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MATHEMATICAL MODELLING, VARIATIONAL FORMULATION AND NUMERICAL SIMULATION OF THE ENERGY TRANSFER PROCESS IN A GRAY PLATE IN THE PRESENCE OF A THERMAL RADIANT SOURCE

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RESUMO

Neste trabalho o processo de transferência de energia em uma placa rígida, cinzenta e opaca, aquecida por uma fonte radiante térmica, é considerada. A fonte é tratada como um corpo negro esférico, com raio a ($a \to 0$) e geração de calor uniforme, posicionado acima da placa. É construído um modelo matemático supondo que a transferência de calor de/para a placa ocorre por radiação térmica. O modelo matemático obtido é não-linear. É apresentado um princípio variacional equivalente, o qual é empregado na simulação de alguns casos particulares.

ABSTRACT

~~In this work~~ the energy transfer process in a gray, opaque and rigid plate, heated by an external thermal radiant source, is considered. The source is regarded as a spherical black body, with radius a ($a \to 0$) and uniform heat generation, placed above the plate. A mathematical model is constructed, assuming that the heat transfer from/to the plate takes place by thermal radiation. The obtained mathematical model is nonlinear. ~~It is presented~~ a suitable variational principle which is employed for simulating some particular cases. is presented

1. INTRODUCTION

The radiant interchange is an energy transfer mechanism which is always present when two bodies are separated by a nonopaque medium. Nevertheless, the radiative transfer is neglected in most of energy transfer phenomena considered for engineering calculations.

However, for bodies at high temperature levels and/or bodies in space, the radiative interchange plays the main role in the energy transfer process and can not be neglected.

Usually, when the radiative interchange can not be neglected, it is considered under very restrictive assumptions in order to reduce (drastically) the mathematical complexity of the problem. The most common assumption consists of assuming uniform temperature for the considered bodies, approximating a partial differential equation (subjected to nonlinear boundary conditions) by an algebric equation.

The subject of this work is the energy transfer process in a thin circular plate which is heated by the thermal radiation coming from a source (a small body). The purpose is to estimate how the presence of a small body, with a high rate of heat generation, may affect the temperature distribution in another one in atmosphere-free space.

A scheme of the problem is shown in figure 1. The source will be modelled as a spherical black body, with a constant heat generation rate Q and radius a ($a \to 0$) while the plate will be assumed to be rigid, opaque and gray, possessing constant emittance ε and constant thermal conductivity k. It will be also assumed that there exist no atmosphere, in such a way that the heat transfer from/to the plate takes place only by thermal radiation.

The circular plate will be represented by the open set Ω, with boundary $\partial\Omega$, given by

$$\Omega \equiv \left\{ x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \in \mathbb{R}^3 \text{ such that } \; x^2+y^2<R^2 \; , \; -1<z<0 \right\} \qquad (1)$$

while the radiant source will be represented by the set Γ, with boundary $\partial\Gamma$, given by

$$\Gamma \equiv \left\{ x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \in \mathbb{R}^3 \text{ such that } \; x^2+y^2+(z-H)^2<a^2 \; , \; a<<H \right\} \qquad (2)$$

in which $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ ($\mathbf{i}\equiv(1,0,0)$; $\mathbf{j}\equiv(0,1,0)$ and $\mathbf{k}\equiv(0,0,1)$) is the usual orthonormal basis associated to the considered rectangular Cartesian coordinate system (x,y,z).

It will be assumed that the subset of $\partial\Omega$ given by $x^2+y^2=R^2$ is insulated.

The objective of this work is the mathematical modelling and the simulation of the above mentioned phenomenon. The mathematical model will be represented by a partial differential equation (governing the conduction heat transfer inside the plate) subjected to nonlinear boundary conditions (which represent the coupling between the radiative transfer and the conduction heat transfer on body boundary). This model, based on classical theories, will be modified (taking into account the physical reality) in order to give rise to a more adequated description, that have an equivalent minimum principle. This minimum principle will be employed for simulating some particular cases using a finite element approximation.

A comparison between the usual "uniform temperature approximation" and the results obtained from the finite element approximations will be carried out in order to show that, in some cases, the uniform temperature approximation is not a good choice.

2. MODELLING THE THERMAL RADIANT SOURCE

The thermal radiant source will be regarded as a spherical black body which dissipates heat at a constant rate Q and is not affected by the presence of the plate. This black body is represented by the set Γ, with isothermal boundary $\partial\Gamma$, in which it is assumed that $a \ll H$.

Since the source is a black body, the emitted thermal radiant energy is diffusely distributed. Hence, the amount of energy that, leaving $\partial\Gamma$, reaches a given subset $\partial\Omega^{*}$ ($\partial\Omega^{*} \subset \partial\Omega$) is given by (Sparrow and Cess 1978)

$$S = \int_{Y\in\partial\Gamma} \left[\int_{X\in\partial\Omega^{*}} \sigma\, T_s^4\, \hat{K}(X,Y)\, dA \right] dA \tag{3}$$

where the kernel $\hat{K}(X,Y)$ is given by

$$\hat{K}(X,Y) = \begin{cases} \dfrac{[(X-Y)\cdot m]\ [(Y-X)\cdot n]}{\pi\ [(X-Y)\cdot(X-Y)]^2} & \begin{cases}\text{if the points } X\in\partial\Omega \text{ and } Y\in\partial\Gamma \text{ can be} \\ \text{connected by a straight line that} \\ \text{does not intersect } \Gamma \text{ or } \Omega\end{cases} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

in which n is the unit outward normal at $X\in\partial\Omega$, m is the unit outward normal at $Y\in\partial\Gamma$ and T_s is the temperature on $\partial\Gamma$ (constant).

Since $\partial\Omega^{*}$ is an arbitrary subset, we may define the field s as follows

$$s = \hat{s}(X) = \int_{Y\in\partial\Gamma} \sigma\, T_s^4\, \hat{K}(X,Y)\, dA \qquad \text{for all } X\in\partial\Omega \tag{5}$$

The field $s=\hat{s}(X)$ represents the incident thermal radiant energy, per unit time and unit area, on $\partial\Omega$. In order to obtain an explicit relation between s and the position on $\partial\Omega$ we shall carry out the above integral (under the assumption $a \ll H$).

A generic point Y on $\partial\Gamma$ may be represented as follows

$$Y = Hk + am \tag{6}$$

in which m is the unit outward normal at $Y\in\partial\Gamma$.

Defining the subset $\partial\Omega_U \subset \partial\Omega$ as

$$\partial\Omega_U \equiv \left\{ (x,y,z)\in\partial\Omega \text{ such that } z=0 \right\} \tag{7}$$

we have, from (4), that

$$\hat{K}(X,Y) \equiv 0 \quad \text{for} \quad Y\in\partial\Gamma,\ X\in\partial\Omega-\partial\Omega_U \tag{8}$$

and, consequently,

$$s = \hat{s}(X) \equiv 0 \quad \text{for any } X\in\partial\Omega-\partial\Omega_U \tag{9}$$

A generic point X on $\partial\Omega_U$ may be represented as

$$X = xi + yj \tag{10}$$

while the unit outward normal on $\partial\Omega_U$ is given by

$$n = k \tag{11}$$

Taking into account that $a \to 0$ (or $Y \to Hk$) we have, from (4), the kernel $\hat{K}(X,Y)$, for $X \in \partial\Omega_U$, given as follows

$$\hat{K}(X,Y) = \begin{cases} \dfrac{[(xi+yj-Hk)\cdot k][(Hk-xi-yj)\cdot m]}{\pi(H^2+x^2+y^2)^2} & \text{if} \quad (xi+yj-Hk)\cdot m > 0 \\ 0 & \text{if} \quad (xi+yj-Hk)\cdot m \le 0 \end{cases} \tag{12}$$

Hence, defining the angle λ as

$$\lambda = \text{arc cos} \left[\frac{m\cdot(xi+yj-Hk)}{(x^2+y^2+H^2)^{1/2}}\right], \quad 0 \le \lambda \le \pi \tag{13}$$

and representing dA as

$$dA = a^2 \sin\lambda \, d\lambda \, d\phi \tag{14}$$

we may express the field s, on $\partial\Omega_U$, as follows (considering (12))

$$s = \hat{s}(X) = \int_0^{2\pi} \int_0^{\pi/2} \sigma T_s^4 \left[\frac{H \cos\lambda}{\pi(H^2+r^2)^{1/2}}\right] a^2 \sin\lambda \, d\lambda \, d\phi \tag{15}$$

or, evaluating the above integral,

$$s = \hat{s}(X) = \begin{cases} \sigma\, T_s^4 \left(\dfrac{H\, a^2}{(H^2+r^2)^{3/2}}\right), & \text{for } 0 \le r \le R\ ,\ z=0 \\ 0\ , & \text{otherwise} \end{cases} \tag{16}$$

in which the radial variable r is defined by

$$r = (x^2+y^2)^{1/2} \tag{17}$$

Denoting by Q the heat generation (per unit time) and taking into account that the source is a black body we have (from Stefan--Boltzmann Law)

$$Q = \sigma\, T_s^4\, 4\, \pi\, a^2 \tag{18}$$

and, therefore, equation (16) may be written as follows

$$s = \hat{s}(X) = \begin{cases} \dfrac{Q}{4\pi} \left(\dfrac{H}{(H^2+r^2)^{3/2}}\right), & \text{for } 0 \le r \le R\ ,\ z=0 \\ 0\ , & \text{otherwise} \end{cases} \tag{19}$$

3. MODELLING THE HEAT TRANSFER IN THE PLATE

Since the plate is assumed to be rigid and opaque, the energy transfer mechanism, inside Ω, is the conduction heat transfer. Hence, the steady-state energy transfer in Ω will be governed by (Holman 1976)

$$\Delta\, T = 0 \quad \text{in } \Omega \tag{20}$$

in which "Δ" denotes the "Laplacian" and the field T represents the absolute temperature (Callen 1960).

