IC/92/339

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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MIRAMARE-TRIESTE

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ABSTRACT

An overdamped particle moving in a periodic potential, and subject to a constant force and a stochastic force (i.e., $\dot{x} = -\sin(2\pi x) + B + \Gamma(t), \Gamma(t)$ is a white noise) is considered. The mobility of the particle, $\frac{d<x(t)>}{dt}$, is investigated. The stochastic resonance type of behaviour is revealed. The study of the SR problem can thus be extended to systems with periodic force.

MIRAMARE – TRIESTE
October 1992

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It has been well accepted that noise may play rather active role in producing coherent motion under proper nonlinear conditions. In this respect stochastic resonance (SR) is taking central focus in the present stage. Up to now, major portion of publications dealing with the SR problem have considered periodically forced bistable systems. [1]-[10]. However, periodic force is not the necessary ingredient for showing the SR effect. In this paper we will investigate the SR phenomenon without periodic force. The mechanism for the novel type of SR systems will be described, and some possible applications will be briefly introduced.

We consider an overdamped particle moving in a periodic potential. The dynamics can be characterized by the following differential equation

$$\dot{x} = -\sin(2\pi x) \tag{1}$$

Without external force the system eventually approaches one of the equilibrium states $x = 0, \pm 1, \pm 2, ... \pm n, ...$ Let us apply a constant force to push the system moving towards positive x. The equation is then replaced by

$$\dot{x} = -\sin(2\pi x) + B \tag{2}$$

If B is small the particle can only stay in one of the potential basins. The asymptotic state of the system is still stationary. Therefore, including the external force (now it is constant rather than periodic) breaks the symmetry of the system with respect to positive and negative x, and makes the condition favorable to the motion to right, this symmetry breaking however does not result in a coherent motion towards right. As $t \to \infty$ the asymptotic mobility of the system (i.e., $\dot{x}$) remains to be zero.

Now we add a stochastic force to (2) and replace it by a Langevin equation

$$\dot{x} = -\sin(2\pi x) + B + \Gamma(t) \tag{3}$$

$$< \Gamma(t) >= 0 \quad , \quad < \Gamma(t)\Gamma(t') >= 2D\delta(t-t') \tag{4}$$

With this equation we ask how the noise influnces the mobility of the particle, and how to get the largest mobility rate by properly applying noise.

For simplifying the analysis and focusing our attention on the conceptional essence of the problem we restrict on the case

$$B, \ D \ << \ 0 \tag{5}$$

Assuming the probability for the particle to stay in the nth ($n = 0, \pm 1, \pm 2, ...$) basin at time t to be $P_n(t)$, the evolution of the probability distribution can be described by the master equations

$$\dot{P}_n(t) = -2(\gamma_r + \gamma_l)P_n(t) + \gamma_r P_{n-1}(t) + \gamma_l P_{n+1}(t) \quad , \quad n = 0, \pm 1, \ldots \tag{6}$$

where γ_r and γ_l represent the transition rates from any basin to the nearest right and left basins, respectively, which can be explicitly given, under the condition (5), as

$$\gamma_{r,l} = 2\exp(-\frac{1}{\pi D} \pm \frac{B}{2D}) \tag{7}$$

The following generating function

$$g(S,t) = \sum_{n=-\infty}^{+\infty} P_n(t) S^n \tag{8}$$

can be introduced to transform the infinite coupled master equations to a single partial differential equation

$$\frac{\partial g(S,t)}{\partial t} = [-2(\gamma_r + \gamma_l) + \gamma_r S + \frac{\gamma_l}{S}]g(S,t) \tag{9}$$

which has an exact solution

$$g(S,t) = f(S)\exp[(-2(\gamma_r + \gamma_l) + \gamma_r S + \frac{\gamma_l}{S})t] \tag{10}$$

where $f(S)$ can be obtained, provided the initial distribution is given

$$f(S) = \sum_{n=-\infty}^{\infty} P_n(0)S^n \tag{11}$$

At time t the probability distribution $P_n(t)$ are given by the power expansion coefficients of $g(S,t)$ (note, the expansion now includes negative powers, and thus the formula $n!P_n(t) = \frac{\partial^n g(S,t)}{\partial S^n}_{|S=0}$ does not work).

From Eq.(10) the mobility rate can be easily calculated as

$$\begin{aligned}\theta &= \frac{d < x(t) >}{dt} = \frac{d}{dt}[\sum_{n=-\infty}^{\infty} nP_n(t)] \\ &= \frac{d}{dt}\frac{\partial g(S,t)}{\partial S}_{|S=1} = \gamma_r - \gamma_l \\ &\approx \frac{B}{D}\lambda \end{aligned} \tag{12}$$

with $\lambda = 2\exp(-\frac{1}{\pi D})$. Thus, the mobility is simply equal to the difference between the right and the left transition rates. In Eq.(12) θ is explicitly given by the product of the two physically meaningful factors. The factor $\frac{B}{D}$ indicates the destructive influnce of noise on the asymmetry of the system. The constant force B produces the asymmetry of the deterministic system, while noise definitely reduces the level of the asymmetry for the stochastic system and makes the system more homogeneous. However, the asymmetry of the system is not sufficient, in the deterministic case, for producing a coherent motion towards to positive x. Without noise the asymptotic solution is still stationary, the coherent motion to the right is stopped by the dissipation of the system. The other factor in Eq.(12), namely, λ, plays central role in producing coherent motion which increases as D increases, and reduces to zero as $D \to 0$. Therefore, noise plays twofold roles. On one hand, it stimulates coherent motion in responding to the asymmetric condition of the system, this motion does not exist in deterministic case.. On the other hand, noise reduces the asymmetry of the system which is the root of the coherent motion. The competition of these two seemly opposite roles leads to the SR type of behavior. For small D the first tendency dominantes, the $\theta - D$ response curve must have positive slope. For large D, the second tendency dominantes, then the $\theta - D$ curve goes down. Finally we certainly get a peaked response curve. From Eq.(12) it is obvious that θ takes the maximal value

$$\theta = \frac{2\pi B}{e} \tag{13}$$

at $D_m = \frac{1}{\pi}$ which is the solution of the equation $\frac{d\theta(D)}{dD} = 0$. Equation (13) is, of course, unreasonable since the all above procedure is valid under the condition (5). At so large D ($D = \frac{1}{\pi}$) this condition is broken, and then the quantitative result is not correct. Nevertheless, the qualitative stochastic resonance behavior is no doubt because we have, precisely,

$$\lim_{D \to 0} \frac{d\theta(D)}{dD} > 0$$

and

$$\lim_{D \to \infty} \frac{d\theta(D)}{dD} < 0$$

.

In this short paper, we have got the following two results. First we have revealed SR effect in a system rather different from the systems investigated so far in the SR problem. Thus the application of SR study can be extended to a large new class of systems. The system (3) has a wide application in the problem of a particle moving in a lattice subject to strong dissipation. Actually, a system similar to Eq.(3) has been investigated long ago in the Josephson junction problem where the x space is periodic (i.e., $x \pm n$ for any integer n make no difference from x). [11]-[13] The result new in this paper is that we are able to show the SR behavior in this type of systems.

Second, the investigation of our novel model helps us to gain deeper understanding on the mechanism underlining the SR effect which is the competition of the twofold roles played by noise: stimulating the coherent motion in responding to the asymmetry of the system on one hand; and destroying the asymmetry of the system and making the system homogeneous on the other hand.

ACKNOWLEDGMENTS

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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