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ABSTRACT

We consider a class of strongly degenerate linear elliptic equation. The boundedness and the Holder regularity of the weak solutions in the weighted Sobolev–Hardy spaces will be studied.

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1 Introduction

Let Ω be a domain in R^n, we consider the second order, linear elliptic equations of the form

$$Lu = D_i(a_{ij}(x)D_ju + a_iu) + b_iD_iu + cu = -D_if_i + g$$

whose coefficients and data are measurable functions. This problem has been studied extensively by (among many other) Stampacchia, Ladyzenskaya and Uraltseva, Moser and Trudinger. Especially, in [TR] we can find a considerably general setting on this subject. By assumming that

$$\lambda^{-1} \in L^t(\Omega), \quad \Lambda \in L^s(\Omega), \quad |a|^2\lambda^{-1}, |b|^2\lambda^{-1}, c \in L^s(\Omega)$$

for some s, t such that $\frac{1}{s} + \frac{1}{t} \leq (resp. <)\frac{2}{n}$, Trudinger was able to prove the existence (resp. the boundedness and Holder regularity) of the weak solutions in suitable weighted Sobolev spaces. His work generalized those of the other authors.

Motivating by the Sobolev-Hardy inequalities obtained in [Kuf],[D0],[D1], we will extend the results of the authors cited above to a larger class of equations which, in fact, includes those considered in [LU],[TR]. Namely, we allow the coefficients and data of the equation become strongly degenerate and singular near the boundary $\partial\Omega$ of Ω. Our conditions are more general and even violate those of these authors. However, by combining the improved imbedding inequalities and modifying the Moser's technique, we are able to obtain the existence, boundedness and regularity of the solutions in Sobolev-Hardy spaces.

In section 2, we improve some inequalities in [D1] and then prove the technical estimates. Next, we define the weighted Sobolev-Hardy spaces in which the solutions will be considered. Section 3 is devoted to the global boundedness, the maximum principle and the existence of the solutions. The local boundedness will be treated in section 4 and these results will be applied in section 5 to get the Holder continuity of the bounded solutions.

For simplicity, we will deal only with $n \geq 3$ in this paper. However, it is not difficult to see that our proofs can be extended to the case $n = 2$ by using the Trudinger's inequality for the singular situation as in [D1]. Moreover, we can also use the same technique to treat the degenerate, singular quasilinear elliptic equations of divergence form.

2 PRELIMINARIES

In this section, we are going to establish the main tools of this work. Almost all of them originate from the general Sobolev - Hardy inequalities derived in [D1]. Motivated by these results, we determine the function spaces that are associated with our class of degenerate equations.

Throughout of this paper we will consider a domain Ω in R^n whose boundary is of class C^2.

DEFINITION 1.1: For any sufficiently small positive real number δ, we put $\Omega_\delta = \{x \in \Omega : d(x) > \delta\}$, where $d(x)$ is the distance from x to $\partial\Omega$. Then (see [GT]) d is continuously differentiable on Ω when δ is small. We denote by d_Ω a C^1 -extension of d into Ω. This means $d_\Omega(x) = d(x)$ for all $x \in \Omega \backslash \Omega_\delta$. Put

$$S_0 = \{u = v|_\Omega : v \in C_0^1(R^n), \qquad v|_{\partial\Omega} = 0\}$$

We have the following improved Sobolev - Hardy inequality in [D1] (see [D1], corollary 1.1. p.423).

LEMMA 1 *Let $m \in (1,n), \alpha \in [0,m]\backslash\{m-1\}, p \in [m, nm/(n-m)]$ and*

$$\gamma \in \left[\frac{p\alpha}{m} - n + \frac{n-m}{m}p, 0\right]$$

Then there exists a positive constant C such that for any $u \in S_0$ we have

$$\left(\int_\Omega |u|^p d_\Omega^\gamma\, dx\right)^{1/p} \le C\left(\int_\Omega |Du|^m d_\Omega^\alpha\, dx\right)^{1/m} \qquad (1)$$

