IC/92/385

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

CONVERGENCE RATES AND FINITE-DIMENSIONAL APPROXIMATIONS FOR NONLINEAR ILL-POSED PROBLEMS INVOLVING MONOTONE OPERATORS IN BANACH SPACES

Nguyen Buong

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

MIRAMARE-TRIESTE

IC/92/385

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

CONVERGENCE RATES AND FINITE-DIMENSIONAL APPROXIMATIONS FOR NONLINEAR ILL-POSED PROBLEMS INVOLVING MONOTONE OPERATORS IN BANACH SPACES

Nguyen Buong *
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The purpose of this paper is to investigate convergence rates for an operator version of Tikhonov regularization constructed by dual mapping for nonlinear ill-posed problems involving monotone operators in real reflexive Banach spaces. The obtained results are considered in combination with finite-dimensional approximations for the space. An example is considered for illustration.

MIRAMARE – TRIESTE
November 1992

* Permanent address: Institute of Computer Science, Lieugiai–Badinh, P.O. Box 634, BoHo, Hanoi 10000, Vietnam.

1. Introduction

Let X be a real reflexive Banach space having the property: X and X^* are strictly convex and weak convergence and convergence of norms of any sequence follow its strong convergence; X^* - its dual. For the sake of simplicity norms of X and X^* we denote by one symbol $\|.\|$. We write (x^*, x) instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let A be a monotone continuous and bounded operator with domain $D(A) = X$ and range $R(A) \subseteq X^*$ and let f_0 be a fixed element of $R(A)$.

Without additional conditions on the structure of A, as strongly or uniformly monotone property, the problem

$$A(x) = f_0 \tag{1.1}$$

is an ill-posed one. By this we mean that solutions of (1.1) do not depend continuously on the data (A, f_0). To solve it we have to use stable methods. An widely used and effective method is Tikhonov regularization that consists of minimizing a some functional depending on a parameter. For the class of problems involving monotone operators there esists another version of Tikhonov regularization that consists of solving the equation

$$A_h(x) + \alpha U^s(x) = f_\delta, \tag{1.2}$$

where $(A_h\ f_\delta)$ are approximations for $(A,\ f_0)$ such that A_h are monotone

$$\|A_h(x) - A(x)\| \le h\|x\|,\ \forall\ x \in X,$$

$$\|f_\delta - f_0\| \le \delta,$$

with well-known levels $(\delta,\ h) \to 0$. The parameter α is called parameter of regularization. Here U^s is a dual mapping of X satisfying the condition

$$\langle U^s(x),\ x\rangle = \|x\|^s,\ \|U^s(x)\| = \|x\|^{s-1},\ s \geq 2.$$

If $s = 2$, the algorithm (1.2) was considered in [2, 14]. In the case of Hilbert space $U^s(x) = I$, I denotes an identical operator, it was studied in [3]. If s is arbitrary number such algorithm was investigated in [15] under the conditions

$$\langle U^s(x) - U^s(y),\ x - y\rangle \geq m\|x - y\|^s,\ m > 0, \tag{1.3}$$

$$\|U^s(x) - U^s(y)\| \leq\ C(R)\|x - y\|^\vartheta,\ 0 < \vartheta \leq 1, \tag{1.4}$$

where $c(R)$, $R > 0$, is a positive increasing function on $R = max\{\|x\|,\ \|y\|\}$.

It is indicated in [2, 15] that if A_h are monotone and hemi-continuous Equation (1.2) has a unique solution, henceforth denoted $x_\alpha^{h\delta}$, and if $h/\alpha,\ \delta/\alpha \to 0$ as $\alpha \to 0$ then the sequence $\{x_\alpha^{h\delta}\}$ converges to a solution x_0 of (1.1)

$$\|x_0\| = min_{x\in S_0}\|x\|,$$

where S_0 denotes the set of all solutions of (1.1) ($S_0 \neq \emptyset$). Moreover, the solution $x_\alpha^{h\delta}$ may be approximated by a solution of the finite-dimensional problem

$$A_h^n(x) + \alpha U^{sn}(x)\ =\ f_\delta^n, \tag{1.5}$$

where $A_h^n = P_n^* A_h P_n$, $U^{sn}(x) = P_n^* U^s P_n(x)$, and $f_\delta^n = P_n^* f_\delta$, P_n denotes a projection from X onto its subspace X_n, P_n^* is the adjoint of P_n. For each $\alpha > 0$ Equation (1.5) has a unique solution $x_\alpha^{h\delta n}$ and if