Taking into account the axial symmetry, equation (20) may be expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}=0 \quad \text{for} \quad 0 \le r < R \ , \quad -h < z < 0 \qquad (21)$$

Since the surface $x^2+y^2=R^2$ is insulated, we have

$$\frac{\partial T}{\partial r}=0 \qquad \text{at} \quad r=R,\ -h < z < 0 \qquad (22)$$

The boundary condition on the surface z=-h arises when the conduction heat flux is imposed to be equal to the radiative heat flux. This boundary condition is mathematically represented by

$$k\,\frac{\partial T}{\partial z}=\varepsilon\,\sigma\,T^4 \qquad \text{at} \quad z=-h,\ 0 \le r < R \qquad (23)$$

in which k is the thermal conductivity, ε $(0<\varepsilon\le1)$ is the emittance and σ is the Stefan-Boltzmann constant (Sparrow and Cess 1978).

On the surface z=0 we must take into account the effect of the source modelled in the previous section. The heat loss, per unit time and unit area, will be given by the difference between the emitted and the absorbed thermal radiant energy (denoted here by $\bar{s}$).

Since the plate is assumed to be gray we have

$$\bar{s}=\varepsilon\,s=\frac{\varepsilon Q}{4\pi}\left(\frac{H}{(H^2+r^2)^{3/2}}\right)>0 \quad \text{at} \quad z=0,\ 0 \le r < R \qquad (24)$$

and, consequently, at z=0

$$-k\,\frac{\partial T}{\partial z} = \varepsilon\,\sigma\,T^4 - \frac{\varepsilon Q}{4\pi}\left[\frac{H}{(H^2+r^2)^{3/2}}\right] \quad \text{at} \quad z=0 \tag{25}$$

Combining (21), (22), (23) and (25) we have the mathematical model for describing the considered phenomenon.

4. AN INDISPENSABLE CAUTION

When Q=0 it is easy to see that the problem (21)+(22)+(23)+(25) admits only the trivial solution (T≡0). It is physically expected that, when Q>0, the solution T will be a nonnegative valued field.

Although physically expected, this reality is ensured only if, together with (21)+(22)+(23)+(25), the following inequality is imposed

$$T \geq 0 \quad \text{for} \quad -h \leq z \leq 0 \quad , \quad 0 \leq r \leq R \tag{26}$$

Aiming to illustrate the necessity of imposing (26), let us consider the following (limit) case

$$Q = 68\,\pi\,H^2\,\sigma \quad , \quad H\to\infty \tag{27}$$

In such situation, when ε=1 and σ=k we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = 0 \quad \text{for} \quad -1 < z < 0\ ,\ 0 \le r < R$$

$$\frac{\partial T}{\partial r} = 0 \quad \text{at} \quad r=R$$

$$\sigma\,\frac{\partial T}{\partial z} = \sigma\,T^4 \quad \text{at} \quad z=-1 \tag{28}$$

$$-\sigma\,\frac{\partial T}{\partial z} = \sigma\,T^4 - 17\,\sigma \quad \text{at} \quad z=0$$

When regarded under a strictly mathematical point of view, the above problem admits the following (real) solutions

(a) → $T = z + 2$ for $-1 \le z \le 0$, $0 \le r \le R$ (29)

(b) → $T = 3.2697\,z + 1.6250$ for $-1 \le z \le 0$, $0 \le r \le R$ (30)

The solution (a) is admissible, because $T \ge 0$ in the considered domain, while the solution (b) is not admissible, because T assumes negative values in the considered domain.

Solving (28), together with the restriction (26), we have only the admissible solution given by (29).

In order to preserve the physical meaning of the mathematical model, we shall work with the following mathematical problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = 0 \quad \text{for} \quad 0 \le r < R\ ,\ -h < z < 0$$

$$\frac{\partial T}{\partial r} = 0 \quad \text{at} \quad r=R$$

$$k\,\frac{\partial T}{\partial z} = \varepsilon\,\sigma\,T^4 \quad \text{at} \quad z=-h \tag{31}$$

$$-k\,\frac{\partial T}{\partial z} = \varepsilon\,\sigma\,T^4 - \frac{\varepsilon Q}{4\pi}\left[\frac{H}{(H^2+r^2)^{3/2}}\right] \quad \text{at} \quad z=0$$

$$T \ge 0 \quad \text{for} \quad 0 \le r \le R\ ,\ -h \le z \le 0$$

5. SOLUTION'S UNIQUENESS FOR (31)

In order to prove that the solution ($T \geq 0$) to (31) is unique, let us assume that T_1 ($T_1 \geq 0$) and T_2 ($T_2 \geq 0$) are two solutions to (31).

Hence, the difference ($T_1 - T_2$) satisfies the Laplace equation and, consequently, will assume its maximum and its minimum on the boundary $\partial\Omega$ (John 1982). In addition, since $\partial T/\partial n = 0$ at $r=R$, the maximum and the minimum occur at $z=-h$ or at $z=0$. At a point of $\partial\Omega$, in which ($T_1 - T_2$) assumes its minimum, we must have a nonpositive exterior normal derivative. Thus, at this point

$$\frac{\partial}{\partial n}(T_1 - T_2) \leq 0 \quad \Leftrightarrow \quad \varepsilon\,\sigma\,(T_1^4 - T_2^4) \geq 0 \tag{32}$$

and, at a point of $\partial\Omega$ in which ($T_1 - T_2$) assumes its maximum, we must have a nonnegative exterior normal derivative. Thus, at this point

$$\frac{\partial}{\partial n}(T_1 - T_2) \geq 0 \quad \Leftrightarrow \quad \varepsilon\,\sigma\,(T_1^4 - T_2^4) \leq 0 \tag{33}$$

Since T_1 and T_2 are nonnegative, we conclude, from (32), that the minimum of ($T_1 - T_2$) is nonnegative and, from (33), that the maximum of ($T_1 - T_2$) is nonpositive. Consequently, we must have $T_1 \equiv T_2$ and, therefore, the solution to (31) is unique.

6. VARIATIONAL FORMULATION

The field T which satisfies (31) is the one that minimizes the following functional

$$I(\tilde{T}) = \frac{1}{2}\int_0^R \int_{-h}^0 k \left(\left(\frac{\partial \tilde{T}}{\partial r}\right)^2 + \left(\frac{\partial \tilde{T}}{\partial z}\right)^2 \right) r \, dz \, dr + \frac{1}{5} \varepsilon\sigma \left[\int_0^R |\tilde{T}|^5 r \, dr \right]_{z=0} +$$

$$+ \frac{1}{5} \varepsilon\sigma \left[\int_0^R |\tilde{T}|^5 r \, dr \right]_{z=-h} - \frac{\varepsilon Q}{4\pi} \left[\int_0^R \frac{H}{(H^2+r^2)^{3/2}} \tilde{T} \, r \, dr \right]_{z=0} \qquad (34)$$

The Euler-Lagrange equation and the Natural boundary conditions associated to the above functional are given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = 0 \quad \text{for} \quad 0 \le r < R \;, \; -h < z < 0$$

$$\frac{\partial T}{\partial r} = 0 \quad \text{at} \quad r=R \qquad (35)$$

$$k \frac{\partial T}{\partial z} = \varepsilon \sigma |T|^3 T \quad \text{at} \quad z=-h$$

$$-k \frac{\partial T}{\partial z} = \varepsilon \sigma |T|^3 T - \frac{\varepsilon Q}{4\pi} \left[\frac{H}{(H^2+r^2)^{3/2}}\right] \quad \text{at} \quad z=0$$

In order to demonstrate that (35) and (31) are equivalent, let us take into account that T reaches its minimum at z=0 or at z=-h. Hence, one of the following assertions must hold

$$\text{(a)} \; \frac{\partial T}{\partial z} \le 0 \;\; \text{at a point } r \text{ in which } T=T_{min}, \text{ at } z=0$$

$$\text{(b)} \; \frac{\partial T}{\partial z} \ge 0 \;\; \text{at a point } r \text{ in which } T=T_{min}, \text{ at } z=-h \qquad (36)$$

If (b) holds we conclude, from the boundary condition at z=-h, that $T \ge 0$ at z=-h and, therefore, $T \ge 0$ in $\bar{\Omega}$. On the other hand, if (a) holds, we conclude, from the boundary condition at z=0, that $T > 0$ at z=0 (once that $Q > 0$) and, therefore, $T \ge 0$ in $\bar{\Omega}$. Hence, the field T is everywhere nonnegative and, thus,

$$|T|^3 T \equiv T^4 \quad \text{at} \quad z=0 \text{ and at } z=-h \qquad (37)$$

Consequently, a solution to (35) is also a solution to (31). Taking into account that (31) admits only one solution, we conclude that the field T, which makes extremum the functional I, is the solution to (31).

Since I is a convex functional, the above mentioned extremum is a minimum (Brezis 1983).

7. THE UNIFORM TEMPERATURE APPROXIMATION

The uniform temperature approximation is an usual assumption in engineering calculations and will be used here for posterior comparisons with other approximations.

