3 \xi^2$. Correspondingly, the slope grows, at first linearly, from zero up to a limiting value $dY/d\theta = 2\alpha$ at the limit of the boundary layer [obtained from (42) after the complete decay of the exponentials].

An alternative method of integration in the very short time domain is given in the Appendix, together with a comparison of the various approximations in the range $0 \leqslant \theta \leqslant 1$.

We now turn to the external solution, for $\theta \geqslant \theta_0$. In this region, the function $\alpha^{-1}\Phi(\alpha^{-2}\theta; \alpha^2)$ of eq. (35) is expanded in powers of α, keeping θ fixed [and putting to zero the exponentially small terms $\exp(-n\alpha^{-2}\theta)$]:

$$\alpha^{-1}\Phi(\alpha^{-2}\theta; \alpha^2) = - \frac{\alpha}{(1 + \theta - \alpha^2)^{3/2}}$$

$$= - \frac{\alpha}{(1 + \theta)^{3/2}} \left[1 + \frac{3}{2}\frac{\alpha^2}{(1 + \theta)} + \frac{15}{8}\frac{\alpha^4}{(1 + \theta)^2} + \dots\right]$$

$$\equiv \sum_n \alpha^n \varphi_n(\theta) . \tag{43}$$

We seek an external solution in the form:

$$Y_e(\theta) = \sum_\nu \alpha^n Y_e^{(n)}(\theta) , \tag{44}$$

and substitute this form into the equation derived from (35) and (43):

$$Y_e(\theta) \ddot{Y}_e(\theta) = \sum_n \alpha^n \varphi_n(\theta) . \tag{45}$$

The solution of this equation through order α^2 is easily found:

$$Y_e(\theta) = [b_0\ (1+\theta) + a_0] + \alpha\left[4\ (1+\theta)^{1/2} + b_1(1+\theta) + a_1\right]$$

$$+ \alpha^2\left[4\ (1+\theta)\ ln(1+\theta) - 4\ (1+\theta) + 16\ (1+\theta)^{1/2} + b_2(1+\theta) + a_2\right]$$

$$+ O(\alpha^3)\ . \tag{46}$$

The integration constants a_n, b_n are determined by matching the external solution to the internal one at the point θ_0. The latter is is chosen as follows:

$$\alpha^2 \ll \theta_0 \ll 1\ . \tag{47}$$

In this way, θ_0 is clearly outside the region of rapid variation in the boundary layer, but still corresponds to a time much shorter than τ_d. θ_0 can thus be considered as a "post-initial condition" for eq. (45), and will be chosen as:

$$\theta_0 = \alpha\ \tau_0\ , \tag{48}$$

where τ_0 is a constant of order one, whose exact value will prove irrelevant.

We now substitute $\xi = \alpha^{-2}\theta$ into the internal solution, expand it in powers of α for fixed θ (annulling the exponentially small terms), and then substitute (48):

$$Y_i(\theta_0) = 1 + \alpha^3\left(\frac{2\theta_0}{\alpha^2} - 2\right) + \alpha^5\left[-\frac{\theta_0^2}{2\alpha^4} + \frac{\theta_0}{\alpha^2} - \frac{1}{2}\right] + \ldots$$

$$= 1 + \alpha\left[2\theta_0 - \frac{1}{2}\theta_0^2 + \frac{1}{4}\theta_0^3\right] + \alpha^3\left[-2 + \theta_0 - \frac{3}{4}\theta_0^2\right] + \ldots$$

$$= 1 + 2\alpha^2\ \tau_0 + O(\alpha^3)\ . \tag{49}$$

[In going from the first to the second form, we retained all terms through order α^7 in the first line].

In the external solution (46) we substitute $\theta = \alpha\tau_0$ and expand the result for fixed τ_0:

$$Y_e(\theta_0) = a_0 + b_0 + \alpha\,(b_0\tau_0 + 4 + a_1 + b_1)$$

$$+ \alpha^2\,(2\tau_0 + b_1\tau_0 + 12 + a_2 + b_2)\ . \qquad (50)$$

The same operations are performed on the first derivatives of Y_i, Y_e. The matching condition then requires:

$$Y_e(\theta_0) = Y_i(\theta_0)\ , \qquad \dot{Y}_e(\theta_0) = \dot{Y}_i(\theta_0)\ . \qquad (51)$$

These equations allow us to determine all the integration constants a_n, b_n (we also note that τ_0 cancels out from these equations!).

Eqs. (51) could also be considered from a different point of view. The values $Y_i(\theta_0) = 1 + O(\alpha)$, $\dot{Y}_i(\theta_0) = 2\alpha + O(\alpha^3)$ play the role of a "post-initial condition" for the long-time differential equation (45). We note, in particular, that the "post-initial slope" is different from zero [see Fig. 1]. This finite slope also represents the correct initial value to be adopted in the white noise limit (where the boundary layer shrinks down to zero). The consideration of a coloured noise allows us to determine this finite slope consistently; had we started directly with the white noise model, the choice of the initial slope would remain largely arbitrary.

Having fixed the free constants by eq. (51), we find the following form for the external solution through order α^2:

$$Y_e(\theta) = 1 + 4\alpha\,[\,\sqrt{1+\theta} - 1\,]$$

$$+ \alpha^2\,[\,4\,(1+\theta)\,ln(1+\theta) - 12\,(1+\theta) + 16\sqrt{1+\theta} - 4\,],\quad \theta > \alpha\ . \qquad (52)$$

The form of this solution is quite interesting. We note that, to order α, the solution exactly coincides with the RW2 solution (22) (when written in the appropriate dimensionless variables), describing a subdiffusive behaviour: $\Gamma(t) \sim t^{½}$. However, the next term $(\sim \alpha^2)$ has a quite different behaviour. Clearly, the α-ordering is not significant for $\theta \gg 1$, because the coefficients of the powers of α are growing functions of time. Thus, for long times, the dominant behaviour is given by the term $\Gamma(t) \sim 4\alpha^2\, \theta\, ln(1+\theta)$, which grows faster than θ: we thus have a ***superdiffusive behaviour*** for these times. Presumably, the higher order terms in α would contain even faster growing coefficients: thus, the α-expansion breaks down for very long times. A comparison with the numerical solution of eq. (35) [Fig. 2] shows that the solution (52) is a good approximation in the range $\alpha \leqslant \theta \lesssim \alpha^{-2}$. Beyond, the local slope of the solution (52) goes through a minimum and begins to grow logarithmically, thus exhibiting a (fictitious) superdiffusive behaviour.

