BIFURCATION THEORY FOR TOROIDAL MHD INSTABILITIES

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Abstract. Using a general representation of magneto-hydrodynamics in terms of stream functions and potentials, proposed earlier, a set of reduced MHD equations for the case of toroidal geometry had been derived by an appropriate ordering with respect to the inverse aspect ratio. When all dissipative terms are neglected in this reduced system, it has the same linear stability limits as the full ideal MHD equations, to the order considered. When including resistivity, thermal conductivity and viscosity, we can apply bifurcation theory to investigate nonlinear stationary solution branches related to various instabilities. In particular, we show that a stationary solution of the internal kink type can be found.

1. EXACT REPRESENTATION AND THE REDUCED EQUATIONS OF TOROIDAL MHD

1.1. General form of the representation

We consider a simple one-fluid model, described by the following equations:

$$d\mathbf{V}/dt = -\nabla p + \mathbf{j}\times\mathbf{B} \quad + \ (\text{friction and source terms})$$
$$d\rho/dt = -\rho\nabla.\mathbf{V} \quad + \ (\text{mass source})$$
$$\mathbf{E} + \mathbf{V}\times\mathbf{B} = \eta\mathbf{j}$$
$$\mathbf{B} = \nabla\times\mathbf{A} \ , \quad \mathbf{E} = \partial\mathbf{A}/\partial t \ -\nabla\Phi_E + \mathbf{E}_{loop} \ , \quad \nabla\times\mathbf{B} = \mu_o\mathbf{j}$$
$$p = \text{const}\,.\rho T \ , \quad 3/2\ \{\partial p/\partial t + \nabla.(p\mathbf{V})\} + p\nabla.\mathbf{V} = \nabla.\mathbf{q}_H + S_H$$

Here $\mathbf{E}_{loop}$ is that part of the electric field which is related to

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changes of the magnetic field outside the torus (induced loop voltage), so that $\nabla \times \mathbf{E}_{loop} = 0$, $\nabla . \mathbf{E}_{loop} = 0$ inside the torus.

We introduce a suitably chosen time-independent reference field $\mathbf{B}_o$ (e.g. a toroidal equilibrium field) and use general curvilinear coordinates ξ^1, ξ^2, ξ^3 (e.g. flux coordinates related to $\mathbf{B}_o$). In terms of the covariant basis $\nabla\xi^i$ (i = 1,2,3) and the associated contravariant basis $\mathbf{e}_3 = J\,\nabla\xi^1 \times \nabla\xi^2$ etc. , denoting J the Jacobian and $g_{ik} = (\mathbf{e}_i . \mathbf{e}_k)$ the metric tensor and assuming that the g_{ik} do not depend on ξ^3, the vector potential and magnetic field may be written in the form

$$\mathbf{A} = -\psi\, \mathbf{e}_3 / g_{33} - \nabla U \times \mathbf{e}_3 - \nabla \Phi_A$$

$$\mathbf{B} = (\Delta U - \Lambda\psi)\mathbf{e}_3 - \nabla \partial U/\partial\xi^3 - \nabla\psi \times \mathbf{e}_3/g_{33}$$

with $\Lambda = (1/J)\{\partial(g_{32}/g_{33})/\partial\xi^1 - \partial(g_{31}/g_{33})/\partial\xi^2\}$

The velocity field is expressed in terms of the reference field,

$$\mathbf{V} = V_{\|}\mathbf{B}_o/B_o + \mathbf{V}_\perp ,$$

$$\mathbf{V}_\perp \times \mathbf{B}_o = \nabla\Phi_V + (\Delta u - \Lambda a)\mathbf{e}_3 - \nabla(\partial u/\partial\xi^3) - \nabla a \times \mathbf{e}_3/g_{33}$$

With these definitions, a system of equations for the scalar functions is obtained [2,3], which is completely equivalent to the original vector equations given above.

1.2. Reduced MHD equations

In reference 3 a reduced system of toroidal MHD equations has been obtained. We summarize here some important results needed in section 2.

The reference field is supposed to be a given axisymmetric equilibrium field, $\mathbf{B}_o = \mathbf{B}_{eq} = I_{eq}(\psi_{eq})\nabla\zeta - \nabla\psi_{eq} \times \nabla\zeta = \{\nabla\zeta + q(\psi_{eq})\nabla\theta\} \times \nabla\psi_{eq}$ where ζ is the toroidal angle about the symmetry axis, ψ_{eq} the poloidal flux and q the safety factor. We use coordinates (r,θ,ζ) with $r = \{2R_a \int_0^{\psi_{eq}} \{q_{eq}/I_{eq}\}d\psi\}^{1/2}$, where R_a is the radius of the magnetic axis at equilibrium.