The uniform temperature approximation may be obtained from the functional I. Aiming to this, we shall assume that

$$\tilde{T} = \bar{A} = \text{constant} \quad \text{for} \quad 0 \le r \le R \ , \ -h \le z \le 0 \qquad (38)$$

Considering (38) the functional I becomes

$$I[\tilde{T}] = f(\bar{A}) = \frac{1}{5}\varepsilon\sigma(2|\bar{A}|^5) \int_0^R r\ dr - \frac{\varepsilon Q}{4\pi} H\bar{A} \int_0^R \frac{1}{(H^2+r^2)^{3/2}}\, r\ dr \qquad (39)$$

and, hence, the constant $\bar{A}$ is the (unique) solution to

$$\frac{df}{d\bar{A}} = 0 \iff \varepsilon\sigma|\bar{A}|^3\bar{A}\, R^2 + \frac{\varepsilon Q}{4\pi}\left[\frac{H}{(H^2+R^2)^{1/2}} - 1\right] = 0 \qquad (40)$$

being given as follows

$$\bar{A} = \left(\frac{Q}{4\pi\sigma R^2}\right)^{1/4}\left[1 - \frac{1}{(1+(R/H)^2)^{1/2}}\right]^{1/4} \quad . \tag{41}$$

B. DIMENSIONLESS FORMULATION

Defining α, β, γ, ξ, η and θ as follows

$$\alpha = \frac{k}{\varepsilon\sigma h}\left(\frac{Q}{4\pi\sigma R^2}\right)^{-3/4} , \quad \beta = \frac{R}{H} , \quad \gamma = \frac{R}{h} \tag{42}$$

$$\xi = \frac{r}{h} , \quad \eta = \frac{z}{h} , \quad \theta = T\left(\frac{Q}{4\pi\sigma R^2}\right)^{-1/4} \tag{43}$$

we may write problem (35) as follows

$$\begin{gathered} \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) + \frac{\partial^2\theta}{\partial\eta^2} = 0 \quad \text{for} \quad 0 \le \xi < \gamma \ , \ -1 < \eta < 0 \\ \frac{\partial\theta}{\partial\xi} = 0 \quad \text{at} \quad \xi=\gamma \\ \alpha\frac{\partial\theta}{\partial\eta} = |\theta|^3\theta \quad \text{at} \quad \eta=-1 \\ -\alpha\frac{\partial\theta}{\partial\eta} = |\theta|^3\theta - \left[\frac{\beta^2}{(1+\xi^2(\beta/\gamma)^2)^{3/2}}\right] \quad \text{at} \quad \eta=0 \end{gathered} \tag{44}$$

and express the functional I as

$$\begin{aligned} I(\tilde{\theta}) = \frac{\alpha}{2}\int_{-1}^{0}\int_{0}^{\gamma}\left(\left(\frac{\partial\tilde{\theta}}{\partial\xi}\right)^2+\left(\frac{\partial\tilde{\theta}}{\partial\eta}\right)^2\right)\xi\,d\xi\,d\eta + \frac{1}{5}\left[\int_0^{\gamma}|\tilde{\theta}|^5\,\xi\,d\xi\right]_{\eta=0} + \\ + \frac{1}{5}\left[\int_0^{\gamma}|\tilde{\theta}|^5\,\xi\,d\xi\right]_{\eta=-1} - \left[\int_0^{\gamma}\frac{\beta^2}{(1+\xi^2(\beta/\gamma)^2)^{3/2}}\,\tilde{\theta}\,\xi\,d\xi\right]_{\eta=0} \end{aligned} \tag{45}$$

9. SIMULATION OF SOME PARTICULAR CASES (USING FINITE ELEMENTS)

In order to present some typical results, we shall approximate the field θ by the following piecewise linear field (see figures 3 and 4)

$$\tilde{\theta} = (\theta_{2j+1} - \theta_{2j-1}) \left(\frac{\xi - \xi_{2j-1}}{\Delta\xi_j}\right) - (\theta_{2j+2} - \theta_{2j+1})\,\eta + \theta_{2j-1}$$

$$\text{for} \quad \xi_{2j-1} \le \xi \le \xi_{2j+1}\,, \quad \left(\frac{\xi - \xi_{2j-1}}{\Delta\xi_j}\right) \le \eta \le 0$$

$$\tilde{\theta} = (\theta_{2j+2} - \theta_{2j}) \left(\frac{\xi - \xi_{2j-1}}{\Delta\xi_j}\right) - (\theta_{2j} - \theta_{2j-1})\,\eta + \theta_{2j-1}$$

$$\text{for} \quad \xi_{2j-1} \le \xi \le \xi_{2j+1}\,, \quad -1 \le \eta \le \left(\frac{\xi - \xi_{2j-1}}{\Delta\xi_j}\right)$$

$$j=1,2,3,4,5,6,\ldots,(N-2)/2 \qquad (46)$$

in which N is the number of nodes, θ_i is the approximation (for θ) at the node i and

$$\Delta\xi_j = \xi_{2j+1} - \xi_{2j-1} = \xi_{2j+2} - \xi_{2j} \qquad (47)$$

$$\xi_1 = \xi_2 = 0\,,\ \xi_{N-1} = \xi_N = \gamma\,,\ \xi_{2j-1} = \xi_{2j}\,,\ \xi_{2j+1} = \xi_{2j+2} = \xi_{2j-1} + \Delta\xi_j \qquad (48)$$

Inserting (46) into (45) and carrying out the integrations, the functional I becomes the following function

$$f(\theta_1, \theta_2, \theta_3, \ldots, \theta_N) = \sum_{j=1}^{M} \left(A_j + B_j + C_j + D_j + E_j + F_j \right) \qquad (49)$$

in which M=(N-2)/2 and

$$A_j = \frac{\alpha}{12}\left[\left(\frac{\theta_{2j+1}-\theta_{2j-1}}{\Delta\xi_j}\right)^2+(\theta_{2j+2}-\theta_{2j+1})^2\right](2\xi_{2j+1}^2-\xi_{2j+1}\xi_{2j-1}-\xi_{2j-1}^2) \quad (50)$$

$$B_j = \frac{\alpha}{12}\left[\left(\frac{\theta_{2j+2}-\theta_{2j}}{\Delta\xi_j}\right)^2+(\theta_{2j}-\theta_{2j-1})^2\right](\xi_{2j+1}^2+\xi_{2j+1}\xi_{2j-1}-2\xi_{2j-1}^2) \quad (51)$$

$$C_j = \gamma^2\left[\theta_{2j+1}(1+(\beta/\gamma)^2\xi_{2j+1}^2)^{-1/2}-\theta_{2j-1}(1+(\beta/\gamma)^2\xi_{2j-1}^2)^{-1/2}\right] \quad (52)$$

$$D_j = \frac{\gamma^3}{\beta}\left(\frac{\theta_{2j+1}-\theta_{2j-1}}{\Delta\xi_j}\right) \mathrm{Ln}\left[\frac{\xi_{2j-1}+((\gamma/\beta)^2+\xi_{2j-1}^2)^{1/2}}{\xi_{2j+1}+((\gamma/\beta)^2+\xi_{2j+1}^2)^{1/2}}\right] \quad (53)$$

$$E_j = \frac{1}{5}\left(\frac{\Delta\xi_j}{\theta_{2j+1}-\theta_{2j-1}}\right)^2\left[\left(\frac{|\theta_{2j+1}|^7-|\theta_{2j-1}|^7}{7}\right)-\right.$$
$$\left.-\left(\frac{|\theta_{2j+1}|^5\theta_{2j+1}-|\theta_{2j-1}|^5\theta_{2j-1}}{6}\right)\left(\frac{\theta_{2j-1}\xi_{2j+1}-\theta_{2j+1}\xi_{2j-1}}{\Delta\xi_j}\right)\right] \quad (54)$$

$$F_j = \frac{1}{5}\left(\frac{\Delta\xi_j}{\theta_{2j+2}-\theta_{2j}}\right)^2\left[\left(\frac{|\theta_{2j+2}|^7-|\theta_{2j}|^7}{7}\right)-\right.$$
$$\left.-\left(\frac{|\theta_{2j+2}|^5\theta_{2j+2}-|\theta_{2j}|^5\theta_{2j}}{6}\right)\left(\frac{\theta_{2j}\xi_{2j+1}-\theta_{2j+2}\xi_{2j-1}}{\Delta\xi_j}\right)\right] \quad (55)$$

The minimum of the function f is reached when $\theta_1, \theta_2, \ldots, \theta_N$ are the roots of the following nonlinear system of equations

$$\frac{\partial f}{\partial \theta_i} = 0 \,, \qquad i=1,2,3,4,5,\ldots,N \quad (56)$$

Some selected results are presented in figures 5, 6, and 7 (each one consisting of three parts).

Figure 5 (α=1.0, β=1.0 and γ=100.0 → part A // α=1.0, β=10.0 and γ=100.0 → part B // α=1.0, β=100.0 and γ=100.0 → part C) present results obtained with M=17 and $\Delta\xi_j$ given as follows

$$\Delta\xi_j = \begin{cases} \gamma/75.0 & j=1,2 \text{ and } 3 \\ \gamma/25.0 & j=4,5,6 \text{ and } 7 \\ \gamma/12.5 & j=8,9,10,\ldots,16 \text{ and } 17 \end{cases} \qquad (57)$$

Figure 6 presents results, obtained with M=10 and $\Delta\xi_j=\gamma/10=$ =constant, for α=1.0, β=10.0 and γ=1.0 → part A // γ=10.0 → part B // γ=100.0 → part C.

Figure 7 presents results obtained (for α=0.1, β=10.0 and γ=100.0) with M=5 and $\Delta\xi_j=\gamma/5$=constant (part A), M=10 and $\Delta\xi_j=\gamma/10$=constant (part B) and with M=17 and $\Delta\xi_j$ given by (57) (part C).

In all these figures the upper curve represents θ vs ξ for η=0 while the lower one represents θ vs ξ for η=-1. The dashed line represents the constant temperature approximation (which depends only on β).

In all the situations, system (56) was solved with the aid of a Newton-Raphson scheme (Ciarlet 1982).

10. FINAL REMARKS

The considered phenomenon is motivated by the practical situations in which a small body, at very high temperature, is placed near another body.

The obtained results show that usual approximations, like constant temperature, may give rise to unrealistic results.

The presented approach is mathematically accurate and provides approximations only in the physically admissible range.