The asymptotic behaviour for very long times can be obtained analytically as follows. The differential equation (35), combined with (43), reduces, for $\theta \gg 1$, to the asymptotic form:

$$Y_a(\theta)\, \ddot{Y}_a(\theta) = -\,\alpha\, \theta^{-3/2}\ , \qquad \theta \gg 1\ . \qquad (53)$$

This asymptotic equation was also quoted (without derivation) in RW1, but their conclusion (also given without any detail) is incorrect: it will be discussed in Sec. V.

From eq. (53) it is obvious that, as $\theta^{-3/2} \to 0$ for $\theta \to \infty$, the second derivative of $Y_a(\theta)$ must vanish in this limit. [An exceptional solution of a different kind, discussed in Sec. V, will not be considered in the forthcoming analysis]. Hence, $Y_a(\theta)$ must tend toward a *linear function in θ*. The ultimate asymptotic behaviour is thus *purely diffusive* (this is also confirmed by the numerical solution). The long-time form of the solution of eq. (53) can be found as follows (we use a method explained in Ref. 15, Chap. 4.3). We assume the form:

$$Y_a(\theta) = C + D\,\theta + e(\theta)\ . \qquad (54)$$

The unknown function $e(\theta)$ must be subdominant for long times:

$$\frac{e(\theta)}{C + D\,\theta} \to 0 \qquad \text{as}\ \ \theta \to \infty\ . \qquad (55)$$

Substituting (54) into (53), we find:

$$[\ C + D\,\theta + e(\theta)\ ]\ \ddot{e}(\theta) = -\ \alpha\,\theta^{-3/2}. \qquad (56)$$

Neglecting $e(\theta)$ in the bracketted term, we obtain an equation that can be integrated by quadratures, with the result:

$$e(\theta) = \frac{2\alpha}{C}\sqrt{\theta} + \frac{2\alpha}{\sqrt{CD}}\left(1 + \frac{D\,\theta}{C}\right)\ tan^{-1}\left(\sqrt{\frac{D\,\theta}{C}}\right) + b_1\theta + b_0, \qquad (57)$$

where b_0 and b_1 are integration constants. b_1 is determined by the condition that the secular term proportional to θ in the limit $\theta \to \infty$ should vanish in (57) in order to satisfy condition (55). The constant b_0 combines additively with C in eq. (54); we take advantage of this feature by setting C = 1 and keeping b_0 and D as the only adjustable constants in the solution. (57) is now rewritten as:

$$e(\theta) = b_0 + 2\alpha\left\{\sqrt{\theta} + \frac{1}{\sqrt{D}}\ tan^{-1}(\sqrt{D\,\theta}) + \sqrt{D}\,\theta\ \left[tan^{-1}(\sqrt{D\,\theta}) - \frac{\pi}{2}\right]\right\}. \qquad (58)$$

The constants b_0 and D are determined by matching the asymptotic solution to the external solution at the point $\theta = \theta_*$:

$$Y_a(\theta_*) = Y_e(\theta_*)\ , \qquad \dot{Y}_a(\theta_*) = \dot{Y}_e(\theta_*)\ . \qquad (59)$$

The choice of the point θ_* requires an analysis performed in the next section. We therefore postpone the final determination of the constants to Sec. VI.

V. DIFFUSIVE OR SUBDIFFUSIVE BEHAVIOUR ?

Rax & White[7] very briefly mention the complete model (5) with a white noise for the collisions, and they quote the asymptotic equation (53) [the result is given without derivation]. *Next, they quote [very briefly, and even* more briefly in Ref. 8] the following solution (in our present notation):

$$Y_{RW}(\theta) = E\,\theta^{1/4} . \tag{60}$$

By substitution into eq. (53), it is checked that (60) does indeed satisfy the equation, with:

$$E = \frac{4}{\sqrt{3}}\sqrt{\alpha} . \tag{61}$$

Eq. (60) is an exact solution of the nonlinear asymptotic equation (53): it represents a ***strongly subdiffusive process***, in contrast to eq. (54). The question thus arises about its status and its relation to the diffusive solution.

We *first check the initial condition.* Clearly, (60) does not satisfy the initial condition in $\theta = 0$, eq. (30): $Y_{RW}(0) = 0$, $\dot{Y}_{RW}(0) = \infty$; neither can it be matched to a "post-initial condition" at any $\theta_0 \gg 1$. Indeed, the single constant E ***must*** have the value (61): it is not an adjustable parameter. Hence, clearly, (60) *is not a solution of the initial value problem* (28)-(31), (34).

It is interesting to study in some more detail the properties of the asymptotic equation (53). We first perform the following transformations:

a) Change the unknown function into $Z(\theta) = \theta^{-1/4} Y_a(\theta)$.

b) Change the variable θ into $u = ln\,\theta$.

Considering Z as a function of u and writing $Z' = dZ/du$, the differential equation (53) is transformed into:

$$Z\,Z'' - \frac{1}{2}\,Z\,Z' - \frac{3}{16}\,Z^2 = -\,\alpha\ . \qquad (62)$$

The advantage of this form is the absence of any u-dependent coefficient. The nonlinear equation (62) admits the exact (constant) solutions:

$$Z_{\pm}(u) = \pm\, Z_0 = \pm\,\frac{4}{\sqrt{3}}\,\sqrt{\alpha}\ . \qquad (63)$$

The positive, physical, solution corresponds to (60), (61).

Besides this particular solution, we may look for a solution which is growing in u, i.e., $|Z(u)| \to \infty$ for $u \to \infty$; such a solution satisfies for large u the linearized equation:

$$Z_a'' - \frac{1}{2}\,Z_a' - \frac{3}{16}\,Z_a = 0\ . \qquad (64)$$

whose general solution is:

$$Z_a(u) = C\,e^{-u/4} + D\,e^{3u/4}\ , \qquad (65)$$

or, in terms of θ:

$$Z_a(\theta) = C\,\theta^{-1/4} + D\,\theta^{3/4}\ .$$

Clearly, this is precisely the dominant part of the diffusive solution $Y_a(\theta)$ determined in eq. (54).

