

UWThPh-1992-46
September 10, 1992

AT 9200616

Saddle point solutions in Yang-Mills-dilaton theory

Piotr Bizon¹
Institut für Theoretische Physik
Universität Wien
Boltzmannngasse 5, A-1090 Vienna, Austria
e-mail: bizon@awirtp.bitnet

Abstract

The coupling of a dilaton to the $SU(2)$ -Yang-Mills field leads to interesting non-perturbative static spherically symmetric solutions which are studied by mixed analytical and numerical methods. In the abelian sector of the theory there are finite-energy magnetic and electric monopole solutions which saturate the Bogomol'nyi bound. In the nonabelian sector there exist a countable family of globally regular solutions which are purely magnetic but have zero Yang-Mills magnetic charge. Their discrete spectrum of energies is bounded from above by the energy of the abelian magnetic monopole with unit magnetic charge. The stability analysis demonstrates that the solutions are saddle points of the energy functional with increasing number of unstable modes. The existence and instability of these solutions are "explained" by the Morse-theory argument recently proposed by Sudarsky and Wald.

¹On leave from Institute of Physics, Jagellonian University, Reymonta 4, 30-059 Cracow, Poland

1 Introduction

As it is well known, the Yang-Mills (YM) equations are scale invariant which excludes globally regular (i.e. non-singular with finite energy) static solutions [1,2]. The usual method for circumventing this nonexistence result is to introduce a Higgs field. The coupling of the Higgs field has two effects. First, it breaks scale invariance. Second, a nonabelian gauge group G gets spontaneously broken to a subgroup H . If the homotopy group $\pi_{D-1}(G/H)$ is nontrivial (where D is the number of space dimensions), then the coupled YM-Higgs theory has topologically stable solutions. A prominent example is the 't'Hooft-Polyakov monopole [3] in the $SU(2)$ -YM theory with a triplet Higgs field.

A spontaneously broken gauge theory may admit another class of globally regular solutions if $\pi_D(G/H)$ is nontrivial. This homotopy group is isomorphic to the group of loops in the configuration space (i.e. space of static, finite energy configurations). Nontriviality of $\pi_D(G/H)$ means that there are noncontractible loops passing through the vacuum. The argument, due to Taubes [4] and Manton [5], of how such noncontractible loops lead to a nontrivial solution runs as follows. Consider all loops starting and ending at the vacuum in a fixed homotopy class. On each loop there is a configuration of maximal energy and the infimum of these energies gives a saddle point of the energy functional (and therefore the static solution). Due to the noncompactness and infinite-dimensionality of the configuration space, this argument is obviously not rigorous, and to actually prove that the mini-max procedure converges is a difficult technical problem. Static solutions corresponding to saddle points of the energy functional were called sphalerons to emphasize that, in contrast to solitons, they are unstable. The existence of a sphaleron was first shown by Taubes [4] in the $SU(2)$ -YM theory with a triplet Higgs field and by Manton [5] in the $SU(2)$ -YM theory with a complex doublet Higgs field.

Although sphalerons were originally discovered in spontaneously broken gauge theories, it should be stressed that the Higgs mechanism is by no means necessary for the existence of a sphaleron. Actually, this is already clear in the $SU(2)$ -YM theory with a complex doublet Higgs, where the gauge group $SU(2)$ is completely broken and the homotopy group relevant for constructing a sphaleron is $\pi_3(SU(2)) \simeq Z$. Thus, in this case the role of the Higgs field is just to break the scale invariance while the gauge group itself has a nontrivial third homotopy group. This suggests that a sphaleron may exist in the $SU(2)$ -YM theory coupled to other fields (of attractive force), provided that the coupling: i) breaks scale invariance, and ii) does not alter the topology of the configuration space of pure $SU(2)$ -YM theory.

In this paper I consider a simple example of a coupling which satisfies these two requirements, namely the coupling of a dilaton. The dilaton, ϕ , is a real (massless) scalar field which couples to other matter fields (with lagrangian L_m) through the term $e^{-2a\phi}L_m$, where a is the dilaton coupling constant. I will show that static spherically symmetric $SU(2)$ -YM-dilaton equations have globally regular solutions with the following properties:

- a) there exist a countable family of solutions X_n ($n \in N$),
- b) the energy $E[X_n]$ increases with n and is bounded from above,
- c) the solution X_n has exactly n unstable modes.

This family of solutions is in striking analogy to the Bartnik-McKinnon (BM) solutions [6] of the Einstein- $SU(2)$ -YM equations, which have the same properties a)-c). In both cases the solution X_1 may be interpreted as a sphaleron (for BM solutions this was first observed by Mazur [7]; see also [8]).