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✱ List of Symbols ✱

Roman Characters (and bold roman)

a - radius of the spherical source

H - distance between the source and the plate

h - plate thickness

I - functional

i, **j** and **k** - orthonormal basis

k - thermal conductivity

K - kernel, related to the angle factor

$\hat{K}(X,Y)$ - K regarded as a function of X and Y

m and **n** - unit outward normals

N - number of nodes

Q - heat generation per unit time

r - radial variable → $r = (x^2+y^2)^{1/2}$

R - plate radius

s - incident thermal radiant energy at $X \in \partial\Omega$

$\bar{s}$ - absorbed thermal radiant energy at $X \in \partial\Omega$

T - absolute temperature

T_s - temperature of the surface $\partial\Gamma$

$\tilde{T}$ - admissible temperature field

T_1 and T_2 - solutions to (30)

x, y and z - variables (associated to system (x,y,z))

x', y' and z' - variables (associated to system (x',y',z'))

X and Y - spatial points

Greek Characters

α - dimensionless thermal conductivity

β and γ - dimensionless geometric parameters

ζ and η - dimensionless variables

ζ_i - dimensionless radial position at the node i

θ - dimensionless temperature

$\tilde{\theta}$ - admissible dimensionless temperature

θ_i - approximation for θ at the node i

ε - emittance

ρ, ϕ and λ - variables of the spherical coordinate system

ψ - angle defined by equation (10)

π - 3.141592654

σ - Stefan-Bolzmann constant

$\Delta\xi$ - dimensionless radial increment

Ω - bounded domain, representing the plate

$\bar{\Omega}$ - closure of Ω

Γ - bounded domain, representing the spherical source

$\partial\Omega$ - boundary of Ω

$\partial\Omega^*$ - subset of $\partial\Omega$

$\partial\Gamma$ - boundary of Γ

$\partial T/\partial n$ - exterior normal derivative of T

FIGURES CAPTIONS

Figure 1 - Scheme of the problem.

Figure 2 - A scheme for evaluating $s=\hat{s}(X)$.

Figure 3 - A typical finite element approximation, obtained with $\Delta\xi$=constant.

Figure 4 - The cell j.

Figure 5 - Dimensionless temperature θ versus dimensionless radial position ξ for $\eta=0$ (upper curve) and for $\eta=-1$ (lower curve), obtained with M=17 and $\Delta\xi_j$ given by (57), for $\alpha=1.0$, $\beta=1.0$ (part A), $\beta=10.0$ (part B), $\beta=100.0$ (part C) and $\gamma=100.0$. The dashed line represents the constant temperature approximation.

Figure 6 - Dimensionless temperature θ versus dimensionless radial position ξ for $\eta=0$ (upper curve) and for $\eta=-1$ (lower curve), obtained with M=10 and $\Delta\xi_j$=constant, for $\alpha=1.0$, $\beta=10.0$, $\gamma=1.0$ (part A), $\gamma=10.0$ (part B) and $\gamma=100.0$ (part C). The dashed line represents the constant temperature approximation.

Figure 7 - Dimensionless temperature θ versus dimensionless radial position ξ for $\eta=0$ (upper curve) and for $\eta=-1$ (lower curve), obtained with M=5 and $\Delta\xi_j$=constant (part A), M=10 and $\Delta\xi_j$=constant (part B) and with M=17 and $\Delta\xi_j$ given by (57) (part C), for $\alpha=0.1$, $\beta=10.0$ and $\gamma=100.0$. The dashed line represents the constant temperature approximation.

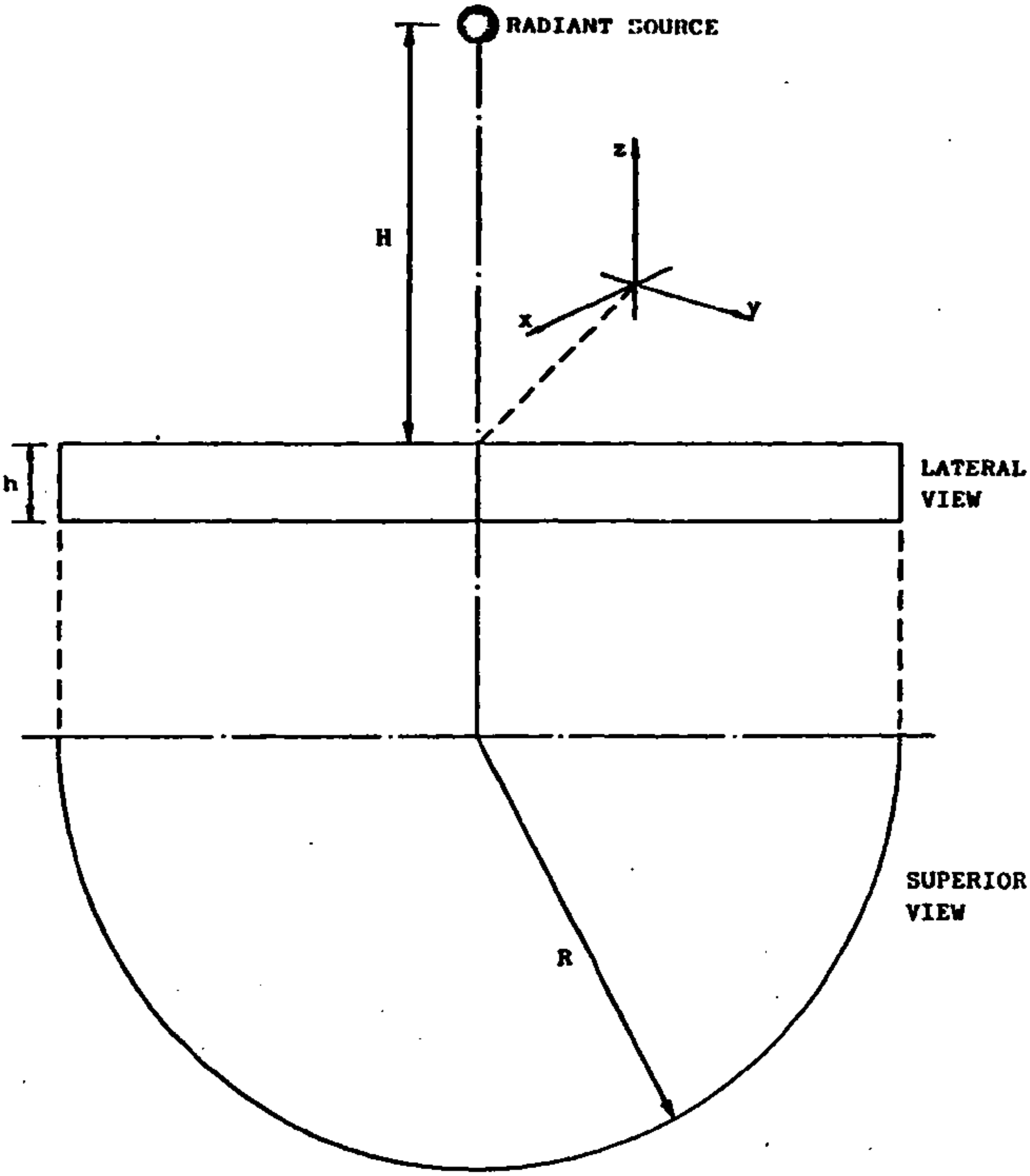
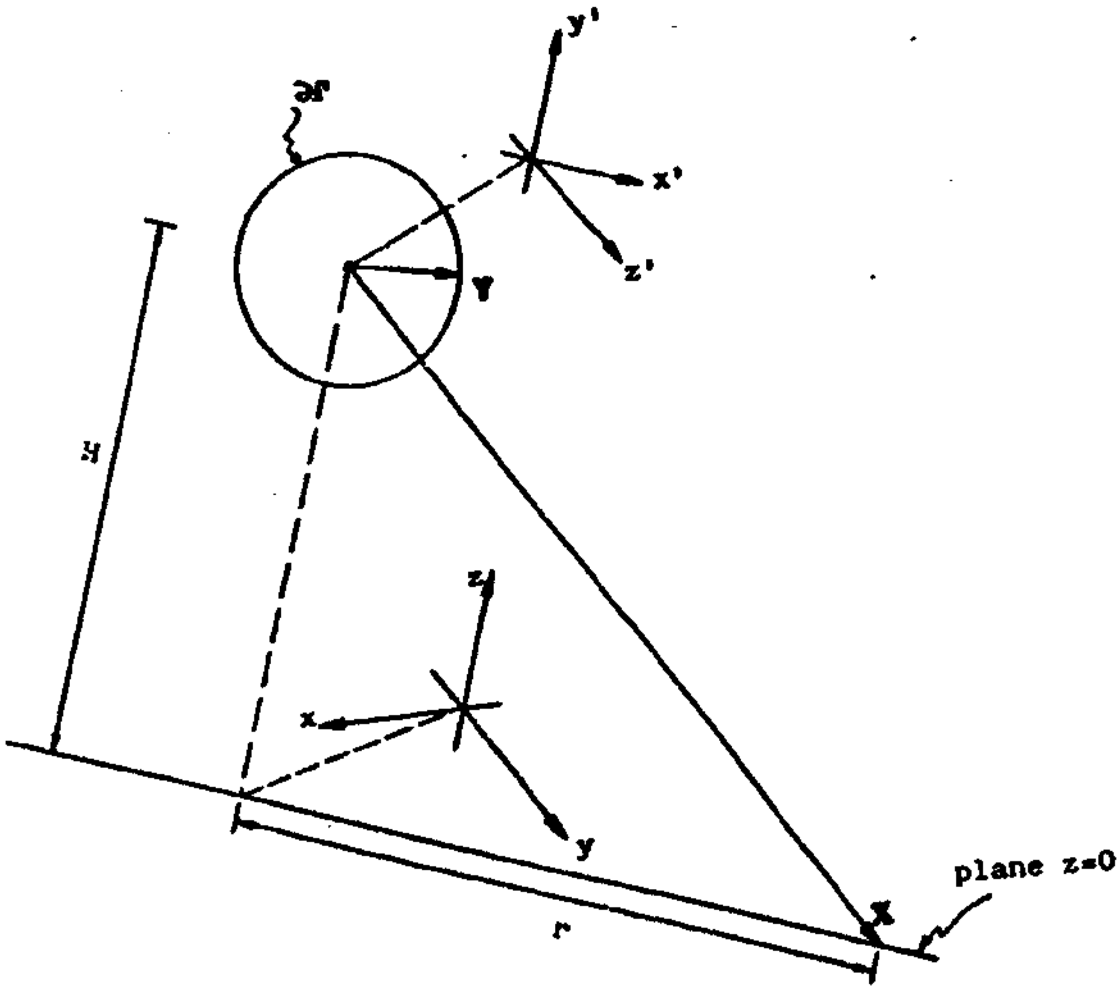