A very interesting picture arises from the study of the linear stability of the "steady-state" solution (63). We rewrite eq. (62) in the form of a set of two first-order differential equations:

$$Z' = V \ , \qquad V' = \frac{1}{2} V + \frac{3}{16} Z - \frac{\alpha}{Z} \ . \tag{66}$$

The only fixed points in the phase plane $(Z,V) \equiv (Z,Z')$ are the two points with coordinates $P_\pm$: $(Z = \pm Z_0 , V = 0)$. From here on, we shall only discuss the physical half-plane $Z \geqslant 0$. We introduce a perturbation to the solution (63):

$$Z_+(u) = Z_0 + s(u) \ . \tag{67}$$

The function s(u) obeys [for $|s(u)| \ll Z_0$] the linearized equation:

$$s'' - \frac{1}{2} s' - \frac{3}{8} s = 0 \ . \tag{68}$$

Its solutions are:

$$s(u) = \sigma_+ \ e^{r_+ u} + \sigma_- \ e^{r_- u} \ , \tag{69}$$

with:

$$r_\pm = \frac{1}{4} (1 \pm \sqrt{7}) \ . \tag{70}$$

The characteristic roots are both real, one positive, the other negative. Thus, the fixed point P_+ is a *saddle point* (or *hyperbolic point)*, presenting in its neighbourhood a stable and an unstable manifold [Fig. 3]. As a result, only initial conditions lying on the stable manifold yield a solution evolving towards (but not attaining) P_+. If the solution starts from any other initial condition, it *never* reaches the subdiffusive regime (60), but rather behaves asymptotically according to the normal diffusive pattern (54). The physical solution, obtained by numerical integration of eq. (62), with the initial condition corresponding to $Y(\theta_0) = 1$, $\dot{Y}(\theta_0) = 2\alpha$ [see eq. (46)] is shown in Fig. 3: it describes a hyperbola-like trajectory that can never reach the saddle point.

We now take advantage of this shape in order to choose the point at which the asymptotic regime (54) sets in. *The point θ_* is defined as the the time at which the trajectory in the (Z,Z') plane most closely approaches the saddle point.* Beyond this point, the trajectory is "repelled" by the fixed point. The time θ_* is determined by the condition: $Z'(u_*) = 0$, or, returning to the function Y and the variable θ:

$$\theta_* \, \dot{Y}(\theta_*) = \frac{1}{4} \, Y(\theta_*) \, . \qquad (71)$$

VI. THE ASYMPTOTIC DIFFUSION REGIME.

We now go back to Eq. (59) and decide to match the asymptotic solution $Y_a(\theta)$ to the external solution $Y_e(\theta)$ at the point θ_* of closest approach to the saddle point. We first determine an approximate analytical expression for θ_* by solving eq. (71) with $Y(\theta) \to Y_e(\theta)$. (It must be verified *a posteriori* that the external solution is a good approximation to the numerical solution up to $\theta = \theta_*$). Using (52), eq. (71) becomes, assuming $\theta_* \gg 1$:

$$\frac{1}{4} - \alpha\left(\sqrt{\theta_*} + 1\right) + \alpha^2\left(5\theta_* - 4\sqrt{\theta_*} - 3\,\theta_* \ln\theta_* - 1\right) = 0\ . \qquad (72)$$

We now temporarily set $\theta_* = r^2/\alpha^2$; we also approximate the slowly varying logarithm by the approximate value $\ln(r^2/\alpha^2) \approx \ln(0.1/\alpha^2)$ [because in the range $10^{-2} < \alpha < 10^{-4}$., r^2 appears to be close to 0.1, as can be verified *a posteriori*]. The resulting second-degree equation is easily solved for r and yields:

$$\theta_* \approx \left\{\frac{1 + 4\alpha + \left[(1+4\alpha)^2 - 4\left(\frac{1}{4} - \alpha - \alpha^2\right)(5 - K + 3\ln\alpha^2)\right]^{1/2}}{2\alpha\,(5 + K + 3\ln\alpha^2)}\right\}^2 , \qquad (73)$$

where $K = -3\ln 0.1 = 15.54$. Noting that in the square root the second term is dominant, this expression can be simplified to:

$$\theta_* \approx \frac{(1+4\alpha)^2}{4\,\alpha^2}\,\frac{-3\ln\alpha^2 - K - 4}{(-3\ln\alpha^2 - K - 5)^2} ,$$

or even more simply to:

$$\theta_* \approx -\frac{1}{12\,\alpha^2 \ln\alpha^2}\ . \qquad (74)$$

This value will be adopted as the matching point to be used in eq.

(59). This equation then yields the following value for the constant in the asymptotic solution:

$$b_0 = \frac{1}{L^2}\left(\frac{D}{12\,L\alpha^2} - 1\right) + 4\alpha\left(\frac{1}{L} - 1\right) - 4\alpha^2 , \tag{75}$$

where:

$$L = \sqrt{-3\ \ln \alpha^2} . \tag{76}$$

The *asymptotic diffusion coefficient* is:

$$D = \frac{\pi^2\ \alpha^2}{4(1 + L^{-1})^2}\left\{1 + \left[1 + \frac{32}{\pi^2}(1 + L^{-1})\ (2\alpha L - \ln(2\alpha L) - 1)\right]^{1/2}\right\}^2 \tag{77}$$

It is easily checked that the right hand side of eq. (77) is positive for any $\alpha < 10^{-2}$.

Our final conclusion is that *the ultimate asymptotic behaviour of the solution is truly diffusive:*

$$\frac{\Gamma(\theta)}{\lambda_\perp^2} \sim b_0 + D\,\theta , \qquad \theta \to \infty . \tag{78}$$

with a positive diffusion coefficient. We have now constructed a complete analytical solution over the whole range of times. It can be summarized as follows:

$$Y(\theta) = \begin{cases} Y_i(\theta) , & \text{for } 0 < \theta < \alpha , \quad \text{eq. (42)} \\ Y_e(\theta) , & \text{for } \alpha < \theta < \theta_* , \quad \text{eq. (52)} \\ Y_a(\theta) , & \text{for } \theta > \theta_* , \quad \text{eq. (54)} \end{cases} \tag{79}$$

As appears from Fig. 4, this analytical solution is an excellent approximation of the numerical solution over the whole range of times.

The dependence of the diffusion coefficient on α is rather complicat-

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ed, but a graph of this function (Fig. 5) shows that the function is of the form:

$$D = \alpha^2 \, d(\alpha) \;, \tag{80}$$

where $d(\alpha)$ is a relatively slowly varying function of α. It turns out that, for small values of α, $d(\alpha)$ is very nearly linear in *ln* α. This scaling shows that the asymptotic diffusion process is much slower than predicted by a quasi-linear scaling (or, *a fortiori*, by a Bohm scaling). Indeed, because of eq. (26), we find $D \sim \beta^4$, whereas the quasilinear scaling would be $D_{QL} \sim \beta^2$.