$$X_n\ \subset\ X_{n+1},\ \forall n,\ P_n x \to x,\ \forall\ x\ \in X\ (\|P_n\| = 1),$$

the sequence $\{x_\alpha^{h\delta n}\}$ converges to $x_\alpha^{h\delta}$, as $n \to +\infty$ (see [14]). It is very important for computation to know convergence rates for the sequence $\{x_\alpha^{h\delta}\}$; whether

$$\lim_{\substack{\alpha,\ h,\ \delta\to 0\\ n\to\infty}} x_\alpha^{h\delta n}\ =\ x_0$$

and convergence rates of the sequence $\{x_\alpha^{h\delta n}\}$.

Note that the above questions were stated and studied in [6, 8] for linear ill-posed problems and in [11, 12] for nonlinear ones for variational method of Tikhonov regularization. These problems were also considered in [4, 5] by the

author for the problems with monotone operators in Hilbert spaces. Here we generalize this results for Banach spaces.

Later, the symbols $\rightharpoonup$ and $\to$ denote weak convergence and convergence in norm, respectively.

In the following section we suppose that all the above conditions are satisfied.

2. Main results.

Firstly, we prove a result above convergence rates for $\{x_\alpha^{h\delta}\}$.

Theorem 2.1. If the following conditions hold: (i) A_h are Fréchet differentiable at a some neighbourhood of x_0 $s-1$-times if $s = [s]$ - the integer part of s, $[s]$-times if $s \neq [s]$,

(ii) there exists a constant $\tilde{L}$ such that

$$\|A_h^{(k)}(x_0) - A_h^{(k)}(x)\|\ \leq\ \tilde{L}\|x_0 - x\|,\ \forall\ x \in\ S(x_0, r),$$

$k = s - 1$ if $s = [s]$, $k = [s]$ if $s \neq [s]$, and if $[s] \geq 3$ $A_h^{(2)}(x_0) = ... = A_h^{(k)}(x_0) = 0$, where $S(x_0, r)$ is a ball with centre x_0 and radius $r > 0$,

(iii) there exist elements z_h such that

$$A_h^{\prime *}(x_0)z_h = U^s(x_0)$$

and if $s = [s]$ then $\tilde{L}\|z_h\| \leq ms!$.

Then, if α is chosen as $\alpha = O((h+\alpha)^\mu)$, $0 < \mu < 1$, we obtain

$$\|x_\alpha^{h\delta} - x_0\| = O((h+\delta)^\theta),\ \theta =\ min\ \{(1-\mu)/(s-1),\ \mu/s\}.$$

Proof. From (1.1) and (1.3) it follows

$$m\|x_\alpha^{h\delta} - x_0\|^s\ \leq\ (\delta + h\|x_0\|)\|x_\alpha^{h\delta} - x_0\| + \alpha\langle U^s(x_0),\ x_0 - x_\alpha^{h\delta}\rangle.$$

From here and condition (iii) of the theorem we can write

$$m\|x_\alpha^{h\delta} - x_0\|^s\ \leq\ (\delta + h\|x_0\|)\|x_\alpha^{h\delta} - x_0\| + \alpha\langle z_h,\ A_h^\prime(x_0)(x_0 - x_\alpha^{h\delta})\rangle. \tag{2.1}$$

If $s = [s]$, since A_h are $s-1$ - times Fréchet differentiable at x_0 and $A_h^{(2)}(x_0) = ... = A_h^{(s-1)}(x_0) = 0$, we have

$$A_h^\prime(x_0)(x_0 - x_\alpha^{h\delta}) = A_h(x_0) - A_h(x_\alpha^{h\delta}) + r_\alpha^{h\delta}$$

with

$$\|r_\alpha^{h\delta}\| \leq \frac{\tilde{L}}{s!}\|x_\alpha^{h\delta} - x_0\|^s.$$