FIGURE 1



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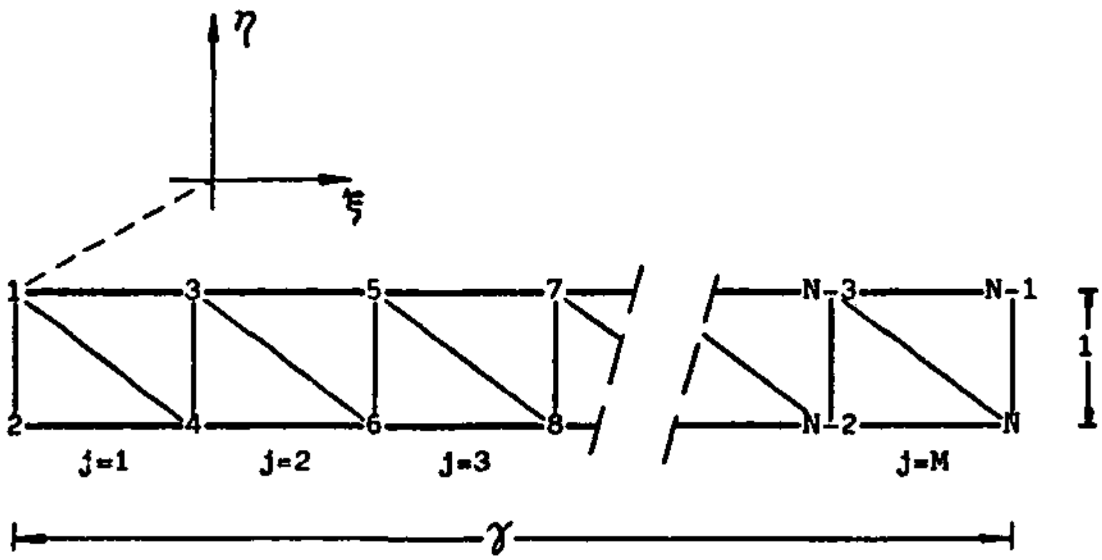

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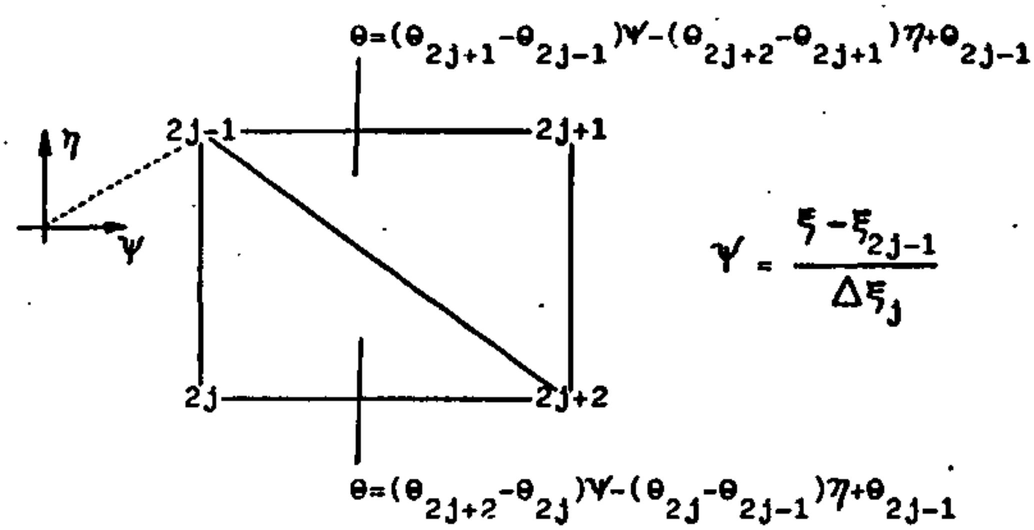


θ=(θ2j+1-θ2j-1)ψ-(θ2j+2-θ2j+1)η+θ2j-1
η
ψ
2j-1
2j+1
2j
2j+2
ψ = (ξ-ξ2j-1)/Δξj
θ=(θ2j+2-θ2j)ψ-(θ2j-θ2j-1)η+θ2j-1