REMARK 1.1: The constant C in the above lemma implicitly depends on Ω (see [Kuf]). When $\partial\Omega$ is smooth enough (say, $\partial\Omega$ is of class C^1), we can prove that C is uniformly bounded in $|\Omega|$, the Lebesgue measure of Ω and $diam(\Omega)$. We remark here this dependence may, in fact, be eliminated in the limiting case when

$$\gamma = \frac{p\alpha}{m} - n + \frac{n-m}{m}p \qquad (2)$$

To this end, we set $R = diam(\Omega)$ and make the coordinate transformation $y = x/R$. Then the transformed domain, say $\hat\Omega$, obviously satisfies $|\hat\Omega| \le 1$ and $diam(\hat\Omega) = 1$. Applying lemma 1 to the function $\hat u(y) = u(x)$ on $\hat\Omega$, we can find a constant C_1 depends only on p, m, n, α but not on $\hat\Omega$ such that

$$\left(\int_{\hat\Omega} |\hat u|^p d_{\hat\Omega}^\gamma\, dy\right)^{1/p} \le C_1\left(\int_{\hat\Omega} |D\hat u|^m d_{\hat\Omega}^\alpha\, dy\right)^{1/m}$$

Transforming back to the original coordinate x and noting the relations:

$$d_{\hat\Omega}(y) = R^{-1} d_\Omega(x), D_y\hat u = RD_x u, dy = R^{-n}dx$$

and using (2) we obtain again (1) with C now replaced by C_1 not depending on Ω.

From lemma 1 we can prove

LEMMA 2 *Let $m \in (1,n), \alpha \in [0,m]\backslash\{m-1\}, p \in [n, nm/(n-m)]$ and $s,t \in (1,\infty]$ such that*

$$(1+\frac{1}{t})\frac{1}{m} - \frac{1}{n} \le \frac{1}{p}(1-\frac{1}{s}) \le (1+\frac{1}{t})\frac{1}{m},$$

where $1/\infty = 0$. Let

$$\gamma \in \left[\frac{p\alpha}{m} - n(1-\frac{1}{s}) + \frac{n}{m}(1+\frac{1}{t})p - p, 0\right].$$

and λ, θ be non negative measurable functions on Ω such that $\lambda^{-1} d_\Omega^\alpha \in L^t(\Omega)$ and $\theta\, d_\Omega^{-\gamma} \in L^s(\Omega)$. Then there exists a positive constant C such that for any $u \in S_0$

$$\left(\int_\Omega |u|^p\theta\, dx\right)^{1/p} \le C\|\theta\, d_\Omega^{-\gamma}\|_{L^s(\Omega)}{}^{1/p}\|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(\Omega)}{}^{1/m}\left(\int_\Omega |Du|^m\lambda\, dx\right)^{1/m}. \qquad (3)$$

PROOF: First we suppose $t = \infty$. If $s = \infty$, (3) is just (1). Suppose $s \in (1,\infty)$. Using the Holder inequality and then lemma 1 for $p_1 = ps/(s-1)$ and $\gamma_1 = \gamma s/(s-1)$ instead of p and γ, there is a constant C such that for any $u \in S_0$

$$\begin{aligned}(\textstyle\int_\Omega |u|^p\theta\, dx)^{1/p} &= \left(\textstyle\int_\Omega |u|^p\theta\, d_\Omega^{-\gamma} d_\Omega^\gamma\, dx\right)^{1/p} \le \|\theta\, d_\Omega^{-\gamma}\|_{L^s(\Omega)}(\textstyle\int_\Omega |u|^{p_1} d_\Omega^{\gamma_1}\, dx)^{1/p_1}\\ &\le \|\theta\, d_\Omega^{-\gamma}\|_{L^s(\Omega)}(\textstyle\int_\Omega |Du|^m d_\Omega^\alpha\, dx)^{1/m}\end{aligned} \qquad (4)$$

provided that

$$\left\{\begin{array}{l} m \le p_1 \le \dfrac{nm}{n-m} \\ \gamma_1 \ge \dfrac{p_1\alpha}{m} - n + \dfrac{n-m}{m}p_1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \dfrac{1}{m} - \dfrac{1}{n} \le \dfrac{1}{p}(1-\dfrac{1}{s}) \le \dfrac{1}{m} \\ \gamma \ge \dfrac{p\alpha}{m} - n(1-\dfrac{1}{s}) + \dfrac{n-m}{m}p \end{array}\right. \qquad (5)$$

Therefore, we get (3) for $t = \infty$. Now suppose $t \in (1,\infty)$. Choose m', α' such that $\frac{1}{m'} = (1+\frac{1}{t})\frac{1}{m}$ and $\alpha' = \frac{m'}{m}\alpha$. By the Holder inequality we have

$$\begin{aligned}\left(\int_\Omega |Du|^{m'} d_\Omega^{\alpha'}\, dx\right)^{1/m'} &= \left(\int_\Omega |Du|^{m'}\lambda^{m'/m}(\lambda^{-1}\, d_\Omega^\alpha)^{m'/m}\, dx\right)^{1/m'}\\ &\le \|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(\Omega)}{}^{1/m}\left(\int_\Omega |Du|^m\lambda\, dx\right)^{1/m}\end{aligned}$$

Therefore we get (3) provided that the condition (5) holds for m', α' instead of m, α which is easily seen to be the conditions of the lemma.