Together with (2.1) this inequality gives us

$$\alpha(m - \frac{\tilde{L}}{s!}\|z_h\|)\|x_\alpha^{h\delta} - x_0\|^s \leq (\delta + h\|x_\alpha^{h\delta} - x_0\| +$$

$$\alpha(\delta + h\|x_0\| + \alpha\|x_\alpha^{h\delta}\|).$$

Using the relation in [10]:

$$a,\ b,\ c \geq 0,\ p > q,\ a^p \leq ba^q + c \Longrightarrow a^p = O(b^{p/(p-q)} + c)$$

we obtain

$$\|x_\alpha^{h\delta} - x_0\| = O((h+\delta)^\theta), \quad \theta = min\ ((1-\mu)^{1/(s-1)},\ \mu/s)$$

If $s \neq [s]$, then

$$\|r_\alpha^{h\delta}\| \leq \frac{\tilde{L}}{([s]+1)!}\|x_\alpha^{h\delta} - x_0\|^{[s]+1}$$

and we have the estimate

$$\alpha(m - \frac{\tilde{L}}{([s]+1)!}\|z_h\|\|x_\alpha^{h\delta} - x_0\|^{[s]+1-s})\|x_\alpha^{h\delta} - x_0\|^s \leq (\delta + h\|x_\alpha^{h\delta} - x_0\| +$$

$$\alpha(\delta + h\|x_0\| + \alpha\|x_\alpha^{h\delta}\|^{s-1}).$$

Since $x_\alpha^{h\delta} \to x_0$ as $h/\alpha,\ \delta/\alpha \to 0$ and $[s] + 1 - s > 0$

$$m - \frac{\tilde{L}}{([s]+1)!}\|z_h\|\|x_\alpha^{h\delta} - x_0\|^{[s]+1-s} > 1/2$$

for sufficiently small α. This remark completes the proof of the theorem. Q.E.D.

If X is a Hilbert space, then $U^s = I$, $s = 2$, $m = 1$, $\vartheta = 1$ and $c(R) = 1$. For the spaces of Lebegue's type $l_p,\ L_p,\ W_m^p,\ p > 1$ we can construct U^s satisfying the conditions (1.3) and (1.4) (see [1]):

$$1 < p < 2 : s = 2,\ m = p - 1,\ c(\rho) = p2^{2p-1}e^pL^{p-1},$$

$$e = max\ \{2^p,\ 2\rho\},\ 1 < L < 3.18,\ \vartheta = p - 1;$$

$$2 < p : s = p,\ m = 2^{2-p}/p,\ c(\rho) = 2^p\rho^{p-2}\{p[p - 1 + max\{\rho,\ L\}]\}^{-1},\ \vartheta = 1.$$

In the case of Hilbert spaces and Banach spaces with $s = 2$ the requirement of zero for Fréchet derivative is redundant. The case $s > 2$ will be illustrated

in next section. Condition (iii) of the theorem was proposed to investigate convergence rates of regularized solutions for variational method of Tikhonov regularization (see [7]). It is easy to see that if A is shaperly known and Fréchet differentiable, we can formulate the theorem by replacing A_h by A. If A is not sufficiently smooth in sense of Fréchet, we have to approximate it by sufficiently smooth A_h.

Theorem 2.2. If the following conditions hold:

(i) conditions (i) and (ii) of Theorem 2.1 at some neighbourhood of S_0,

(ii) $\alpha = \alpha(h,\ \delta,\ n) \to$ such that $h/\alpha,\ \delta/\alpha \to 0$ and

$$\gamma_n * \alpha^{-1} \to 0.$$

Then the sequence $\{x_\alpha^{h\delta n}\}$ converges to x_0.

Proof. By (1.1) and (1.5)

$$A_h^n(x_\alpha^{h\delta n}) - A_h^n(x_0^n) + \alpha(U^{sn}(x_\alpha^{h\delta n}) - U^{sn}(x_0^n)) =$$

$$f_\delta^n - A_h^n(x_0^n) - \alpha U^{sn}(x_0^n) - f^n + P_n^*A(x_0),\ x_0^n = P_nx_0,\ f^n = P_n^*f.$$

Thus

$$\alpha\langle U^s(x_\alpha^{h\delta n}) - U^s(x_0^n),\ x_\alpha^{h\delta n} - x_0^n\rangle \leq (\delta + h\|x_0\|)\|x_\alpha^{h\delta n} - x_0^n\| + \tag{2.2}$$

$$\langle A_h(x_0) - A_h(x_0^n),\ x_\alpha^{h\delta n} - x_0^n\rangle + \alpha\langle U^s(x_0^n),\ x_0^n - x_\alpha^{h\delta n}\rangle.$$