FIGURE 4

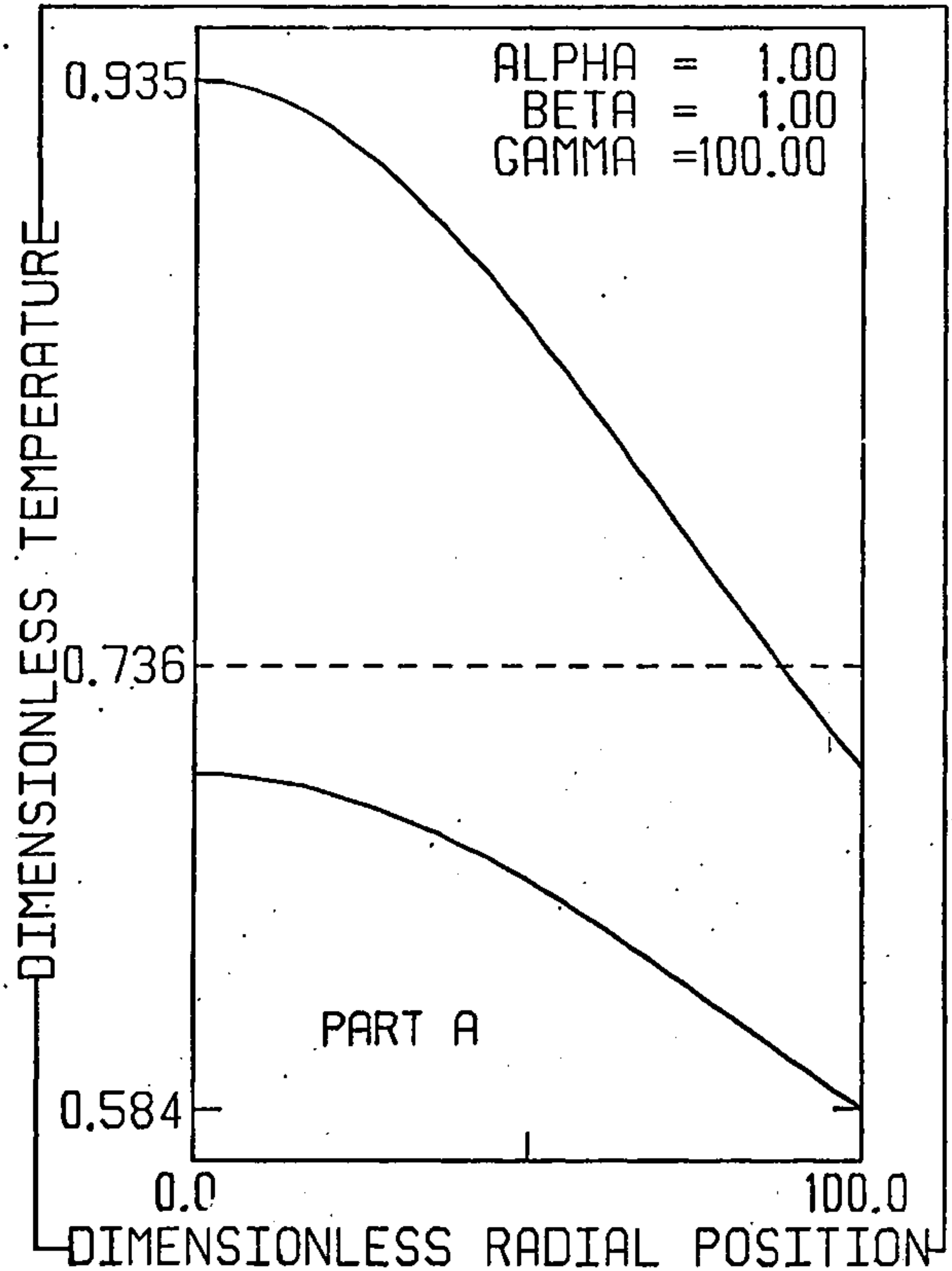


FIGURE 5 PART A

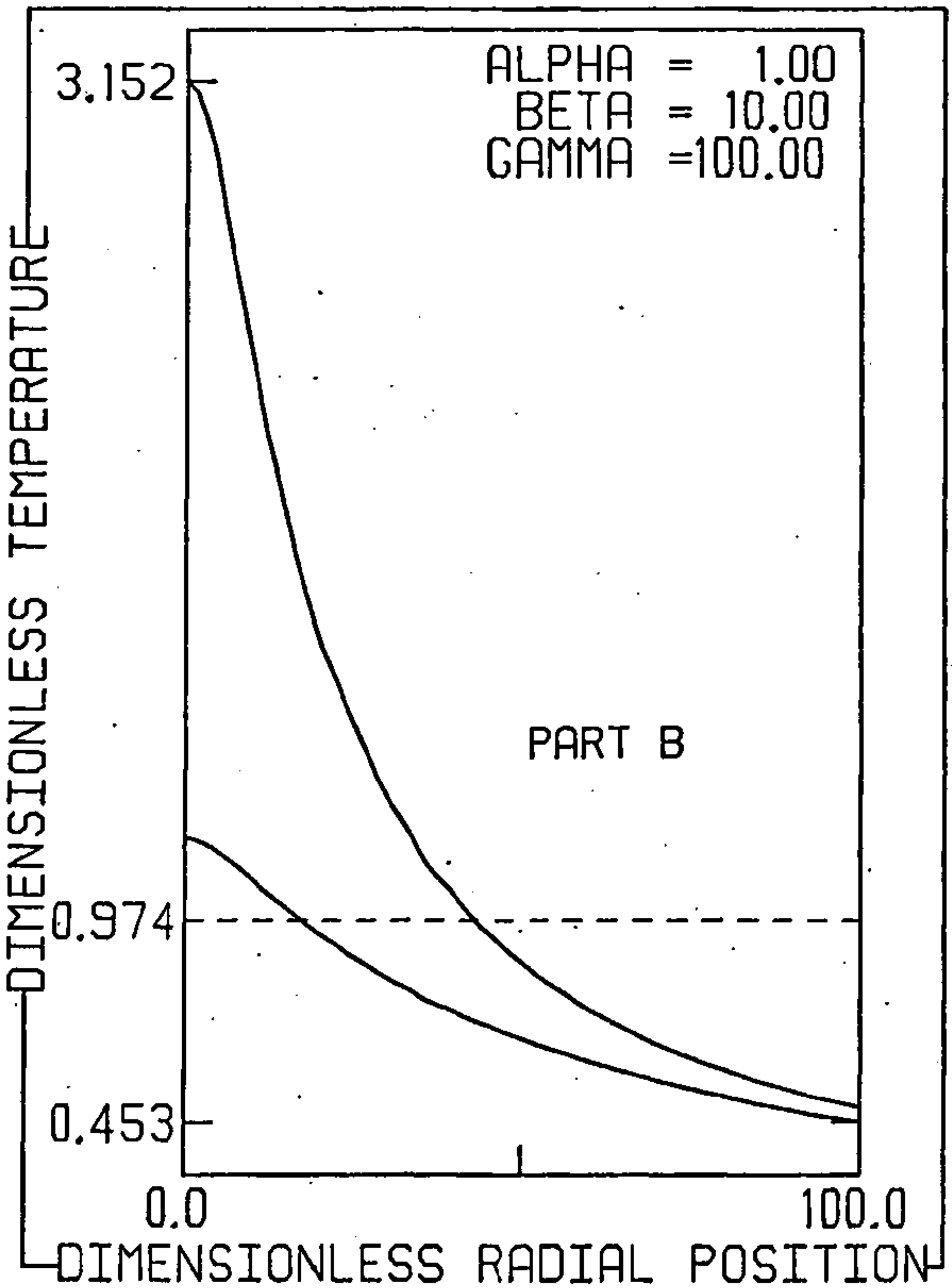


FIGURE 5 PART B

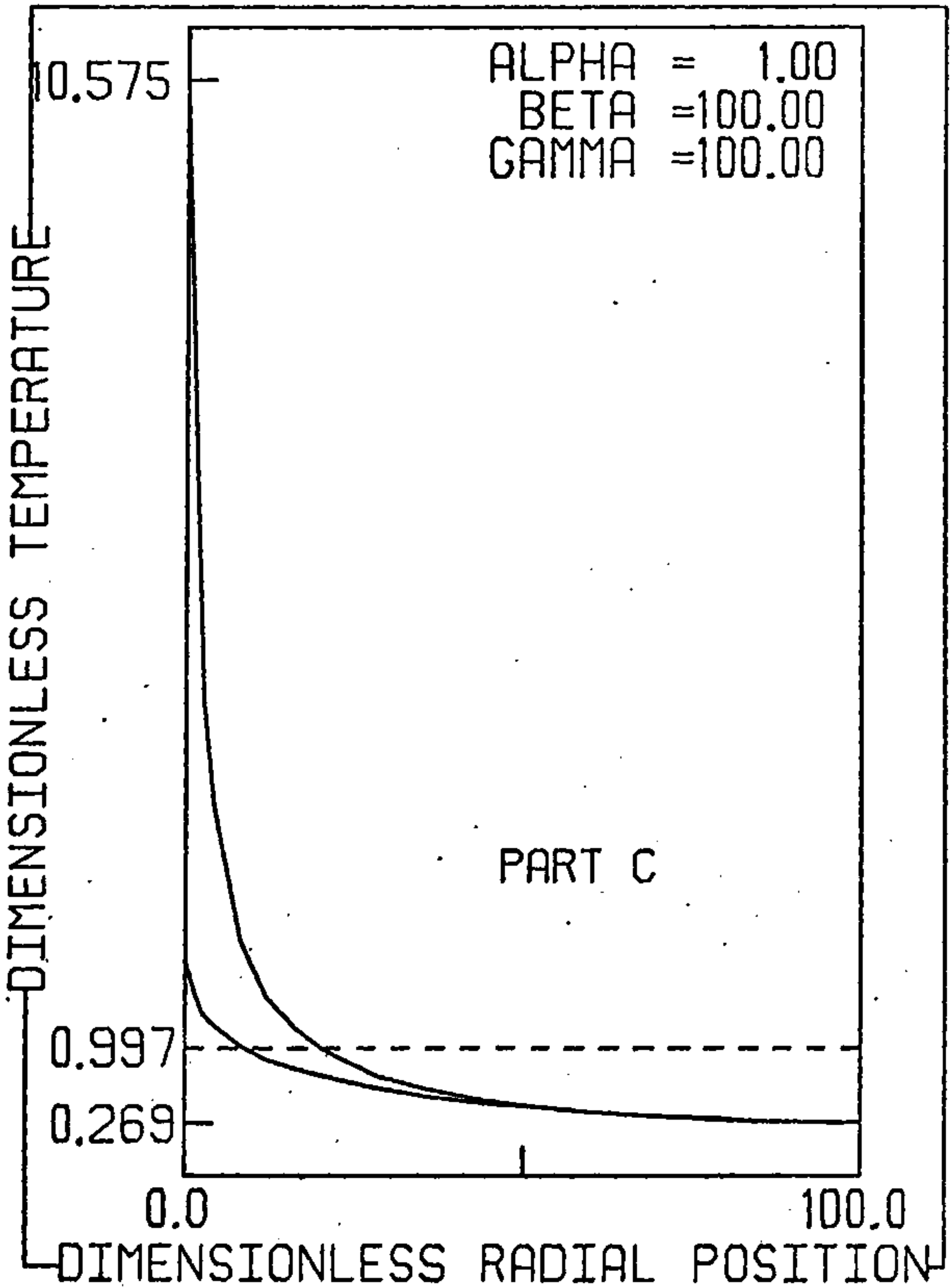
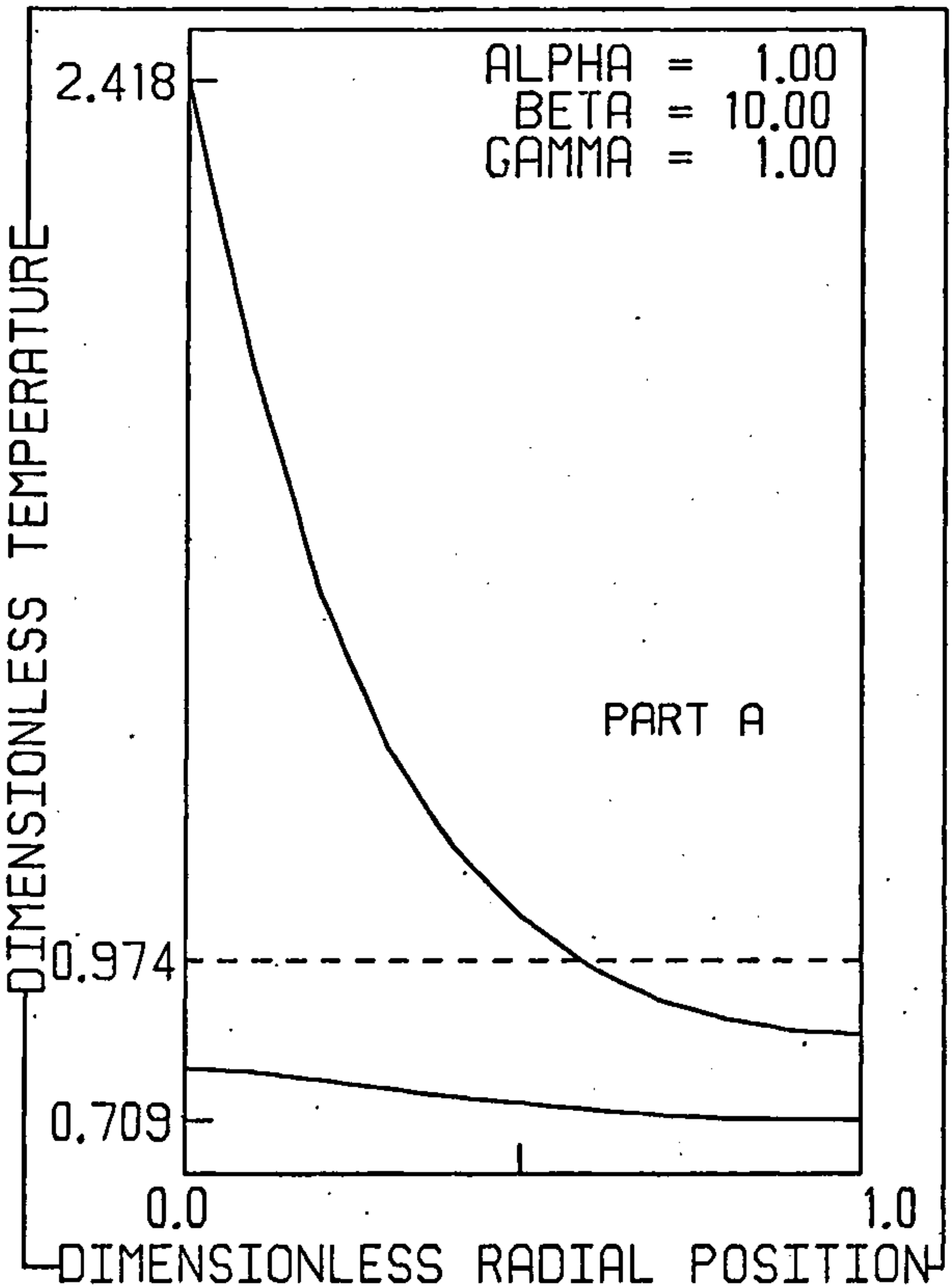