Q.E.D.

REMARK 1.2: From the proof we see that the norms involving λ, θ could be taken on $supp(u)$ and the constants appearing in the estimates, *except (4)*, do not depend on Ω. However, by remark 1.1, we see that constant in (4) can be bounded independently on Ω if

$$\gamma = \frac{p\alpha'}{m'} - n(1-\frac{1}{s}) + \frac{n-m'}{m'}p = \frac{p\alpha}{m} - n(1-\frac{1}{s}) + \frac{n}{m}(1+\frac{1}{t})p - p$$

and so does the constant in (3).

When $p = m = 2$, we can state lemma 2 in a simpler form as follows.

LEMMA 3 *Let $s,t \in (1,\infty], \alpha \in [0,2)\backslash\{1\}$, and γ be such that $\frac{1}{t} + \frac{1}{s} \le \frac{2}{n}$ and $\gamma \in [\alpha - 2 + n\left(\frac{1}{s} + \frac{1}{t}\right), 0]$. Let λ, θ be non negative measurable functions on Ω such that $\lambda^{-1}\, d_\Omega^\alpha \in L^t(\Omega), \theta\, d_\Omega^{-\gamma} \in L^s(\Omega)$. Then there exists a constant C such that for any $u \in S_0$*

$$\int_\Omega |u|^2\theta\, dx \le C\|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(\Omega)}\|\theta\, d_\Omega^{-\gamma}\|_{L^s(\Omega)}\int_\Omega |Du|^2\lambda\, dx \qquad (6)$$

From the considerations in remark 1.2, we have the special case $s = \infty, \gamma = 0$ as

LEMMA 4 *Let $\alpha \in [0,2)\backslash\{1\}$ and $t \in (1,\infty]$ and λ be non negative measurable function on Ω such that $\lambda^{-1}\, d_\Omega^\alpha \in L^t(\Omega)$ and*

$$\frac{1}{t_*} = \frac{1}{2}(1+\frac{1}{t}) - \frac{1}{n} + \frac{\alpha}{2n}$$

Then there exists a positive constant $C(n,\alpha,t)$ such that for any $u \in S_0$

$$\left(\int_\Omega |u|^{t_*}\ dx\right)^{1/t_*} \le C(n,\alpha,t)\|\lambda^{-1}\ d_\Omega^\alpha\|_{L^t(\Omega)}{}^{1/2}\left(\int_\Omega |Du|^2\lambda\ dx\right)^{1/2} \tag{7}$$

REMARK 1.3: For $\alpha = 0$, lemma 4 is stated in [TR]. Moreover, the results of lemma 1 to lemma 4 continue to hold in a more general situation when$u$ may need to be zero only on a part of $\partial\Omega$ (see [D1]) and $\partial\Omega$ may not be of class C^2 (see [D0]).

However, some more generalizations seem to be essential in the local treatments in section 4 of this paper and we would like to present them in detail here.

For a measurable subset S of Ω such that $|S|$ is positive, we write for the integral mean value of u on S as follows.

$$u_S = \frac{1}{|S|}\int_S u\ dx \tag{8}$$

Using the potential estimate and the Holder inequality, it is easy to prove the following extension of lemma 1.1 in [TR]

LEMMA 5 *Let B_R be a ball in R^n and u be a function in $C^1(R^n)$ having $u_S = 0$ for some subset S of B_R such that $|S| \ge \tau|B_R|$ for some $\tau \in (0,1)$. Let λ be a non negative measurable function such that $\lambda^{-1} \in L^t(\Omega)$, with $t \in (1,\infty]$ and $m \in (1,n)$ satisfying*

$$\frac{1}{t_*} = \frac{1}{m}(1+\frac{1}{t}) - \frac{1}{n} \ge \frac{1}{m} - \frac{1}{n} \tag{9}$$