A natural question arises: why do two theories with completely different dynamics have qualitatively the same spectrum of solutions? The answer was recently proposed by Sudarsky and Wald (SW) [9]. They presented a heuristic but convincing argument which accounts for the properties a)-c) (except for the boundedness of energy) in the case of BM solutions. This argument is formulated in the spirit of Morse theory for Hamiltonian systems and exploits the existence of topologically nonequivalent multiple vacua in the $SU(2)$ -YM theory (which is related to the fact that $\pi_3(SU(2)) \simeq Z$). A detailed description of the SW argument will be given in Section 7. Here, let me only note that in the case of the solution X_1 , the SW argument is, in essence, equivalent to the mini-max procedure for noncontractible loops. However, in contrast to the mini-max construction, the SW argument can be naturally extended (admittedly, under additional assumptions) to account also for the existence of solutions X_n with $n > 1$. Although the SW argument was originally formulated in the context of Einstein-YM theory, it is essentially insensitive to the concrete form of coupling, and applies almost without modifications to the YM-dilaton theory. In this sense, SW predicted the existence of solutions found in this paper. On the other hand, the results of this paper give further credence to the SW argument.

The existence of the upper bound for the spectrum may be understood by considering the $U(1)$ sector of the YM-dilaton theory. Surprisingly enough, there are *finite* energy abelian solutions which describe magnetic and electric point monopoles. The finiteness of energy is due to the regulating effect of the dilaton which weakens the short distance singularity. Moreover, these solutions saturate the Bogomol'nyi bound (in the $U(1)$ sector), hence their energies are equal to their charge. It turns out from numerics that the limiting solution X_∞ (whose energy bounds the spectrum from above) corresponds to the abelian magnetic monopole with unit magnetic charge.

The YM-dilaton theory and the Einstein-YM theory may be embedded in a single Einstein-YM-dilaton theory governed by the action

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{G} R + 2(\nabla\phi)^2 + e^{-2\alpha\phi} F^2 \right]. \quad (1)$$

This theory is characterized by a dimensionless parameter $\alpha = a/\sqrt{G}$. When $\alpha \rightarrow \infty$, the action (1) reduces to the YM-dilaton theory. When $\alpha = 0$, the action (1) becomes the Einstein-YM theory (plus trivial kinetic term for the scalar field). Finally, the case $\alpha = 1$ corresponds to the low-energy string theory. It was shown by the author elsewhere [10] that the theory defined by the action (1) has static spherically symmetric (globally regular and black-hole) solutions with properties a)-c), for all values of α . This paper specializes to the limiting case $\alpha \rightarrow \infty$. It seems instructive to consider this case separately, because it involves the essential features of the general case, but has an advantage of being considerably simpler, which allows to obtain some analytical estimates on the parameters of solutions. Also, the non-perturbative effect of the dilaton can be clearly seen in this model.

The paper is organized as follows. In the next Section the field equations are derived and some scaling properties are discussed. In Section 3 the explicit abelian solutions are described. In Section 4 the a priori behaviour of globally regular solutions is obtained. In Section 5 the numerical results are presented and some qualitative properties of solutions are discussed. The deep analogy between these solutions and the BM solution is emphasized. Section 6 is devoted to the stability analysis. Finally, in Section 7 the SW argument is summarized and some possibilities of proving rigorously the existence of numerical solutions are suggested.

2 Field equations

The dynamics of the $SU(2)$ -YM field coupled to a dilaton is defined by the action

$$S = \int d^4x \left[2(\nabla\phi)^2 + e^{-2a\phi} F^2 \right], \quad (2)$$

where $F = dA + eA \wedge A$ is the YM curvature of the $SU(2)$ connection A and ϕ is the dilaton. Hereafter, for convenience I put the coupling constants $a = e = 1$, which is equivalent to choosing a/e as the unit of length and $1/ae$ as the unit of energy.

The field equations derived from (2) are

$$D(e^{-2\phi} *F) = 0, \quad (3)$$

$$\nabla^2\phi + \frac{1}{2}e^{-2\phi} F^2 = 0, \quad (4)$$

where D is the $SU(2)$ covariant derivative.

I wish to find static spherically symmetric solutions to these equations that are globally regular i.e. non-singular and with finite energy.

The most general spherically symmetric $SU(2)$ connection has the form [11]

$$A = a\tau_3 dt + b\tau_3 dr + (w\tau_1 + d\tau_2)d\vartheta + (\cot\vartheta\tau_3 + w\tau_2 - d\tau_1)\sin\vartheta d\varphi, \quad (5)$$

where a , b , w and d are functions of (r, t) and τ_i ($i = 1, 2, 3$) are generators of $su(2)$ Lie algebra. Using the residual gauge freedom the radial gauge $b = 0$ can be imposed. When the connection is static, i.e. a , w and d depend only on r , one can also set $d = 0$ by a constant gauge transformation. Hence, the general static spherically symmetric $SU(2)$ connection is described by two functions: the electric potential $a(r)$ and the magnetic potential $w(r)$. Now, I assume further that $a \equiv 0$. Actually, this is not a restriction because one can show, following the argument given in [12], that there are no globally regular solutions with nonzero electric field.