FIGURE 5 PART C



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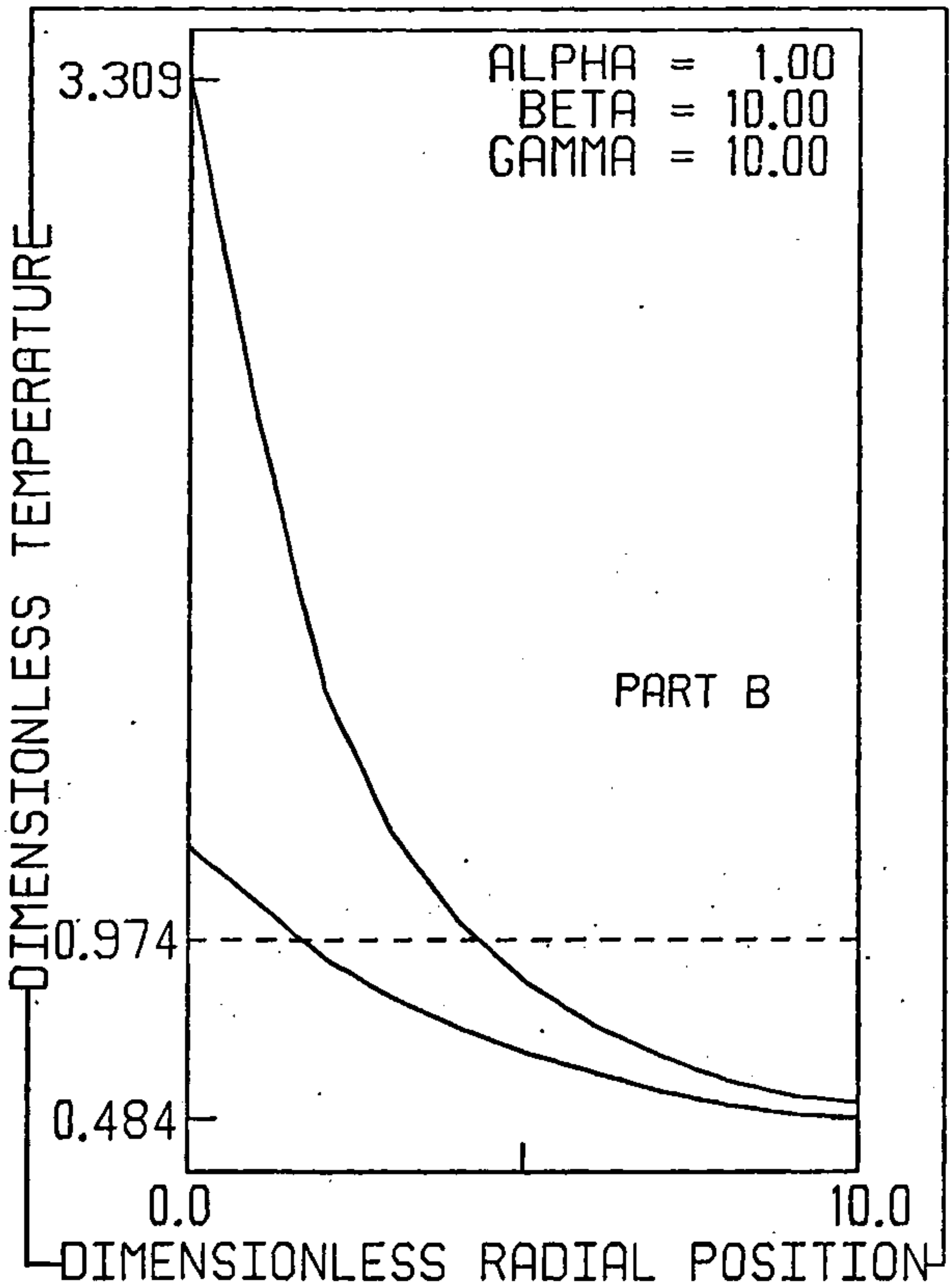

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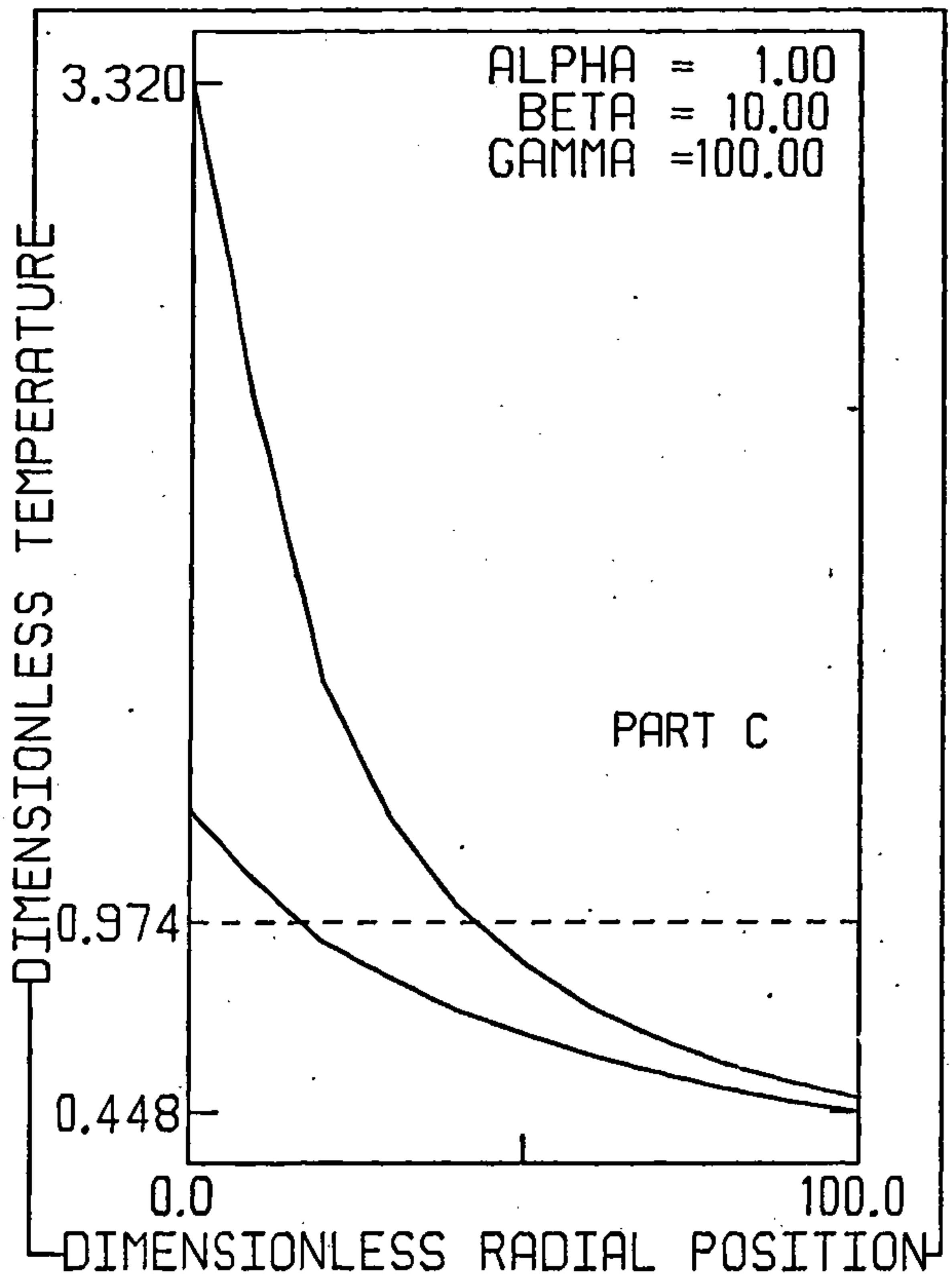