Our results may now be compared to experimental data. We first switch back to dimensional quantities, using eqs. (26), (29):

$$\Gamma(t) \sim \lambda_\perp^2 \left[b_0 + D \left(\chi_\parallel/\lambda_\parallel^2\right) t\right] \equiv \hat{B} + \hat{D}\, t \;. \tag{81}$$

The dimensional diffusion coefficient is thus:

$$\hat{D} = \delta^{-2} D \chi_\parallel \;. \tag{82}$$

We make use of the experimental data obtained by Antoni & Ortolani[9] on the ETA-BETA II reversed-field pinch experiment, which are particularly complete for our purpose. From the global data (T = 50 eV, $n = 5\cdot10^{20}$ m^{-3}, B = 1 T) we estimate the collisional diffusion coefficients by calculating the classical thermal diffusivities [5] with the result: $\chi_\parallel = 7.27\cdot10^5$ m^{2}sec^{-1}, $\chi_\perp = 0.0676$ m^{2}sec^{-1}. The upper limit of the measured magnetic field fluctuation intensity is $\beta \approx 5\cdot10^{-2}$. The correlation lengths $\lambda_\parallel$, $\lambda_\perp$, were measured and found to have comparable values, thus $\delta \approx 1$. With these values we find $\alpha = 1.25\cdot10^{-3}$; equation (76) then yields the following value for the diffusion coefficient:

$$\hat{D} = 6.05 \times 10^{-5} \, \delta^{-2}\chi_\parallel = 44 \text{ m}^2\text{sec}^{-1} \;, \tag{83}$$

which is in excellent agreement with the value $D_\perp \approx 50$ m^2 sec^{-1} obtained by

Antoni & Ortolani [9] (see also Ref. 4). Let us also note the strong anomalous enhancement factor with respect to the perpendicular collisional diffusion coefficient:

$$\frac{\hat{D}}{\chi_\perp} = 651 \; .$$

It is also interesting to note that, for $\lambda_\parallel = 5$ cm, we find a diffusion time: $\tau_d = 6.87 \times 10^{-8}$ sec.

VII. CONCLUSIONS.

We analyzed in detail a model of diffusion of charged particles in a completely stochastic magnetic field. Our model differs from the RW2 model in two important respects:

a) We admit arbitrary values for the ratio $\delta = \lambda_{\parallel}/\lambda_{\perp}$ of the correlation lengths of the magnetic fluctuations (whereas RW consider primarily the case $\lambda_{\perp} \to \infty$).

b) We admit only small values of the parameter $\alpha = \beta^2\delta^2/2$, or, correspondingly, of the parameter $R = \sqrt{2\alpha}$. We are thus considering a "*quasilinear regime*" in the sense defined by Kadomtsev & Pogutse[10] and Isichenko[11].

c) We consider a coloured noise for the collisional fluctuation modelling, with a finite correlation time τ_c (whereas RW treat the white noise limit $\tau_c \to 0$).

Considering point a), it may be noted that the experimental data of Antoni & Ortolani [9] exclude the value $\delta = 0$, at least in the reversed-field pinch experiments (they correspond more closely to $\delta \approx 1$). Hence, the *subdiffusive behaviour* $\Gamma(t) \sim \sqrt{t}$, though being an exact solution of the model in the limit $\lambda_{\perp} \to \infty$, could not be observed in real experiments.

The analytic solution obtained for our model yields a very detailed picture of this doubly stochastic diffusion process. In a first stage, the mean-square displacement grows ballistically like t^2. Its slope grows very quickly from zero to about 2α in a time-span $0 \leqslant \theta \lesssim \alpha$. Next, the slope starts decreasing monotonously. The mean-square displacement has a rather complicated, but monotonously growing behaviour, which is transiently superdiffusive. In the final, asymptotic stage, the slope tends toward a constant positive limit, and the mean-square displacement approaches a truly diffusive regime, $\Gamma(t) \sim \hat{D}\, t$, with a positive diffusion coefficient.

It is particularly rewarding to note that the value found for this coefficient agrees very well with the experimental values[9].

In conclusion, comparing our results with those of Rax & White[7,8], we find that their results are mathematically correct, but do not correspond to the physical problem.

A) Their result $\Gamma(t) \sim t^{1/2}$ is obtained in the limit $\lambda_\perp \to \infty$ (*i.e.*, $\delta \to 0$), which is not realized in physical experiments.

B) Their result $\Gamma(t) \sim t^{1/4}$, obtained for arbitrary $\lambda_\perp$, is an exceptional, unstable solution, which cannot be reached from the physical initial condition $\Gamma(0) = \dot{\Gamma}(0) = 0$. Whenever the system starts from an initial condition not lying on the stable manifold of the saddle point, the mean-square displacement ends up growing proportionally to time, i.e. in a truly diffusive manner.

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APPENDIX.

We consider here an alternative method for the determination of the short-time solution, which exploits the properties of the function $\Phi(x;\eta)$ in eq. (29). This method does not require the scaling (34). Consider the function:

$$T(\theta;\eta) = \sqrt{1 + \eta\, \psi(\eta^{-1}\theta)} \ . \qquad \text{(A.1)}$$

The function $\psi(x)$ was defined in eq. (14). It is easily checked that the right hand side of (28) is proportional to the second derivative of $T(\theta;\eta)$ with respect to θ, hence:

$$Y(\theta)\ \ddot{Y}(\theta) = 4\alpha\ \ddot{T}(\theta;\eta) \ . \qquad \text{(A.2)}$$

For very short times, due to the initial condition (30), $Y(\theta)$ is very close to 1. Eq. (A.2) is then linearized:

$$\ddot{Y}_s(\theta) = 4\alpha\ \ddot{T}(\theta;\eta) \ . \qquad \text{(A.3)}$$

Note that this equation is identical with eq. (20) [written in dimensionless form] which describes the exact behaviour of the solution in the limit $\lambda_\perp \to \infty$. Due to its form, the integration of eq. (A.3) is trivially simple:

$$Y_s(\theta) = 4\alpha \left[T(\theta;\eta) - 1 + \frac{1}{4\alpha} \right] , \qquad (Y_s - 1 \ll 1) \ . \qquad \text{(A.4)}$$

[This solution is, of course, identical to (21)]. The comparison of the various solutions in the interval $0 \leqslant \theta \leqslant 1$ is shown in Fig. 6: The simple solution $Y_s(\theta)$ is remarkably accurate up to times of order 5α.