The purely magnetic YM curvature is

$$F = w'\tau_1 dr \wedge d\vartheta + w'\tau_2 dr \wedge \sin\vartheta d\varphi - (1 - w^2)\tau_3 d\vartheta \wedge \sin\vartheta d\varphi, \quad (6)$$

where prime denotes differentiation with respect to r . For F given by this ansatz and for $\phi = \phi(r)$, the equations (3) and (4) reduce to the following system

$$(e^{-2\phi} w')' + \frac{1}{r^2} e^{-2\phi} w(1 - w^2) = 0, \quad (7)$$

$$(r^2 \phi')' + 2e^{-2\phi} \left[w'^2 + \frac{(1 - w^2)^2}{2r^2} \right] = 0. \quad (8)$$

These equations may also be derived from the variation of the energy functional

$$E[w, \phi] = 4\pi \int_0^\infty T_{00} r^2 dr, \quad (9)$$

where T_{00} is the local energy density

$$4\pi T_{00} = \frac{1}{2}\phi'^2 + e^{-2\phi} \left[\frac{1}{r^2} w'^2 + \frac{(1 - w^2)^2}{2r^4} \right]. \quad (10)$$

Let me make two remarks which will be useful in the subsequent discussion. First, note that, in general, the energy functional E is extremized only against variations with $\delta\phi(\infty) = 0$. However, it is also useful to consider more general variations for which $\delta\phi(\infty) \neq 0$. Then, the variation of energy around a solution has the form

$$\delta E = D \delta\phi(\infty), \quad (11)$$

where $D = \lim_{r \rightarrow \infty} r^2 \phi'$ is the dilaton charge. To avoid confusion I want to emphasize that the "surface term" on the right side of eq.(11) is not of the Regge-Teitelboim type (in particular it cannot be cancelled by adding correction to energy) but it is rather a term which appears in variational problems with a free end.

Second, note that the equations (7) and (8) have a "scaling" symmetry. Namely, if $w(\tau)$ and $\phi(\tau)$ are solutions so are

$$\begin{aligned} w_\lambda(\tau) &= w(e^\lambda \tau), \\ \phi_\lambda(\tau) &= \phi(e^\lambda \tau) + \lambda. \end{aligned} \quad (12)$$

Under this transformation the energy scales as follows

$$E[w_\lambda, \phi_\lambda] = e^{-\lambda} E[w, \phi]. \quad (13)$$

The existence of this "scaling" symmetry excludes, via Derrick's argument, nontrivial static finite energy solutions with vanishing dilaton charge D (nota bene this also follows immediately from eq.(8)). However, when $D \neq 0$, Derrick's argument doesn't apply because for the variation induced by the transformation (12) $\delta\phi(\infty)$ is nonzero, and therefore, as follows from (11), the energy is not extremized. Hereafter, I will assume that all solutions satisfy $\phi(\infty) = 0$, which can always be set by the transformation (12). This choice sets the scale of energy in the theory.

3 Abelian solutions

The equations (7) and (8) have two explicit abelian solutions. The first one is the vacuum solution

$$w = \pm 1, \quad \phi = 0 \quad (14)$$

for which the energy has the global minimum $E = 0$.

The second solution is

$$w = 0, \quad \phi = \ln\left(1 + \frac{1}{r}\right) \quad (15)$$

and its YM curvature is

$$F = -\tau_3 d\vartheta \wedge \sin \vartheta d\varphi. \quad (16)$$

which corresponds to the Dirac magnetic monopole with unit magnetic charge. There is also an electrically charged abelian solution related to (15) by the duality rotation :

$$\tilde{F} = e^{-2\phi} * F = \frac{1}{(1+r)^2} \tau_3 dr \wedge dt, \quad \tilde{\phi} = -\phi. \quad (17)$$

These solutions have very interesting properties. The dilaton dramatically changes the properties of $U(1)$ point monopoles (electric and magnetic). Without a dilaton, the energy density of a point monopole diverges at $r = 0$ as $T_{00} \sim 1/r^4$, whereas in the present case $T_{00} \sim 1/r^2$. Thus,

although the solution (15) is singular at $r = 0$, its total energy is finite and equals one! This result may be viewed as non-perturbative cancellation of two infinite self-energies: positive one of the point magnetic monopole and negative one of the dilaton.

Since this solution will play an important role in the discussion of nonabelian solutions, it is useful to see how one can obtain it in a systematic way. Namely, in the $w \equiv 0$ sector, the energy (9) can be written as

$$E[\phi] = \int_0^\infty \left(r\phi' + \frac{1}{r}e^{-\phi} \right)^2 dr + e^{-\phi} \Big|_0^\infty . \quad (18)$$

Thus, if $\phi(0) = \infty$ (and $\phi(\infty) = 0$), the energy is bounded from below by the value of magnetic charge (here set equal to one) and attains a global minimum $E = 1$ on solutions of the first order Bogomol'nyi-type equation

$$r^2\phi' + e^{-\phi} = 0 . \quad (19)$$

Solutions of this equation automatically satisfy the eq.(8) with $w = 0$. Elementary integration of this equation gives the solution (15). A detailed discussion of Bogomol'nyi inequalities in the Maxwell-dilaton theory, without an assumption of spherical symmetry, will be given elsewhere [13].

4 Boundary conditions

Now I will specify the boundary conditions for the globally regular solutions. They are determined by the requirement that

- i) the local energy density be finite for all r

$$T_{00} < const < \infty \quad (20)$$

which imposes boundary conditions for w and ϕ at $r = 0$, and

- ii) the energy be finite

$$E < \infty \quad (21)$$

which imposes boundary conditions for w and ϕ at infinity.