Then we have

$$\|u\|^m_{L^{t_*}(B_R)} \le C(n,t,\tau)\|\lambda^{-1}\|_{L^t(B_R)}\int_{B_R}\lambda|Du|^m\ dx \tag{10}$$

DEFINITION 1.2: Let B be a ball in R^n and Ω be a subdomain of B. We say Ω is ***a regular subdomain*** of B if $|\Omega| \ge \tau|B|$ for some $\tau \in (0,1)$ and $T = B \cap \partial\Omega$, the part of $\partial\Omega$ lying inside B, is of class C^2. We denote by d_T a C^1 extension of the distance function from a point $x \in B$ to T. We put

$$S_0(T) = \{u = v|_\Omega : v \in C_0^1(R^n), v = 0 \text{ on } T\}$$

Now let u be in $S_0(T)$ and p, m, α, γ be as in lemma 1. Extending u to be zero inside $B\backslash\Omega$, we can then apply (10) to the function $v = ud_T^{\alpha/m}$ and obtain.

$$\begin{aligned}\left(\int_\Omega |ud_T^{\alpha m}|^{nm/(n-m)}\ dx\right)^{(n-m)/n} &\le C(B,\tau)\int_\Omega |Dv|^m\ dx \\ &\le C(B,\tau)\left\{\int_\Omega |Du|^m d_T^\alpha\ dx + \int_\Omega |u|^m d_T^{\alpha-m}\ dx\right\}\end{aligned} \tag{11}$$

where τ is the contant appearing in the definition 1.2 . For the second integral on the right side, we can apply the generalized Hardy-Littlewood inequality in [Kuf] (see [Kuf] p.28, theorem 8.4) and notice that the arguments there requires only that the function *u be zero on the boundary part to which the distance function d_T is considered*, we then have

$$\int_\Omega |u|^m d_T^{\alpha-m}\ dx \le C(\Omega)\int_\Omega |Du|^m d_T^\alpha\ dx$$

This and (11) give

$$\|ud_T^{\alpha/m}\|_{L^{nm/(n-m)}(\Omega)} \le C(\Omega,\tau)\int_\Omega |Du|^m d_T^\alpha\ dx$$

By using the interpolation methode as in [D1] (corollary 1.1) we obtain again lemma 1 for the functions in $S_0(T)$. This also implies

PROPOSITION 1 *The conclusion of lemma 1 to lemma 4 still hold for the function $u \in S_0(T)$ with d_Ω replaced by d_T. The constants in those lemmas now depend also on the parameter τ in the definition 1.2.*

We are ready now to define the function spaces that will be used in the studies of our equation.

DEFINITION 1.3: Let $A(x) = (a_{ij}(x))$ be an $n \times n$ symmetric measurable real matrix function which is positive definite almost everywhere on Ω. For $u, v \in C_0^\infty(R^n)$, we put

$$\begin{aligned}(u,v)_\theta &= \int_\Omega \theta uv\ dx, \\ (u,v)_A &= \int_\Omega a_{ij}D_iuD_jv\ dx \\ (u,v)_{A,\theta} &= (u,v)_A + (u,v)_\theta\end{aligned}$$

We also put

$$S(A,\theta,\Omega) = \{u = v|_\Omega : v \in C_0^\infty(R^n), \qquad (v,v)_{A,\theta} < \infty\}$$

$$S_0(A,\theta,\Omega) = S(A,\theta,\Omega) \cap S_0$$

and for $u \in S(A,\theta,\Omega)$, set

$$\|u\|_{A,\theta} = (u,v)_{A,\theta}^{1/2}$$

Then we denote by $W(A,\theta,\Omega)$ and $W_0(A,\theta,\Omega)$ the completions of $S(A,\theta,\Omega)$ and $S_0(A,\theta,\Omega)$ respectively under the norm $\|.\|_{A,\theta}$. The results of the previous lemmas motivate us to consider the matrix A and the function θ satisfying the following conditions.

(A.1) Let $\lambda(x), \Lambda(x)$ denote the minimum, maximum eigenvalues of the matrix $A(x)$. We assume that they are positive measurable functions on Ω and for any real vectors $\zeta \in R^n$ and $x \in \Omega$, we have

$$\lambda(x)|\zeta|^2 \le a_{ij}(x)\zeta_i\zeta_j \le \Lambda(x)|\zeta|^2 \tag{12}$$

(A.2) There exist real numbers $r, t \in (1, \infty]$ and $\alpha \in [0,2)\backslash\{1\}$ such that

$$\alpha - 2 + n\left(\frac{1}{r} + \frac{1}{t}\right) \leq \gamma \leq 0 \tag{13}$$

Suppose that $\lambda^{-1} d_\Omega^\alpha \in L^t(\Omega)$ and $\theta\, d_\Omega^{-\gamma} \in L^s(\Omega)$.