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CAPTIONS OF FIGURES.

Fig. 1. The inner solution $Y_i(\theta)$ [a] and the slope $\dot{Y}_i(\theta)$ [b] for very short times, within the boundary layer. The analytical solution coincides perfectly with the numerical solution. $\alpha = 0.1$. [This exceptionally high value is chosen in order to enhance the characteristic features].

Fig. 2. The inner solution $Y_i(\theta)$ and the external solution $Y_e(\theta)$ [a] (and the corresponding slopes [b]) for intermediate times. Beyond these times the external solution (dashed line) starts deviating from the numerical one. $\alpha = 0.1$. [This exceptionally high value is chosen in order to enhance the characteristic features].

Fig. 3. Phase portrait of the solutions Z(u). P is the physical (positive) hyperbolic fixed point. The straight segments entering or leaving it are, respectively, its stable and unstable manifolds. The dashed line is the solution of the physical initial value problem. $\alpha = 0.01$.

Fig. 4. The complete solution $Y(\theta)$ [a] and its slope [b] for long times. The numerical solution is shown as a continuous line. The dotted line is the external solution. The dashed line is the asymptotic solution. The curve [b] for the slope has been enlarged in the interesting region in order to clearly exhibit the transition from the external to the asymptotic solution. $\alpha = 0.01$; $\theta_* = 89$.

Fig. 5. The diffusion coefficient, divided by α^2, $d = D/\alpha^2$, as a function of α. The continuous line represents eq. (77). The dots are values obtained by numerical integration (final, stable values of $\dot{Y}(\theta)$ for very large values of θ).

Fig. 6. Comparison of the various solutions in the short-time domain. [a]: $Y(\theta)$, [b]: $\dot{Y}(\theta)$. $\alpha = 0.1$. Continuous line: Numerical solution. Dotted line: inner solution Y_i. Dashed line: external solution Y_e. Dashed-dotted line: short-time solution Y_s. In [a]: the dotted curve merges into the continuous one for small θ; in [b]: the dashed line merges into the continuous one after the boundary layer.

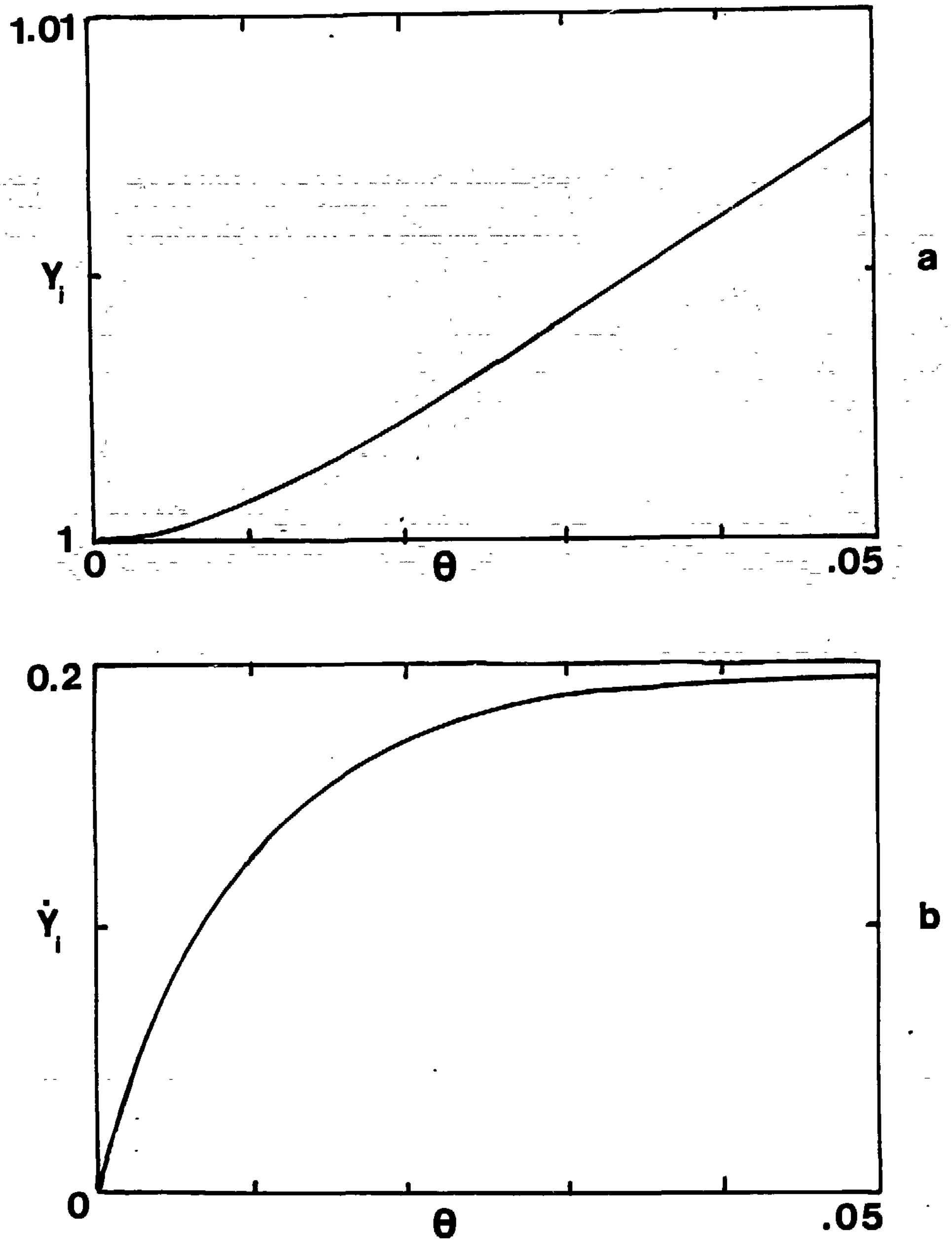
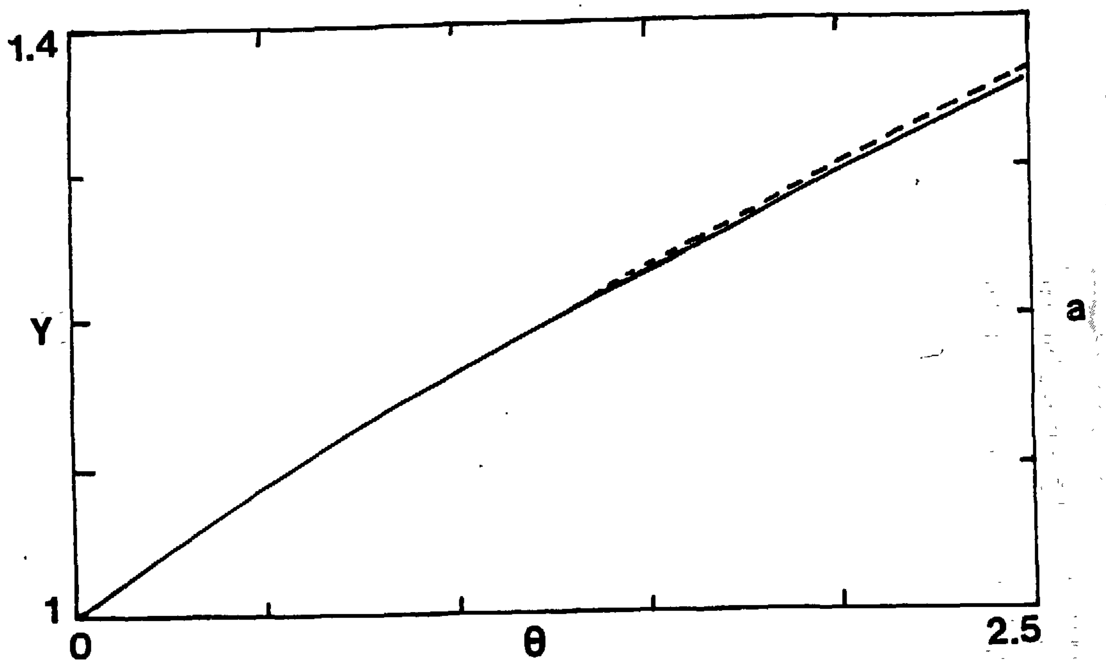