It is easy to construct asymptotic solutions to eqs. (7) and (8) satisfying these boundary conditions. The solution near $r = 0$ is

$$w = 1 - br^2 + O(r^4) , \quad (22)$$

$$\phi = c - 2b^2r^2 + O(r^4) .$$

At $r = \infty$ the asymptotic solution is

$$\pm w = 1 - d/r + O(1/r^2) , \quad (23)$$

$$\phi = \epsilon/r + O(1/r^4) .$$

Here b, c, d , and ϵ are arbitrary constants. All higher order terms in the above expansions are uniquely determined, through recurrence relations, by b and c in (22), and d and ϵ in (23).

Using these boundary conditions one can get the following elementary a priori results for the solutions of eqs.(7) and (8):

Lemma 1. The function w oscillates around zero between -1 and 1 (or $|w| \equiv 1$).

Proof. It follows from eq.(7) that if $w'(r_0) = 0$ then at r_0

$$\text{sgn } w'' = \text{sgn } w(w^2 - 1). \quad (24)$$

This implies that w cannot have local maxima for $w > 1$ and local minima for $w < -1$. Since $w(0) = 1$ and $|w(\infty)| = 1$, this gives $|w| < 1$ for all $r > 0$. Thus from (24), $w''w < 0$, which concludes the proof.

Lemma 2. The function ϕ is monotonically decreasing.

Proof. As above, this follows immediately from the maximum principle applied to eq.(8).

Finally, note that, for the asymptotic behaviour (23) the radial magnetic curvature, $B = \tau_3(1-w^2)/r^2$, falls-off as $1/r^2$, and therefore all globally regular solutions have zero YM magnetic charge.

5 Nonabelian solutions

Let us assume that there exist a 2-parameter family of local solutions defined by the expansion (22). Note that this is a nontrivial statement because the point $r = 0$ is a singular point of the equations (7) and (8), so the formal power-series expansion (22) may have, in principle, a zero radius of convergence. A generic solution with initial data (22) certainly will not satisfy the asymptotic conditions (23) (in fact, the solution may even become singular at some finite distance). The standard numerical strategy, called the shooting method, is to find initial data (b, c) for which the local solution extends to a global solution with the asymptotic behaviour (23). Actually, only b is a nontrivial shooting parameter, since one take arbitrary c and after finding the solution adjust the value of $\phi(\infty)$ to zero using the transformation (12). For generic orbits with $b < b_\infty \simeq 0.3795$ the function w oscillates finite number of times in the region between $w = -1$ and $w = 1$ and then goes to $\pm\infty$. For $b > b_\infty$ all orbits become singular at a finite distance (in a sense that w' becomes infinite).

The numerical results strongly indicate that there exist a countable family of initial data (b_n, c_n) , $n \in \mathbb{N}$, determining globally regular solutions $X_n = (w_n, \phi_n)$. Here the index n labels the number of nodes of the function w . The values of initial data and energies of the first five solutions are given in Table 1. The functions w and ϕ for the first three solutions are graphed in Fig.1 and 2.

Table 1: Initial data (b, c) and energies of the first five globally regular solutions.

n	b	c	E
1	0.26083011	1.711	0.804
2	0.35351804	3.374	0.9659
3	0.3750017	5.158	0.9944
4	0.378754	6.966	0.9992
5	0.379373	8.754	0.99993

The solutions display three characteristic regions. The energy density T_{00} is concentrated in the inner core region $r < R_1$, where R_1 is approximately the location of the first zero of w .

This region decreases with n and shrinks to zero as $n \rightarrow \infty$. In the second region, $R_1 < r < R_2$, where R_2 is approximately the location of the last but one zero of w , the function w slowly oscillates around $w = 0$ with a very small amplitude. In this region the solution is very well approximated by the abelian magnetic monopole (15). This region extends to infinity as $n \rightarrow \infty$. Finally, in the asymptotic region $r > R_2$, the function w goes monotonically to $w = \pm 1$ (hence the YM magnetic charge is gradually screened) and for $r \rightarrow \infty$ the solution tends to the vacuum ($w = \pm 1, \phi = 0$).

Because of these properties, the solutions are in striking resemblance to the BM solutions [6] of the Einstein-YM equations - the dilaton coupling has almost the same effect as the gravitational coupling. In both cases the equilibrium configurations result from a balance between repulsive YM force and attractive, gravitational or dilatonic, force. There are indications that this analogy is even deeper. Below I will discuss two facets of the apparent duality between gravity and dilaton interacting with the YM field.

First, I will show that in the YM-dilaton theory the energy of a static solution can be expressed as a surface integral at spatial infinity. To show this, I will first derive a simple virial identity. Consider a one-parameter family of field configurations defined by

$$\begin{aligned} w_\beta(r) &= w(\beta r), \\ \phi_\beta(r) &= \phi(\beta r). \end{aligned} \quad (25)$$

For this family the energy (9) is given by

$$E[w_\beta, \phi_\beta] = \beta^{-1} I_1 + \beta I_2 \quad (26)$$

where

$$I_1 = \frac{1}{2} \int_0^\infty r^2 \phi'^2 dr, \quad (27)$$

$$I_2 = \int_0^\infty e^{-2\phi} \left[w'^2 + \frac{(1-w^2)^2}{2r^2} \right] dr. \quad (28)$$

Since the energy is extremized at $\beta = 1$, it follows from (26) that on-shell

$$I_1 = I_2. \quad (29)$$

Next, integrating eq.(8) one gets $D = -2I_2$ (D is the dilaton charge defined in Section 2), and therefore eq.(29) yields

$$E = -D. \quad (30)$$

Thus the energy of a static solution can be read-off from the monopole term of the asymptotic expansion (23) of the dilaton field. This is a remarkable property which reminds very much the situation in general relativity and shows a relation between the dilaton field and the conformal degree of freedom of the metric.