If $s = [s]$, we can write

$$A_h(x_0^n) - A_h(x_0) = A_h'(x_0)(P_n - I)x_0 + r_h^n$$

with

$$\|r_h^n\| \leq \frac{\tilde{L}}{s!}\|(I - P_n)x_0\|^s.$$

Then we have

$$\alpha m\|x_\alpha^{h\delta n} - x_0^n\|^s \leq (\delta + h\|x_0\| + \|A_h'(x_0)(I - P_n)x_0\| + \frac{\tilde{L}}{s!}\|(I - P_n)x_0\|^s) \tag{2.3}$$

$$\|x_\alpha^{h\delta n} - x_0^n\| + \alpha\langle U^s(x_0^n),\ x_0^n - x_\alpha^{h\delta n}\rangle.$$

Together with the conditions of the theorem this inequality gives us that the sequence $\{x_\alpha^{h\delta n}\}$ is bounded. Without loss of generality, let $x_\alpha^{h\delta n} \rightharpoonup x_1$ as $h,\ \delta,\ \alpha \to 0$ and $n \to +\infty$. Now, we write the monotone property for $A^n = P_n^*AP_n$ as

$$\langle A^n(x^n) - A^n(x_\alpha^{h\delta n}),\ x^n - x_\alpha^{h\delta n}\rangle \geq 0,\ \forall\ x \in X,\ x^n = P_nx.$$

As $P_n^*P_n^* = P_n^*$, the last inequality may be written by the form

$$\langle A(x^n) - A^n(x_\alpha^{h\delta n}),\ x^n - x_\alpha^{h\delta n}\rangle \ \geq\ 0.$$

From here and (1.5) it implies

$$h\|x_\alpha^{h\delta}\|\|x^n - x_\alpha^{h\delta n}\| + \langle A(x^n) - f_\delta^n,\ x^n - x_\alpha^{h\delta n}\rangle +$$

$$\alpha\|x\|\|x^n - x_\alpha^{h\delta n}\| \ \geq\ 0.$$

Passing $h,\ \delta,\ \alpha \to 0$ and $n \to +\infty$ in this inequality we obtain

$$\langle A(x) - f,\ x - x_1\rangle \ \geq\ 0,\ \ \forall\, x\ \in\ X.$$

By Minty's lemma $x_1 \in S_0$. By (2.2) and (1.3) we also have $\|x_1\| \leq \|x_0\|,\ \forall x_0 \in S_0$. Since S_0 is a convex and closed subset of a strictly convex Banach space X, then $x_1 = x_0$. Consequently, entire sequence $\{x_\alpha^{h\delta n}\}$ converges weakly to x_0. The inequality (2.2) also gives us that $\|x_\alpha^{h\delta n}\| \to \|x_0\|$. Now, the property of X guarantees the strong convergence of the sequence $\{x_\alpha^{h\delta n}\}$ to x_0.
If $s \neq [s]$, then the term $\frac{\tilde{L}}{s!}\|(I - P_n)x_0\|^s$ will be placed by $\frac{\tilde{L}}{([s]+1)!}\|(I - P_n)x_0\|^{[s]+1}$ in (2.3) and the process of proof of the theorem will be entirely repeated. Q.E.D.

Now, we shall prove a result about convergence rates for the sequence $\{x_\alpha^{h\delta n}\}$.

Theorem 2.3. If the following conditions hold:

(i) conditions (i)-(iii) of theorem 2.1 and

(ii) α is chosen as $\alpha = O((h + \delta)^{1/2} + \gamma_n^\theta)$

Then

$$\|x_\alpha^{h\delta n} - x_0\| \ =\ O((h + \delta)^{1/4} + \gamma_n^{\theta/2}).$$