Fig. 1. The inner solution $Y_i(\theta)$ [a] and the slope $\dot{Y}_i(\theta)$ [b] for very short times, within the boundary layer. The analytical solution coincides perfectly with the numerical solution. $\alpha = 0.1$. [This exceptionally high value is chosen in order to enhance the characteristic features].



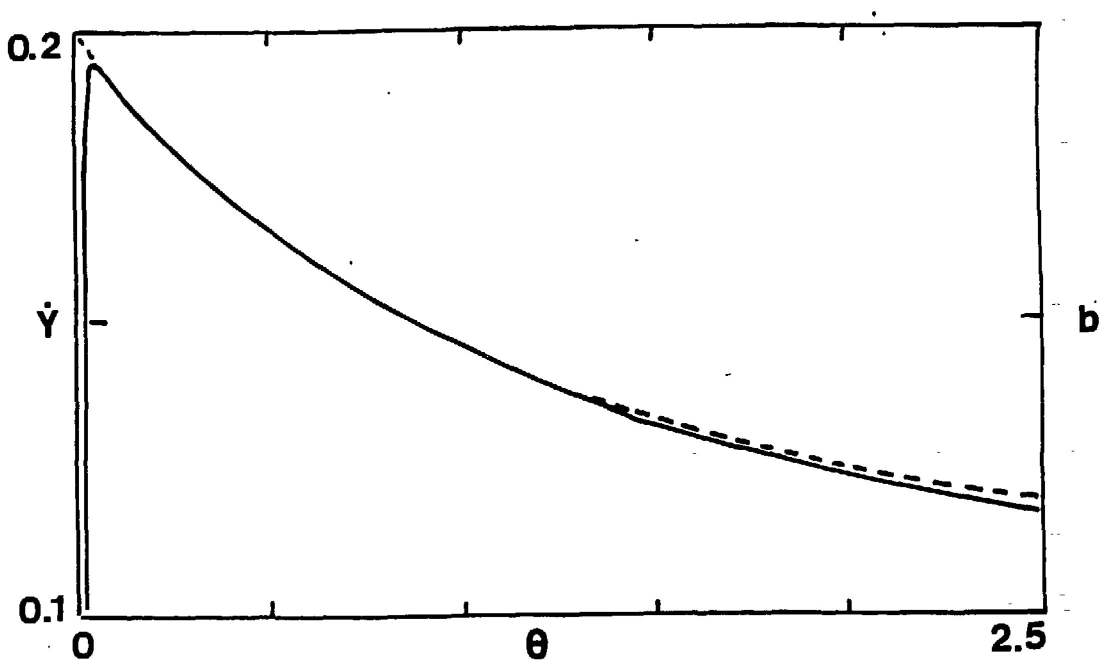
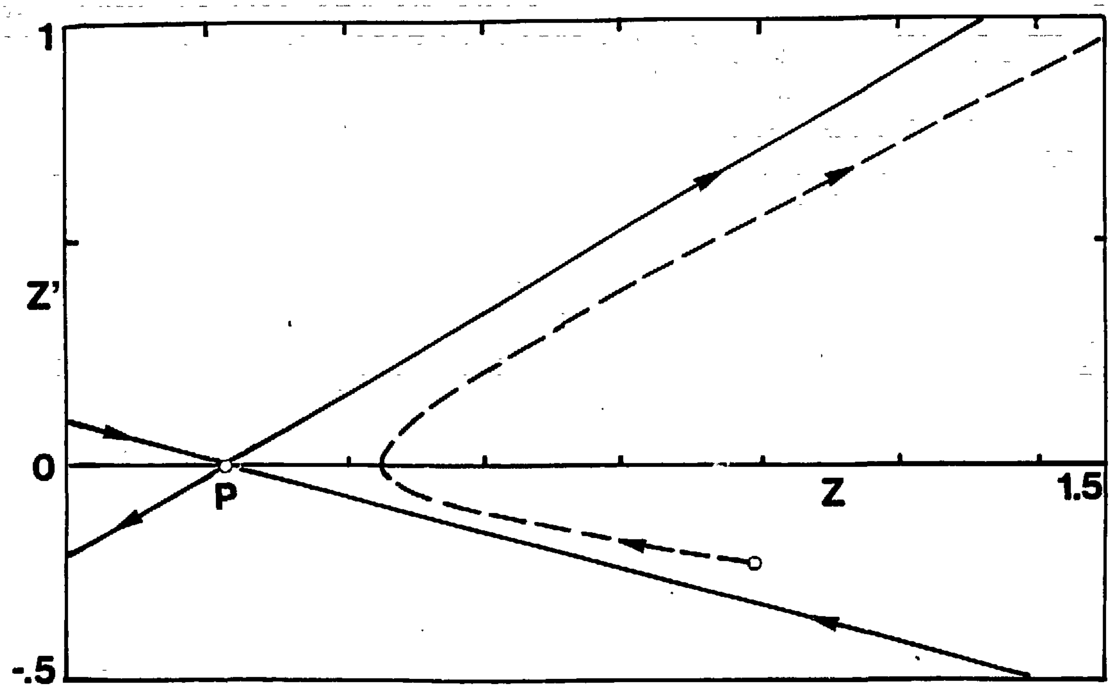