The conditions (A.2) allows us to use the lemmas 3,4 and then to assert that

PROPOSITION 2 *Let (A.1), (A.2) be satisfied. Then*

i/ *The estimates (6),(7) continue to hold for $u \in W_0(A, \theta, \Omega)$.*

ii/ *The norm $\|.\|_{A,\theta}$ of $W_0(A, \theta, \Omega)$ is equivalent to the following norm*

$$\|u\|_* = \int_\Omega a_{ij} D_i u D_j u \, dx \tag{14}$$

iii/ *$W_0(A, \theta, \Omega)$ is continuously imbedded into $L^{t_*}(\Omega)$ and $L^{r_*}(\Omega)$, where*

$$\frac{1}{t_*} = \frac{1}{2}\left(1 + \frac{1}{t}\right) - \frac{1}{n} + \frac{\alpha}{2n} \text{ and } \frac{1}{r_*} = \frac{1}{2}\left(1 - \frac{1}{r}\right)$$

PROOF: i/ is clear. ii/ comes from the estimate (6) and (12) of (A.1). iii/ is a result of (7) and the fact that from (13) we have $t_* \geq r_*$.

Q.E.D.

We note that the conditions (A.1), (A.2) do not impose any restriction on the maximum eigenvalue Λ. However, a weak assumption on Λ such as $\Lambda \in L^1_{loc}(\Omega)$ would be sufficient for the spaces $W(A, \theta, \Omega), W_0(A, \theta, \Omega)$ to be not empty. Furthermore, in local treatments of section 4, some more controls on Λ seems to be necessary. Let us consider the following

(A.3) There exists $s \in (n/2, \infty]$ such that for t, α as in (A.2) we have

$$\alpha - 2 + n\left(\frac{1}{s} + \frac{1}{t}\right) \leq 0 \tag{15}$$

and $\Lambda \in L^s(\Omega)$.

PROPOSITION 3 *Let (A.1),(A.2) and (A.3) be satisfied. We have*

i/ *$W_0(A, \theta, \Omega)$ is continuously imbedded into $L^{s_*}(\Omega)$ where $\frac{1}{s_*} = \frac{1}{2}\left(1 - \frac{1}{s}\right)$.*

ii/ *If $\eta \in W_0(A, \theta, \Omega)$ or $C_0^1(\Omega)$, and $u \in W_0(A, \theta, \Omega)$, then $\eta u \in W_0(A, \theta, \Omega)$.*

PROOF: i/ is clear, since we also have $t_* \geq s_*$. We consider ii/. By the Holder inequality and the ellipticity of A, we have

$$\begin{aligned} &\int_\Omega a_{ij} D_i(\eta u) D_j(\eta u)\, dx \\ &\quad = \int_\Omega [a_{ij}\eta^2 D_i u D_j u + 2a_{ij}\eta u D_j u D_i \eta + a_{ij} u^2 D_i \eta D_j \eta]\, dx \\ &\quad \leq C\left\{\int_\Omega \Lambda \eta^2\, dx \int_\Omega a_{ij} D_i u D_j u\, dx + \int_\Omega \Lambda u^2\, dx \int_\Omega a_{ij} D_i \eta D_j \eta\, dx\right\} \end{aligned}$$

If $\eta \in S_0(A, \theta, \Omega)$, by using the Holder inequality and the result of i/ we can majorize the last term by

$$C\|\Lambda\|_{L^s(\Omega)} \|\eta\|_*^2 \|u\|_*^2$$

In case $\eta \in C_0^1(\Omega)$, we simply bound that term by

$$C\|\Lambda\|_{L^s(\Omega)} \{\sup_\Omega |\eta| + \sup_\Omega |D\eta|\}^2 \|u\|_*^2$$

From the definition of $W_0(A, \theta, \Omega)$ we get ii/.

Q.E.D.

We also need the following technical result.