Secondly, the most striking analogy between our solutions and the BM solutions is their spectrum of energies (see Table 1 and compare with Table I in ref.[14]). In both cases the energies increase with n and are bounded from above by $E = 1$. This cannot be a coincidence, but what distinguishes this particular value of energy which provides the common upper bound? The answer is remarkably simple. The limiting X_∞ solutions (whose energies give upper bounds) of our family and the BM family saturate the Bogomol'nyi inequalities in the abelian sectors of

respective theories and therefore their energies are equal to the unit magnetic charge. To see this, consider first the dilatonic solutions. As was discussed above, the second region $R_1 < r < R_2$, covers the whole space as $n \rightarrow \infty$, since in this limit $R_1 \rightarrow 0$ and $R_2 \rightarrow \infty$. As n grows the amplitude of oscillations of the function w decreases and goes to zero as $n \rightarrow \infty$. Thus, for $n \rightarrow \infty$ the solution X_n tends (nonuniformly) to the (singular) abelian magnetic monopole described in Section 3:

$$X_\infty = (w = 0, \phi = \ln(1 + \frac{1}{r})) . \quad (31)$$

As I have shown in Section 3, in the $U(1)$ sector of the YM-dilaton theory, the static solutions satisfy the Bogomol'nyi inequality $E \geq Q$. The limiting solution X_∞ saturates the bound in the $Q=1$ sector.

The behaviour of the BM solutions (in isotropic coordinates) is similar: as $n \rightarrow \infty$ the YM field tends to the abelian magnetic monopole while the metric develops a horizon and becomes the extremal Reissner-Nordstrom black hole solution with unit magnetic charge. Thus

$$X_\infty^{BM} = (w = 0, ds^2 = -e^{-2U} dt^2 + e^{2U} (dx^2 + dy^2 + dz^2)) , \quad (32)$$

where

$$U = \ln(1 + \frac{1}{r}) . \quad (33)$$

It is well known that this solution saturates the Bogomol'nyi inequality in Einstein-Maxwell theory [15]. Actually, the limiting solutions (31) and (32) can be mapped one into another by the duality transformation $U \leftrightarrow \phi$ and $\alpha \leftrightarrow 1/\alpha$ in the abelian sector of the theory defined by the action (1).

From the content of the last two paragraphs it is clear that to prove rigorously that the energy is bounded from above by one, one needs in the YM-dilaton theory (and in the Einstein-Yang-Mills theory in the case of BM solutions) a sort of Bogomol'nyi inequality with *reversed* sign, $E \leq Q$, which is saturated by the limiting abelian solution. Unfortunately, I wasn't able to find such an inequality. It would be probably easier, but also much less interesting, to find an upper bound which is not sharp (for the BM solutions that was done in ref.[14]). Also, it is not difficult to obtain not strict bounds on initial parameters. For example, multiplying eq.(8) by ϕ , integrating by parts and combining the result with eq.(29) yields the identity

$$\int_0^\infty (\phi - 1) e^{-2\phi} \left[w'^2 + \frac{(1 - w^2)^2}{2r^2} \right] dr = 0 , \quad (34)$$

which implies that $\phi(0) \geq 1$.

6 Stability analysis

In this Section I address the issue of linear stability of the static solutions described above. To that purpose one has to analyse the time evolution of linear perturbations about the equilibrium configuration. I will assume that the time-dependent solutions remain spherically symmetric and the YM field stays within the ansatz (6). This is sufficient to demonstrate instability because unstable modes appear already in this class of perturbations. The spherically symmetric evolution equations are

$$-(e^{-2\phi} \dot{w}) + (e^{-2\phi} w')' + \frac{1}{r^2} e^{-2\phi} w (1 - w^2) = 0 . \quad (35)$$

$$-r^2\ddot{\phi} + (r^2\phi')' + 2e^{-2\phi} \left[w'^2 + \frac{(1-w^2)^2}{2r^2} \right] = 0, \quad (36)$$

where dot denotes differentiation with respect to time t .

Now, I take the perturbed fields: $w(r) + \delta w(r, t)$, and $\phi(r) + \delta\phi(r, t)$, where $(w(r), f(r))$ is a static solution, and insert them into the eqs. (35),(36). Linearizing and assuming the harmonic time-dependence for the perturbations: $\delta w(r, t) = e^{i\sigma t}\xi(r)$ and $\delta\phi(r, t) = e^{i\sigma t}\psi(r)$, one obtains an eigenvalue problem

$$-\xi'' + 2\phi'\xi' + 2w'\psi' - \frac{1}{r^2}(1-3w^2)\xi = \sigma^2\xi, \quad (37)$$

$$-(r^2\psi')' - 4e^{-2\phi} \left[w'\xi' - \frac{1}{r^2}w(1-w^2)\xi - (w'^2 + \frac{(1-w^2)^2}{2r^2})\psi \right] = \sigma^2 r^2 \psi. \quad (38)$$

It is easy to check that if the perturbations satisfy the boundary conditions

$$\begin{aligned} \xi(0) &= 0 & \psi(0) &= \text{const}, \\ \xi(\infty) &= 0 & \psi(\infty) &= 0, \end{aligned} \quad (39)$$

then the above system is self-adjoint, hence eigenvalues σ^2 are real. Instability manifests itself in the presence of at least one negative eigenvalue.