Fig. 2. The inner solution $Y_i(\theta)$ and the external solution $Y_e(\theta)$ [a] (and the corresponding slopes [b]) for intermediate times. Beyond these times the external solution (dashed line) starts deviating from the numerical one. $\alpha = 0.1$. [This exceptionally high value is chosen in order to enhance the characteristic features].



.Fig. 3. Phase portrait of the solutions Z(u). P is the physical (positive) hyperbolic fixed point. The straight segments entering or leaving it are, respectively, its stable and unstable manifolds. The dashed line is the solution of the physical initial value problem. $\alpha = 0.01$.

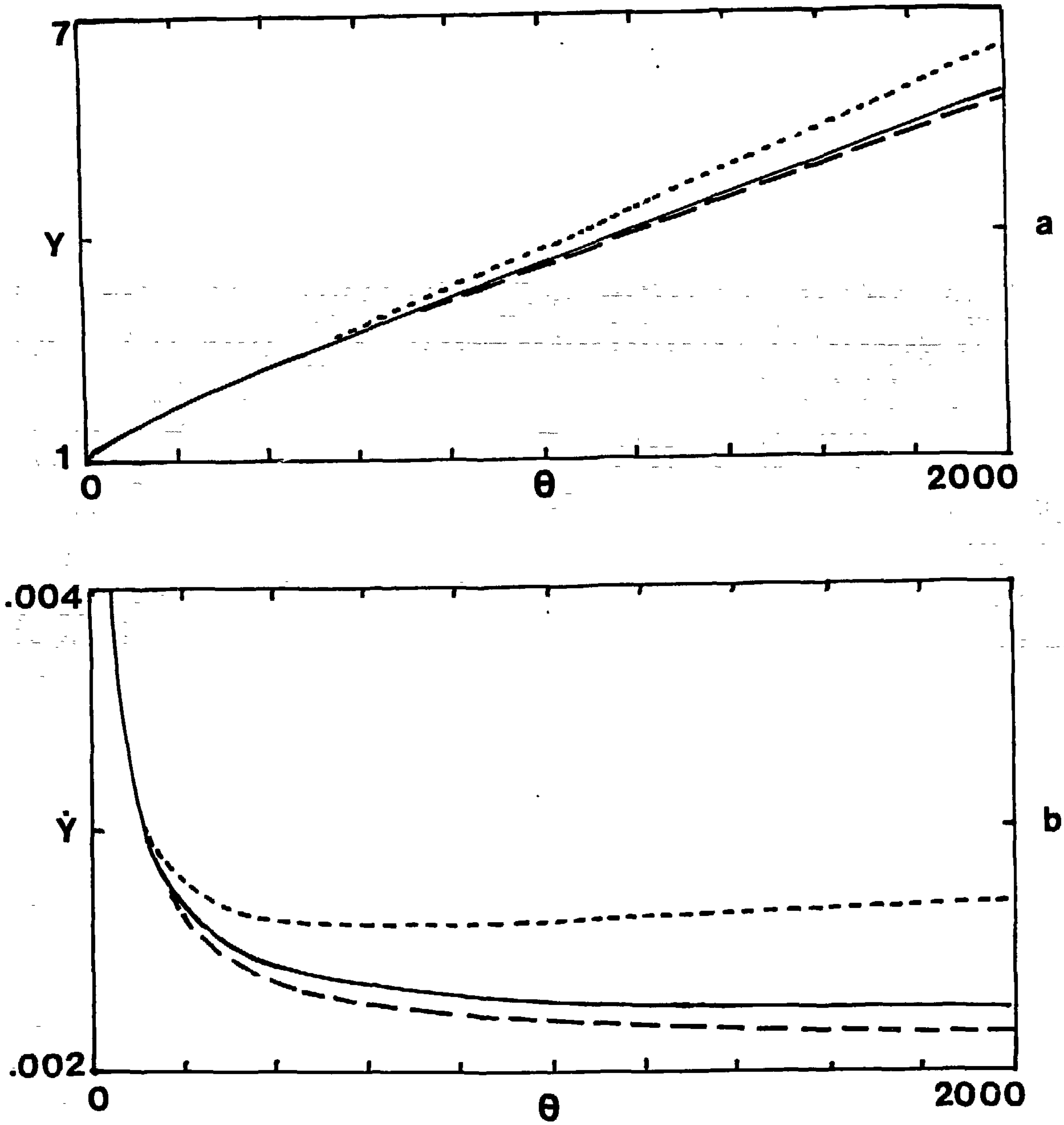


Fig. 4. The complete solution $Y(\theta)$ [a] and its slope [b] for long times. The numerical solution is shown as a continuous line. The dotted line is the external solution. The dashed line is the asymptotic solution. The curve [b] for the slope has been enlarged in the interesting region in order to clearly exhibit the transition from the external to the asymptotic solution.
$\alpha = 0.01$; $\theta_* = 89$.