LEMMA 6 *Let (A.1), (A.2) and (A.3) be satisfied. Let $\eta \in C_0^1(R^n), v \in W(A, \theta, \Omega)$ such that $\eta v \in W_0(A, \theta, \Omega)$. Then for any positive real number ϵ, β, we can find the positive constants $C = C(|\Omega|, \|\lambda^{-1} d_\Omega^\alpha\|_{L^t(\Omega)}, \|\theta\, d_\Omega^{-\gamma}\|_{L^s(\Omega)})$ and $q = q(r, t, \alpha, \gamma)$ such that*

$$\int_\Omega \theta \eta^2 v^2\, dx \leq C\left\{\epsilon \int_\Omega \lambda \eta^2 \left|\frac{Dv}{\beta}\right|^2 dx + (\epsilon^{-1}\beta^2)^q \int_\Omega \eta^2 v^2\, dx + \int_\Omega \lambda |D\eta|^2 v^2\, dx\right\} \tag{16}$$

PROOF: We can assume that $v \in S(A, \theta, \Omega)$ and $\eta v \in S_0(A, \theta, \Omega)$. Let $m \in (1,2), m \neq 2/(2-\alpha)$. We can choose m near to 2 and m depends only on r, t, α, γ such that

$$\frac{1}{r} + \frac{1}{t} < \frac{2}{mr} + \frac{1}{t} < \frac{2}{n} \Rightarrow \frac{1}{r} + \frac{m}{2t} < \frac{m}{n}$$

$$\gamma > \frac{m}{2}\left(\alpha - 2 + n\left(\frac{2}{mr} + \frac{1}{t}\right)\right) = \frac{m\alpha}{2} - m + n\left(\frac{1}{r} + \frac{m}{2t}\right)$$

Putting $\alpha' = m\alpha/2, t' = 2t/m$, we have

$$\frac{1}{r} + \frac{1}{t'} < \frac{m}{n}, \quad \alpha' \in [0, m)\backslash\{m-1\}, \quad \gamma > \alpha' - m + n\left(\frac{1}{r} + \frac{1}{t'}\right) \tag{17}$$

Let us define $\chi = (\eta v)^{2/m}$. Then

$$|D\chi|^m \leq C(m)(\eta v)^{2-m}|D(\eta v)|^m \leq C(m)\{\eta^{2-m}|D\eta|^m v^2 + \eta^2 v^{2-m}|Dv|^m\} \tag{18}$$

Using the simple inequality $a^{2-m}b^m \leq a^2 + b^2$, which holds for any $a, b \geq 0$, we have

$$\begin{aligned} v^2\eta^{2-m}|D\eta|^m \lambda^{m/2} &\leq v^2\{\eta^2 + \lambda|D\eta|^2\} \\ \eta^2 v^{2-m}|Dv|^m \lambda^{m/2} &\leq \eta^2\{(\epsilon^{-1}\beta^2)^q v^2 + \epsilon\lambda\left|\frac{Dv}{\beta}\right|^2\} \end{aligned} \tag{19}$$

where $q = m/(2-m)$. (17) allows us to use lemma 2 to get

$$\int_\Omega \theta\eta^2 v^2 \, dx = \int_\Omega \theta\chi^m \, dx \le C \int_\Omega |D\chi|^m \lambda^{m/2} \, dx \tag{20}$$

where $C = C(|\Omega|)\|\lambda^{-m/2} d_\Omega^{\alpha'}\|_{L^{t'}(\Omega)} \|\theta d_\Omega^{-\gamma}\|_{L^r(\Omega)} = C(|\Omega|)\|\lambda^{-1} \, d_\Omega^\alpha\|_{L^t(\Omega)}{}^{m/2} \|\theta \, d_\Omega^{-\gamma}\|_{L^s(\Omega)}$.
Combining (20),(18) and (19) we get (16).

Q.E.D.

REMARK 1.4: Without the condition (A.3) and setting $\eta \equiv 1$, we see that the above proof can be repeated to give (16) for the functions in $W_0(A,\theta,\Omega)$.

We also notice that when (A.1), (A.2) and (A.3) hold, the spaces W just defined are subspaces of the space of weakly differentiable functions (for a definition, see e.g [GT]). So that the chain rule (see also [GT], theorem 7.8) can be applied here.