To solve the eigenvalue equations (37),(38) with the boundary conditions (39), is a straightforward but tedious numerical problem. I have done that for several lowest-energy static solutions. It turns out that the solution X_1 has exactly one unstable mode of frequency $\sigma^2 \simeq -0.0225$. Each successive static solution picks up one additional unstable mode (I have checked this up to $n = 4$). This is consistent with the fact that the limiting solution X_∞ , given by (31), has infinitely many unstable modes. To prove this, consider the perturbations of X_∞ with $\delta\phi = 0$. Then, eq.(37) reads

$$-\xi'' - \frac{2}{r(1+r)}\xi' - \frac{1}{r^2}\xi = \sigma^2\xi. \quad (40)$$

This equation has infinitely many negative modes because the zero energy solution satisfying $\xi(0) = 0$, has infinitely many nodes as can be seen easily from the asymptotic solution.

The result that the solution X_n has n unstable modes fits very nicely to the interpretation of solutions. In particular, for the interpretation of the solution X_1 as a sphaleron, it is crucial that it has exactly one unstable mode. However, remember that I have considered a restricted class of perturbations, and by doing so some directions of instability might have been suppressed. If there are additional unstable modes outside the ansatz (which I doubt), the interpretation of solutions given by Sudarsky and Wald would have to be revised.

7 Sudarsky and Wald's argument

Sudarsky and Wald have recently proposed a heuristic argument which "explains" the existence and instability of the BM solutions of the Einstein-YM equations. This argument is, in principle, applicable to other theories involving the $SU(2)$ -YM field, which are not scale invariant and possess a stable solution. Below I outline the SW argument in application to the $SU(2)$ -YM-dilaton theory.

Let $\bar{\Gamma}$ be a space of all functions (A_i, ϕ) , defined over R^3 , for which the energy E is finite. Let $\bar{\Gamma}$ be a subspace of $\bar{\Gamma}$ with $\phi(\infty) = 0$. The static solutions are extrema of energy on $\bar{\Gamma}$. One such extremum is the vacuum solution $(A_i, \phi = 0)$ for which energy has a global minimum $E = 0$. Now, the key feature of the $SU(2)$ -YM group is the presence of "large gauge transformations" i.e. topologically inequivalent cross-sections of the YM-bundle, classified by the homotopy group $\pi_3(SU(2)) \simeq Z$. Thus, the energy functional E has a countable set of disconnected global minima corresponding to the trivial vacuum $(A_i, \phi = 0)$ and all large gauge transformations of it. To avoid complications with the group of small gauge transformations G , it is convenient to pass from $\bar{\Gamma}$ to the space of gauge orbits $\Gamma = \bar{\Gamma}/G$.

Now, to apply Morse theory methods in Banach spaces, one needs a sort of compactness condition (like the Palais-Smale condition). A convenient way of implementing such a condition (which is here simply assumed to hold), is to introduce on Γ a Riemannian metric G_{AB} (upper case latin letters denote indices of tensor fields on Γ), such that the flow generated by the vector field $M^A = -G^{AB}\nabla_B E$ carries each point of Γ to a critical point of E . As discussed above, there exist a countable set of global minima of E . Since this set is disconnected, the flow defined by M^A cannot carry all points of Γ to global minima (or local minima if any exist), because this would contradict the connectedness of Γ . Thus, the set, $\Gamma_1 \subset \Gamma$, of points which do not flow to local minima, must contain at least one critical point of E . A critical point on Γ_1 with smallest energy E_1 is a saddle point on Γ with exactly one unstable direction. This is believed to account for the existence of the solution X_1 .

Actually, there is a countable set of local minima of E restricted to Γ_1 , namely X_1 and all large gauge transformations of it. Hence, one can repeat the above argument, replacing Γ by Γ_1 (and assuming that Γ_1 is connected), to predict the existence of a submanifold $\Gamma_2 \subset \Gamma_1$ with a point X_2 , whose energy E_2 minimizes E restricted to Γ_2 . The point X_2 is an extremum of E on Γ , which has the 2-dimensional space of unstable directions. This is believed to account for the existence of the second static solution X_2 . All higher n solutions are predicted by the repetition of this argument.

It seems very unlikely that the SW argument in its present form can be converted into a rigorous proof. However, the same argument can be made for spherically symmetric connections. Then, the powerful methods of equivariant Morse theory are available, and in fact these methods were successfully applied in related problems [16]. In my opinion this is a very promising direction for future research.

Another possibility of proving rigorously the existence of numerical solutions found in this paper is to apply the methods of the dynamical systems theory. This approach was used recently by Smoller and his collaborators [17,18], who proved the existence of the BM family of solutions to the Einstein-YM equations. A similar proof should be possible for the YM-dilaton equations although it might be more difficult, because here the corresponding (nonautonomous) dynamical system is four-dimensional whereas in the Einstein-YM case it is three-dimensional.