Proof. Since

$$\|A_h(x_0) - A_h(x_0^n)\| \ \leq\ h\|x_0\| + \delta + \|f_\delta - A_h(x_0^n)\|,$$

$$\langle U^s(x_0^n),\ x_0^n - x_\alpha^{h\delta n}\rangle \ =\ \langle U^s(x_0^n) - U^s(x_0),\ x_0^n - x_0\rangle + \langle U^s(x_0),\ x_0^n - x_\alpha^{h\delta n}\rangle$$

$$\leq C(R)\|(I - P_n)x_0\|^\theta + \langle U^s(x_0),\ x_0^n - x_\alpha^{h\delta n}\rangle$$

the inequality (2.2) gives

$$\alpha m\|x_\alpha^{h\delta n} - x_0^n\|^s \ \leq\ 2(\delta + h\|x_0\| + \|f_\delta - A_h(x_0^n)\|$$

$$+\alpha C(R)\gamma_n^\theta)\|x_\alpha^{h\delta n} - x_0^n\| + \alpha\langle U^s(x_0),\ x_0^n - x_\alpha^{h\delta n}\rangle.$$

Thus

$$\alpha m\|x_\alpha^{h\delta n} - x_0^n\|^s \ \leq\ 2(\delta + h\|x_0\| + \|f_\delta - A_h(x_0^n)\|$$

$$+\alpha C(R)\gamma_n^\theta)\|x_\alpha^{h\delta n} - x_0^n\| + \alpha\|x_0\|^s\gamma_n + \alpha\langle U^s(x_0),\ x_0 - x_\alpha^{h\delta n}\rangle.$$

If $s = [s]$, as

$$\|f_\delta - A_h(x_0^n)\| \ \leq\ \delta + \|A(x_0) - A_h(P_n x_0)\|$$

$$\leq\ \delta + h\|x_0\| + \|A_h(x_0) - A_h(P_n x_0)\|$$

$$\leq\ O(\delta + h + \|A_h(x_0)\|\gamma_n)$$

and

$$\langle U^s(x_0),\ x_0 - x_\alpha^{h\delta n}\rangle \ =\ \langle z_h,\ A_h'(x_0)(x_0 - x_\alpha^{h\delta n})\rangle$$

$$\leq\ \langle z_h,\ A_h(x_0) - A_h(x_0^n)\rangle + \frac{\tilde{L}}{s!}\|x_\alpha^{h\delta n} - x_0^n\|^s\|z_h\|$$

we have

$$\alpha m\|x_\alpha^{h\delta n} - x_0^n\|^s \ \leq\ O(\delta + h + \alpha\gamma_n^\theta)\|x_\alpha^{h\delta n} - x_0^n\|$$

$$+\alpha O(\delta + h + \gamma_n^\theta + \alpha).$$

If $s \neq [s]$, the left-hand side of this inequality is

$$\alpha\Big(m - \frac{\tilde{L}}{([s]+1))!}\|z_h\|\|x_\alpha^{h\delta n} - x_0^n\|^{[s]+1-s}\Big)\|x_\alpha^{h\delta n} - x_0^n\|^s.$$

Therefore

$$\|x_\alpha^{h\delta n} - x_0^n\| \ =\ O((\delta + h)^{1/4} + \gamma_n^{\theta/2}).$$

Then

$$\|x_\alpha^{h\delta n} - x_0\| \ =\ O((\delta + h)^{1/4} + \gamma_n^{\theta/2}). \qquad Q.E.D.$$

3. Example.

Consider the nonlinear integral equation

$$\int_0^1 k(t,y)F(\varphi(y))dy \ =\ f_0(t), \tag{3.1}$$

where $k(t,y) = t(1 - y)$, if $y - t \leq 0$ and $= y(1 - t)$ othertwise
$F(t)$ is a non-decreasing differential function on $(-\infty,\ +\infty)$ satisfying the condition $|F(t)| \leq a_0 + b_0|t|^{p-1},\ \forall\ t \in R, a_0 + b_0 > 0$ and $f_0 \in L_p[0,\ 1]$.

Define the operator

$$(K\varphi)(t) = \int_0^1 k(t,y)\varphi(y)dy,\ \forall\ \varphi\ \in L_q[0,\ 1],\ p^{-1}+q^{-1}=1,$$

$$F(\phi)(t) = F(\phi(t)),\ \forall\ \phi\ \in L_p[0,\ 1].$$

Suppose that equation (3.1) has solution $u(t)$ which is expressed in the form

$$u(t) = K^* x(t).$$

Then to find a solution of (3.1) we have to solve the following equation

$$Ax\ \equiv\ KFK^*x\ =\ f_0,\ f_0\ \in\ X^*:=L_p[0,\ 1],$$

with the monotone Fréchet differentiable operator A, and $X = L_q[0,\ 1]$.