In order to simplify the present paper we assume the viscosity and momentum source to be given by $\mu_\perp \Delta V + S_{mom}$, the heat flux and source by $q_H + S_H = -\kappa_\perp \nabla(p - p_{eq})$, and $\rho = \rho_o$ = constant.

Normalizing as usual (e.g. lengths by the small radius r_o , etc.) and assuming the following ordering: $\partial/\partial r \sim \partial/\partial\theta \sim \varepsilon^o$, $\partial_{\|} = \partial/\partial\zeta + (1/q)\partial/\partial\theta \sim \varepsilon$, $p,\psi \sim \varepsilon^0$, $\beta \sim \varepsilon$, $\Phi_V \sim \varepsilon$, $a, u \sim \varepsilon^3$, $|I - I_{eq}| \sim \varepsilon^2$, we can use the following reduced toroidal MHD equations of ref.3 :

$$\Delta_p{}^{+}\Phi_V = w \qquad \text{with } \Delta_p{}^{+} = (I^2/R^2)\nabla.(R^2/I^2)\nabla_p$$

$$\frac{\partial w}{\partial t} = \frac{I^2 R_a}{rR^4}\left\{\frac{R^4}{I^3}w\,,\Phi_V\right\} - \beta\,\frac{I^2 R_a}{rR^4}\left\{\frac{R^2}{I},p\right\} + \frac{I^3}{R^4}\left[\frac{\partial}{\partial\zeta}\left(\frac{j_\zeta}{I}\right) + \frac{R_a}{rI}\left\{\psi,\frac{j_\zeta}{R}\right\}\right] + \mu_\perp\Delta_p w + S_{mom}$$

$$\frac{\partial\psi}{\partial t} = \frac{R_a}{rI}\left\{\psi,\Phi_V\right\} + \frac{\partial\Phi_V}{\partial\zeta} + \eta\,\Delta_p{}^{*}\psi + E_o \quad \text{, with } \Delta_p{}^{*} = R^2\nabla.R^{-2}\nabla_p$$

$$\frac{\partial p}{\partial t} = \frac{R_a}{rI}\left\{p,\Phi_V\right\} - \nabla.q_H + S_H\,, \qquad q = -\kappa_\perp\nabla_p p$$

where $\{f,g\} = (\partial f/\partial r)(\partial g/\partial\theta) - (\partial f/\partial\theta)(\partial g/\partial r)$, and we choose boundary conditions $p = \psi = \Phi_V = w = 0$ at $r = 1$.

Similar systems of reduced equations have been discussed in the litterature (for example refs.4,5). The above system is characterized by the use of intrinsic (flux) coordinates of a given equilibrium. Therefore, important toroidal effects are contained in the metric coefficients which are implicitly present in the above equations. As a consequence, stability limits deduced from the above equations are usually very close to the stability limits of the complete MHD equations. In particular, it can be shown that the ideal stability limits obtained by an energy principle derived from the above reduced equations are equal to the corresponding stability limits of the complete equations to order ε^2 [3,9].

2. BIFURCATION PROBLEM FOR A TOROIDAL PLASMA

In fusion plasmas, radial profiles of temperature, density, current density etc. are maintained by external sources against neoclassical and anomalous (turbulent) transport. Restricting ourselves to MHD, we describe equilibria and time-asymptotic (stationary) turbulent states by MHD equations including sources and dissipative terms. For a large class of models describing this type of problems, it has been shown that theorems of bifurcation theory can be applied [6], and typical examples have been worked out explicitly [2,3,7]. In particular, bifurcation of a stationary kink type state in a cylindrical plasma has been demonstrated [2]. Here we wish to present results on the bifurcation of stationary internal kink states in a toroidal fusion plasma described by the above reduced MHD equations. Linearizing these equations for small perturbations of the equilibrium, letting $\partial/\partial t \Rightarrow \gamma$ (growth rate), we *shall obtain* the bifurcation points as the values of an appropriately chosen control parameter at marginal stability ($\gamma = 0$).

We now use the fact that the solution of the linearized ideal MHD equations and the linear growth rate of the ideal internal kink mode have been calculated in ref.8 , where the minimization of the potential energy variation δW has been carried out by an expansion in orders of ε . The results of that work enable us to draw conclusions on the bifurcation of a stationary kink perturbation, as we shall now demonstrate. Assuming $\Phi_{V,eq} = 0$ and linearizing our reduced dissipative MHD equations we get (putting $\tilde{\psi} = \psi - \psi_{eq}$, $\tilde{p} = p - p_{eq}$)