Acknowledgement. I am grateful to Peter Aichelburg for his continuous interest in my work and many discussions. I thank Robert Beig, Gary Horowitz, Max Meinhart, Walter Simon and Robert Wald for useful comments. I would like to acknowledge the hospitality of the Aspen Center for Physics, where part of this work was carried out. This work was supported by the Fundación Federico.

References

- [1] S.Coleman, in *New Phenomena in Subnuclear Physics*, ed. A.Zichichi (Plenum, New York, 1975).
- [2] S.Deser, *Phys.Lett.* *64 B* (1976) 463.
- [3] G.t'Hooft, *Nucl.Phys.* *B279* (1974) 276; A.M.Polyakov, *JETP Lett.* *20* (1974) 194.
- [4] G.H.Taubes, *Comm.Math.Phys.* *86* (1982) 257; 299; *ibid* *97* (1985) 473.
- [5] N.S.Manton, *Phys.Rev.* *D28* (1983) 2019.
- [6] R. Bartnik and J. McKinnon, *Phys. Rev. Lett.* *61* (1988) 41.
- [7] P.O.Mazur, private communication, 1990.
- [8] D.V.Gal'tsov and M.S.Volkov, *Phys.Lett.* *B273* (1991) 255.
- [9] D.Sudarsky and R.M.Wald, *Phys.Rev.* *D46* (1992) 1453.
- [10] P.Bizon, *Saddle point solutions in low energy string theory*, to appear
- [11] E.Witten, *Phys.Rev.Lett.* *38* (1977) 121.
- [12] P.Bizon and O.T.Popp, *Class.Quantum Grav.* *9* (1992) 193.
- [13] P.Bizon, *Bogomol'nyi inequalities in Maxwell-dilaton theory*, to appear
- [14] H.P.Künzle and A.K.M. Masood-ul-Alam, *J.Math.Phys.* *31* (1990) 928.
- [15] G.W.Gibbons and C.M.Hull, *Phys.Lett.* *109B* (1982) 190.
- [16] L.M.Sibner et al., *Proc.Natl.Acad.Sci.* *86* (1989) 8610.
- [17] J.Smoller et al., *Comm.Math.Phys.* *143* (1991) 115.
- [18] J.A.Smoller and A.G.Wasserman, University of Michigan preprint, 1992.

Figure captions

Fig.1 The function w for the first three globally regular solutions.

Fig.2 The function ϕ for the first three globally regular solutions.