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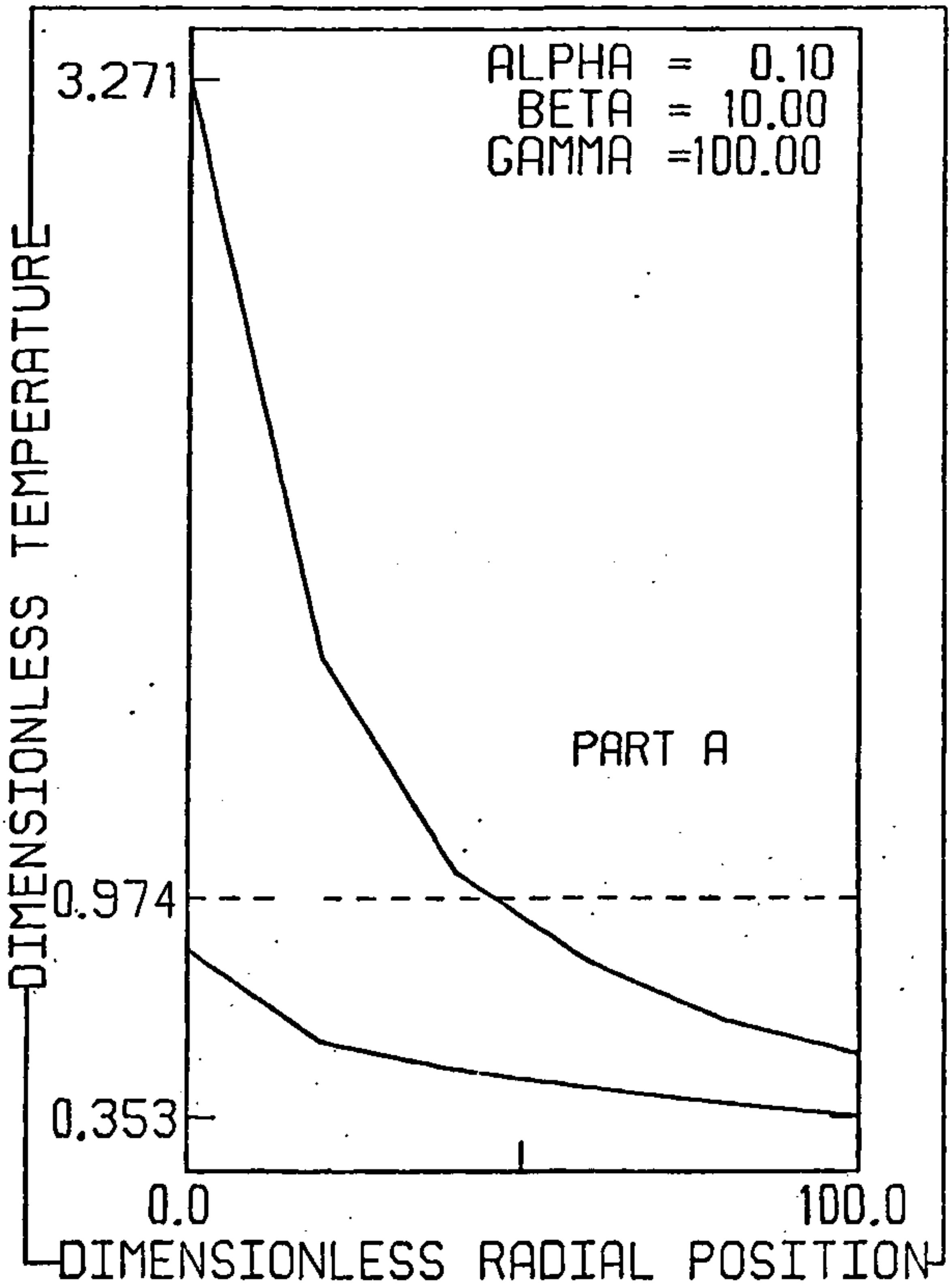

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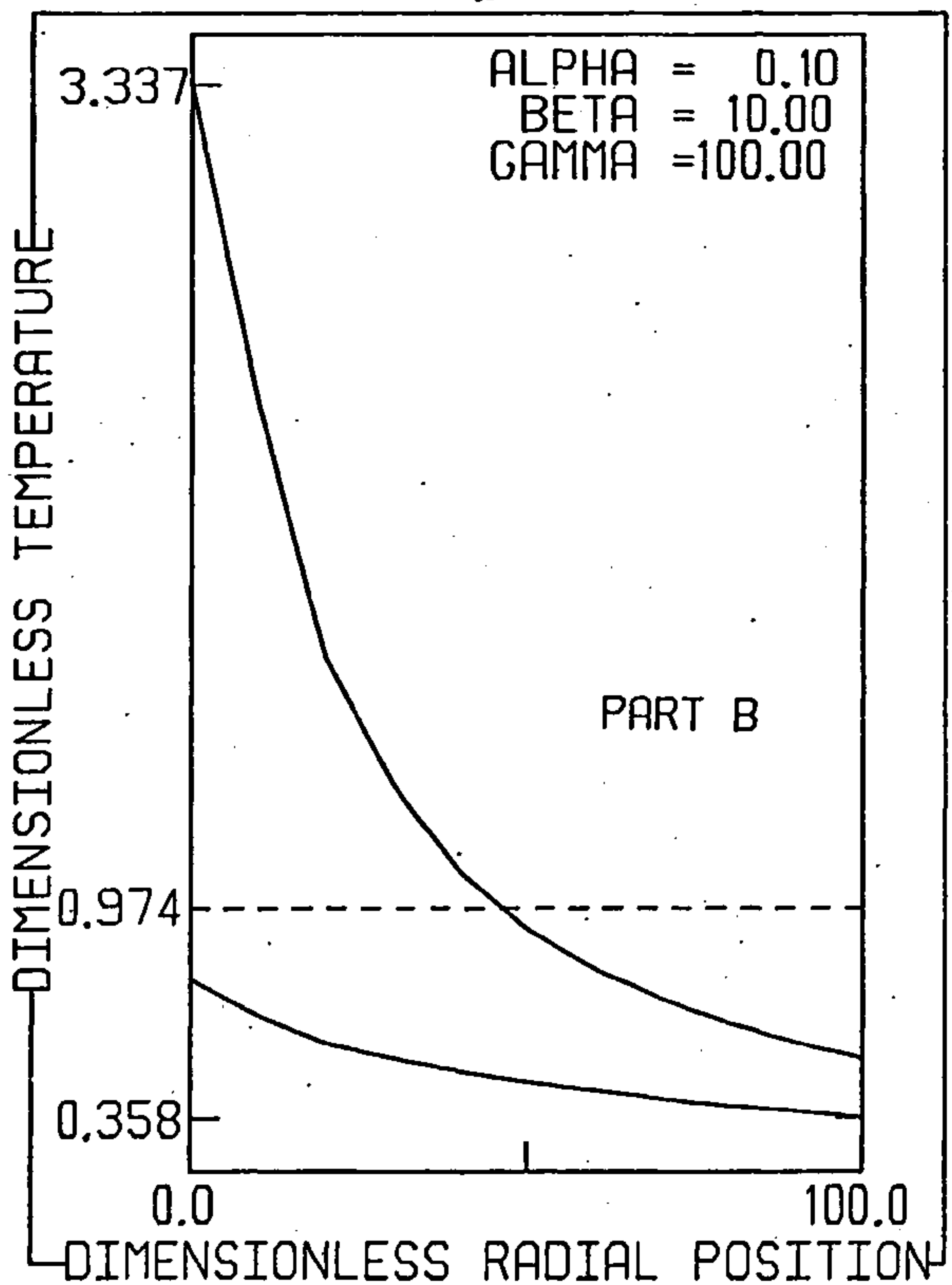

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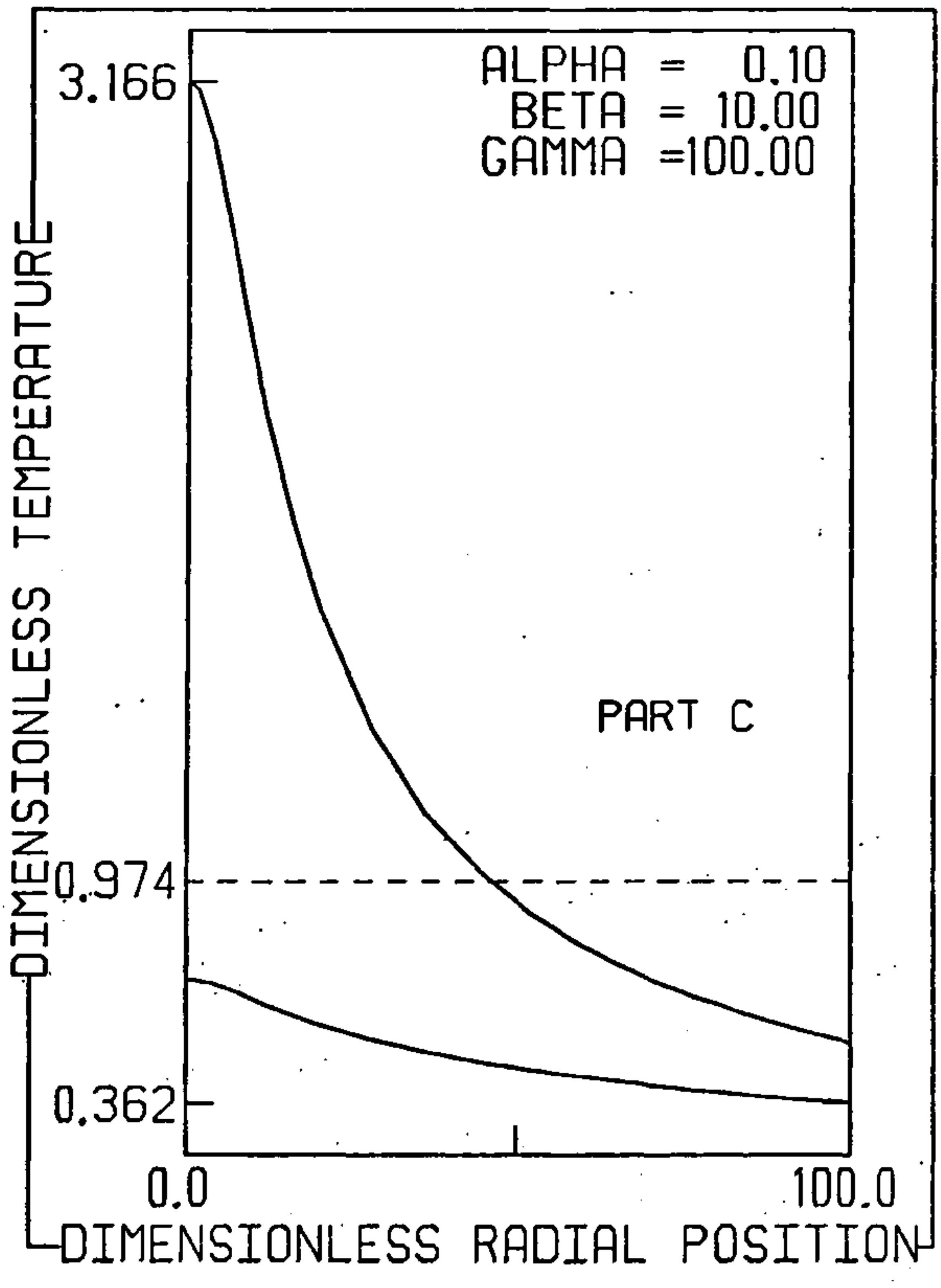

FIGURE 7 PART C

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