$$(\gamma-\mu_\perp\Delta_p)w = -\beta\frac{I^2 R_a}{rR^4}\left\{\frac{R^2}{I}, \tilde{p}\right\} + \frac{I^2}{R^4}\left(\frac{\partial}{\partial\zeta} + \frac{1}{q}\frac{\partial}{\partial\theta}\right)\Delta_p^*\tilde{\psi} + \frac{I^2 R_a}{rR^4}\left\{\tilde{\psi}, \frac{j_{eq,\zeta}}{I}\right\}$$

$$(\gamma-\eta\Delta_p^*)\tilde{\psi} = \frac{\partial\Phi_V}{\partial\zeta} + \frac{1}{q}\frac{\partial\Phi_V}{\partial\theta}$$

$$(\gamma-\kappa_\perp\Delta_p)\tilde{p} = \frac{1}{q}\frac{dp_{eq}}{d\psi_{eq}}\frac{\partial\Phi_V}{\partial\theta}$$

For the sake of simplicity of the present demonstration we assume $\mu_\perp = \eta = \kappa_\perp$ .Since the dissipative coefficients are small, we shall neglect the difference between Δ_p and Δ_p^*, and we apply the operator $(\gamma - \mu_\perp\Delta_p)$ to the equation for w . The derivatives of the equilibrium quantities are characterized by a radial scale of the order of the small plasma radius $r_0=1$, whereas the radial scale of the perturbation is less than the radius r_1 of the magnetic surface where $q = 1$. *Assuming that this scale is much smaller than 1* , we may neglect the variation of the equilibrium quantities (I , $j_{eq,\zeta}$, q) when applying the operator $(\gamma - \mu_\perp\Delta_p)$ to the right hand side of the equation for w . In this way we obtain

$$(\gamma-\mu_\perp\Delta_p)^2 w = -\beta\frac{I^2 R_a}{rR^4}\left\{\frac{R^2}{I}, \frac{1}{q}\frac{dp_{eq}}{d\psi_{eq}}\frac{\partial\Phi_V}{\partial\theta}\right\}$$

$$+ \frac{I^2}{R^4}\left(\frac{\partial}{\partial\zeta} + \frac{1}{q}\frac{\partial}{\partial\theta}\right)\Delta_p^*\left(\frac{\partial}{\partial\zeta} + \frac{1}{q}\frac{\partial}{\partial\theta}\right)\Phi_V + \frac{I^2 R_a}{rR^4}\left\{\left(\frac{\partial}{\partial\zeta} + \frac{1}{q}\frac{\partial}{\partial\theta}\right)\Phi_V, \frac{j_{eq,\zeta}}{I}\right\}$$

Since $w = \Delta_p + \Phi_V$, this represents a homogeneous differential equation for the function Φ_V , for which we have to specify appropriate boundary conditions. γ is then the eigenvalue of the problem if $\mu_\perp$ is given. As is well known, the solution is characterized by the presence of the $q = 1$ surface ($r = r_1$). For $\mu_\perp = 0$ the perturbation drops sharply to zero outside this surface [8]. When $\mu_\perp \neq 0$ but small, there is a narrow dissipative layer about $r = r_1$, and the solution Φ_V must connect to the ideal ($\mu_\perp = 0$) solutions outside this layer. In analogy with the method of ref.2 we introduce eigenfunctions of Δ_p obeying the boundary conditions near r_1:

$$\Delta_p f + \sigma^2 f = 0 .$$

where σ^2 is the eigenvalue. To lowest order in ε the left hand side of the equation for w is now replaced by

$$(\gamma + \mu_\perp \sigma^2)^2 w$$

In this way we have arrived at an equation for Φ_V which is identical with that obtained from the linearized equations except for the fact that γ is now replaced by $(\gamma + \mu_\perp \sigma^2)$. In the range of parameters near the stability limit of the ideal internal kink mode, we therefore obtain approximately

$$(\gamma + \mu_\perp \sigma^2)^2 = (\gamma_{ideal})^2$$

where γ_{ideal} is the expression of the ideal growth rate given in ref.8. This expression depends on physical parameters, as for instance β, which may be chosen as bifurcation parameter.

This result, together with the general mathematical properties of our reduced dissipative MHD equations, allow us to make the following statement:

In the dissipative MHD model discussed here the equilibrium becomes unstable for parameters such that $\gamma_{ideal} = \mu_\perp \sigma^2$. But contrary to the ideal MHD instability which may grow indefinitely, the dissipative case leads to a stationary (attracting) solution of the nonlinear equations: Indeed, due to the form of the dissipative terms (containing a Laplacian), the theorems on bifurcation tell us that a stationary nonlinear solution branch bifurcates at the marginal point defined by $\gamma_{ideal} = \mu_\perp \sigma^2$.

3. CONCLUSION

In the present paper we have given a short demonstration of the bifurcation of a stationary state of the type of the internal kink mode. A more detailed calculation including the discussion of the solution near the q=1 surface will be presented elsewhere.

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