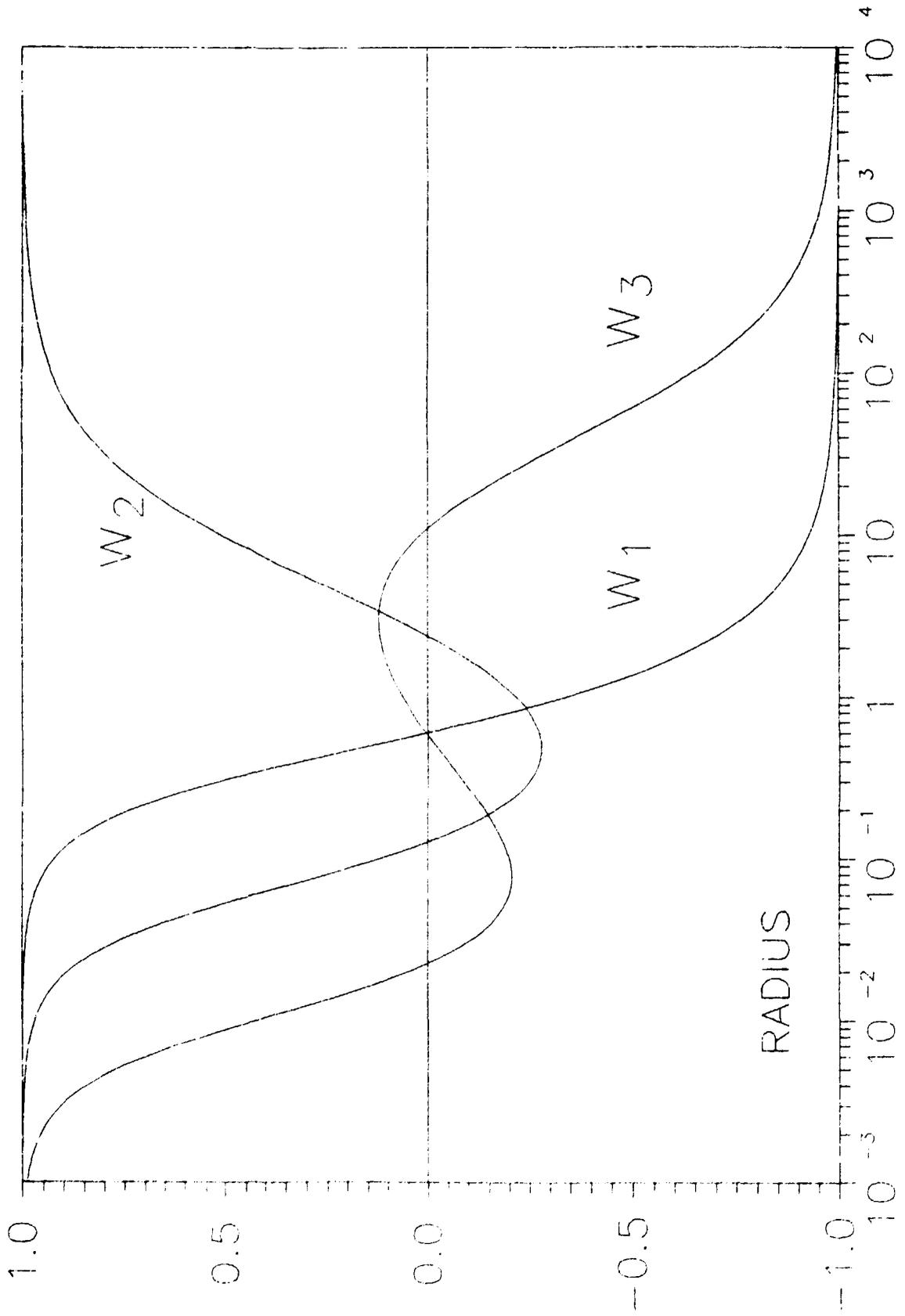


Fig. 1

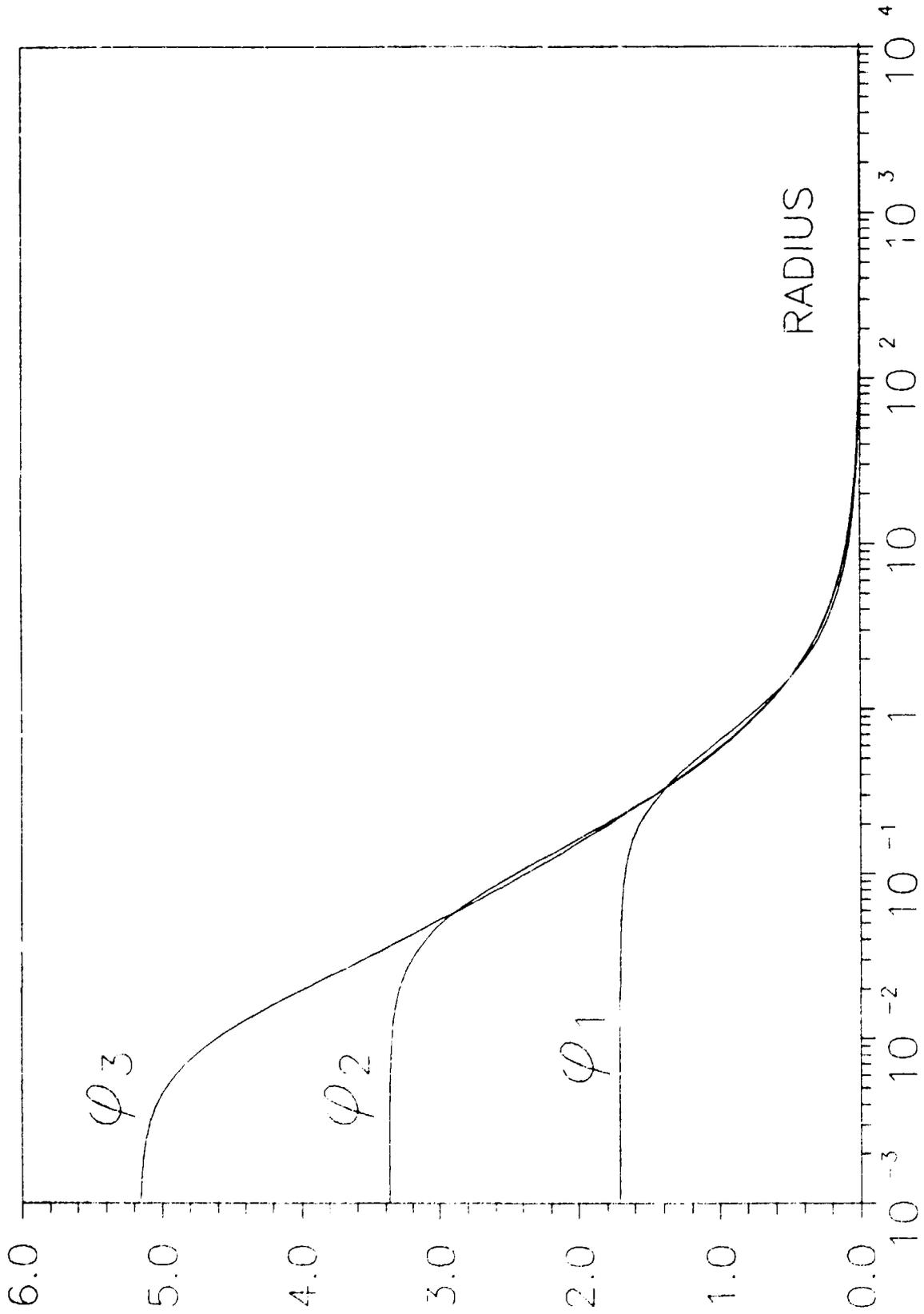


Fig. 2