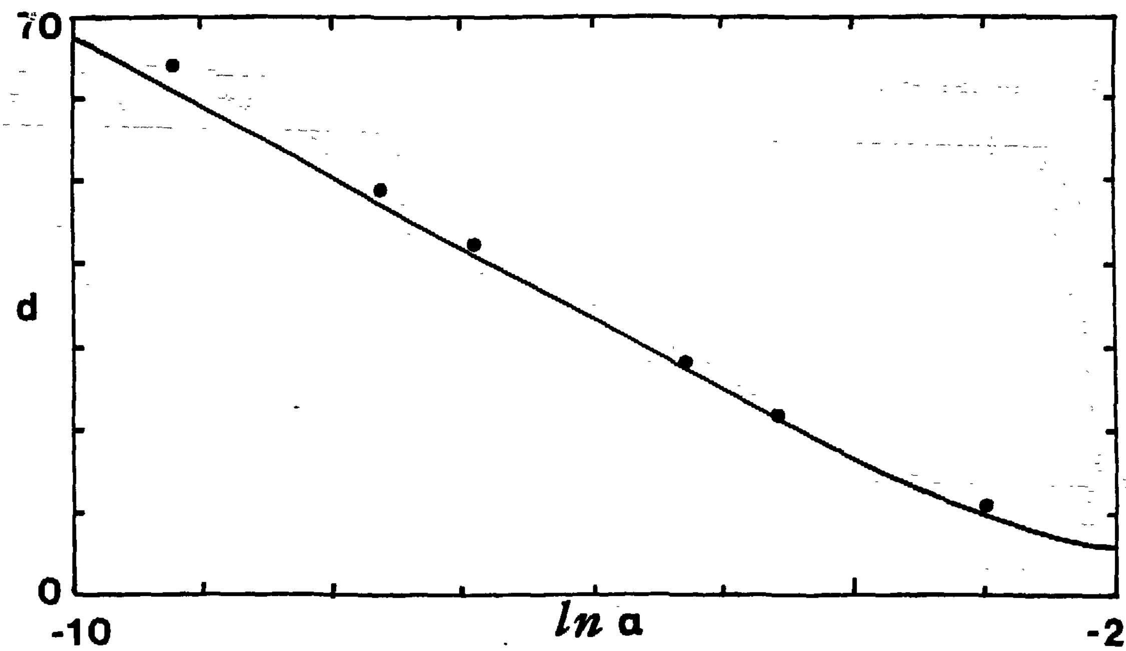


Fig. 5. The diffusion coefficient, divided by α^2, $d = D/\alpha^2$, as a function of α. The continuous line represents eq. (77). The dots are values obtained by numerical integration (final, stable values of $\mathring{Y}(\theta)$ for very large values of θ).

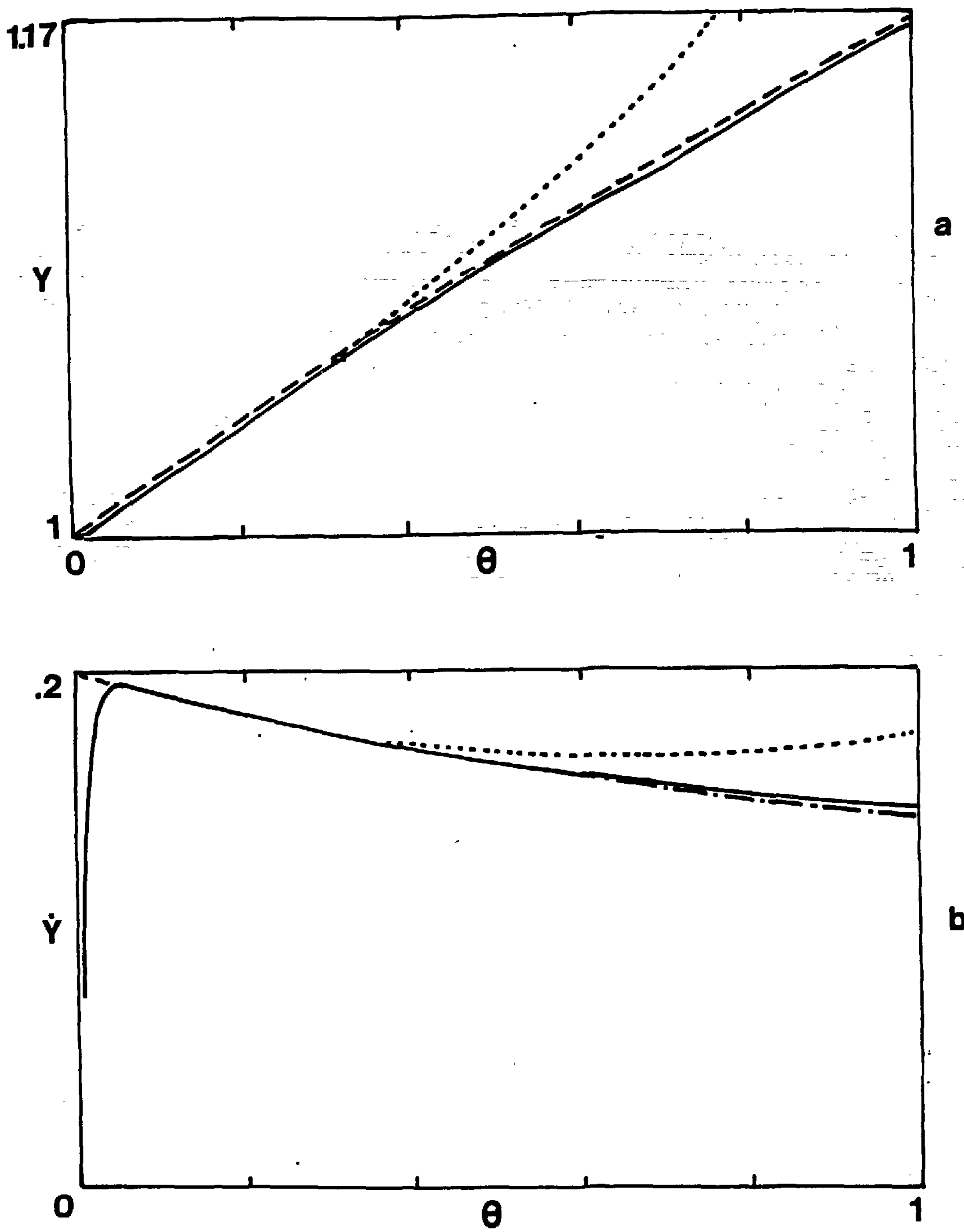


Fig. 6. Comparison of the various solutions in the short-time domain. [a]: $Y(\theta)$, [b]: $\dot{Y}(\theta)$. $\alpha = 0.1$. Continuous line: Numerical solution. Dotted line: inner solution Y_i. Dashed line: external solution Y_e. Dashed-dotted line: short-time solution Y_s. In [a]: the dotted curve merges into the continuous one for small θ; in [b]: the dashed line merges into the continuous one after the boundary layer.