Obviously, from [9] it follows that the problem (3.1) is an ill-posed one. If $2 \le p < +\infty$, then $1 < q \le 2$ and condition (iii) of Theorem 2.1 is described in the form

$$A'^*(x_0)z\ =\ U^s(x_0),$$

where

$$U^s(\varphi)\ =\ \|\varphi\|^{2-q}_{L_q[0,1]}|\varphi(t)|^{q-2}\varphi(t),\ t\ \in\ [0,\ 1]$$

and $A'(x_0). = KF'(K^*x_0)K^*$.

Condition (iv) of Theorem (2.1) is $(q-1)2 \ge \tilde{L}\|z\|$.

If $1 < p \le 2$, then $2 < q < +\infty$, this last condition is superfluous. If $F^{(2)}(K^*x_0) = \dots = F^{(k)}(K^*x_0) = 0$, then $A^{(2)}(x_0) = \dots = A^{(k)}(x_0) = 0$. Function $F(t)$ which satisfies this conditions in fact exists, for instance

$$F(t) = (t-t_0)^5 + (t-t_0)$$

with $u(t) \equiv t_0$.

Acknowledgments

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

References

1. I.J. Alber and A.I. Notik, Geometrical properties of Banach spaces and approximation methods solving nonlinear operator equations. Dokl. Acad. Nauk SSSR **276** (1984), 5, 1033-1037.

2. I.J Alber and I.P. Ryazanseva, On solutions of nonlinear problems involving monotone discontinuous operators. Differential uravnenia SSSR **25** (1979), 2, 331-342.

3. I.J Alber and I.P. Ryazanseva, Regularization of nonlinear equations with monotone operators. J. of Comp. Math. and Math. physics SSSR, **15** (1975), 283-289.

4. Ng. Buong, Tikhonov regularization for nonlinear ill-posed problems involving monotone operators: convergence rates and finite-dimensional approximations. Prepr. Institute of Computer Science, Hanoi, Vietnam **4** (1991).

5. Ng. Buong, Convergence rates of Tikhonov regularization for nonlinear ill-posed problems for monotone perturbations. Prepr. Institute of Computer Science, Hanoi, Vietnam **5** (1991).

6. H.W. Engl and C.W. Groetsch Projection-Regularization for linear operator equations of the first kind. In Special Programs on Inverse Problems, (Proc. of the Center of Math. Anal. **17** (1988), Australial Nat.Univ., 17-31), R.S Anderson and G.N.Neusan eds .

7. H.W Engl, K. Kunisch and A. Neubauer, Convergence rates for Tikhonov regularization of non-linear ill-posed problems, Inverse Problems 1989, 5, p.523-540

8. C.W. Groetsch, On a Regularization-Ritz Method for Fredholm Equations of the First Kind, Journal of Integral Equations, **4** (1982), 173-182.

9. A. Neubauer, Finite-Dimensional Approximations of Constrained Tikhonov-Regula - rized solutions of Ill-Posed Linear Operator Equations Math. of Comp. **48** (1987), 178, 565-583.

10. A. Neubauer, An a-poerioti parameter choice for Tikhonov regularization in Hilbert scales leading to optimal convergence rates. SIAM J. Num. Math. **25** (1988), 1313-1326.

11. A. Neubaucr, Tikhonov Regularization for nonlinear ill-posed problems: optimal convergence rates and finite-dimensional approximations, Inverse problems **5**, (1989), 541-557.

12. A. Neubauer and O. Scherzer, Finite-dimensional approximation of Tikhonov regularized solutions of nonlinear ill-posed problems, Numer. Funct. Anal. and Optim. **11** (1990) (1&2), 85-99.

13. I.P. Ryazanseva, On Galerkin's method solving equations with discontinuous monotone operators. Izvestia VUZ, ser. Math. SSSR, (1978), 7, 68-72.

14. I.P. Ryazanseva, On select of regularizating parameters for equations under monotone perturbations. Izvestia VUZ, ser. Math. SSSR, (1981), 8, 39-43.

15. I.P. Ryazanseva, On one algorithm for solving nonlinear monotone equations with an unknown estimate input errors. J. Math. of Comp. and Math. physics SSSR, **29** (1989), 10, 1572-1576.