The notions of inequality at the boundary will be accordingly defined as follows. Let $u \in W(A,\theta,\Omega)$, we shall write $u^+ = \sup\{u, 0\}, u^- = \inf\{u, 0\}$ and say that $u \le 0 (u \ge 0)$ on $\partial\Omega$ if the function u^+ (u^-) belongs to $W_0(A,\theta,\Omega)$. Moreover, let T be a boundary portion of $\partial\Omega$, we shall say $u \le 0$ (resp. $u = 0, u \ge 0$) on T if u^+ (resp. u, u^-) is the limit in $W(A,\theta,\Omega)$ of a sequence of smooth functions in $S(A,\theta,\Omega)$ that vanish on T. Note that the two notions just defined are coincident when $T = \partial\Omega$. Then we can put

$$\sup_T u = \inf\{L \in R : u - L \le 0 \text{ on } T\}$$

$$\inf_T u = \sup\{L \in R : L - u \le 0 \text{ on } T\}$$

Finally, let Ω be a regular subdomain of a ball B_R in R^n in the sense of the definition 1.2. We also write $W_T(A,\theta,\Omega)$ for the subspace of $W(A,\theta,\Omega)$ which consists of the functions vanishing on T. The weight function d_Ω here is also replaced by d_T given in that definition. Then, from proposition 1 we still have the estimates of the lemmas 3,4 for the functions in the space $W_T(A,\theta,\Omega)$.

3 GLOBAL BOUNDEDNESS, UNIQUENESS AND EXISTENCE THEOREMS

In this section, and the rest of this paper, we will consider the following elliptic operator

$$Lu = -D_i(a_{ij}D_ju + a_iu) + b_iD_iu + cu \tag{21}$$

where the coefficients are real measurable functions given on Ω. The linear elliptic differential equation associated with this operator is

$$Lu = -D_if_i + g \tag{22}$$

Let us consider the operator L that satisfies

(H.1) $A(x) = (a_{ij}(x))$ is a positive definite symmetric measurable matrix function defined on Ω. Let λ, Λ denote the minimum, maximum eigenvalues of (a_{ij}). We assume that λ, Λ are nonnegative measurable functions on Ω and for any real vector $\zeta \in R^n$, we have

$$\lambda(x)|\zeta|^2 \le a_{ij}(x)\zeta_i\zeta_j \le \Lambda(x)|\zeta|^2 \tag{23}$$

(H.2) There exists real numbers $r, s, t \in (1,\infty]$ and $\alpha \in [0,2)\backslash\{1\}$ and a real number γ such that

$$\alpha - 2 + n\left(\frac{1}{s} + \frac{1}{t}\right) < 0 \tag{24}$$

$$\alpha - 2 + n\left(\frac{1}{r} + \frac{1}{t}\right) < \gamma \le 0 \tag{25}$$

and a non negative measurable function θ such that

i) $\lambda^{-1} \, d_\Omega^\alpha \in L^t(\Omega)$ and $\theta \, d_\Omega^{-\gamma} \in L^s(\Omega)$.

ii) $|a|^2\lambda^{-1}, |b|^2\lambda^{-1}, |c| \le \theta$.

iii) $|f|^2\lambda^{-1} \in L^s(\Omega)$ and $|g|^2\theta^{-1} \in L^s(\Omega)$ where $|a|, |b|, |f|$ are respectively the Euclidean norms of the vector functions $(a_i), (b_i), (f_i)$.

Then we can define the notion of solutions as follows

DEFINITION 2.1: We say a measurable function u is a *solution (resp. subsolution, supersolution)* of the equation (22) if for any φ (resp. *nonnegative function* φ) in $W_0(A,\theta,\Omega)$, we have

$$L(u,\varphi) = \int_\Omega \{a_{ij}D_juD_i\varphi + a_iuD_i\varphi + b_i\varphi D_iu + cu\varphi\} \, dx = \int_\Omega (f_iD_i\varphi + g\varphi) \, dx \tag{26}$$

(resp. $\le, \ge$)

At first, using (H.1),(H.2), the Schwartz inequality and the definition of $\|.\|_{A,\theta}$, we easily prove that

PROPOSITION 4 *Let $u \in W(A,\theta,\Omega)$ and $\varphi \in W_0(A,\theta,\Omega)$. Then $L(u,\varphi)$ is finite and there is a general constant C such that*

$$|L(u,\varphi)| \le \|u\|_{A,\theta}\|\varphi\|_{A,\theta}$$

So $L(u,\varphi)$ is a bounded bilinear form on $W_0(A,\theta,\Omega)$ and for any fixed u in $W(A,\theta,\Omega)$, it is a bounded linear functional on $W_0(A,\theta,\Omega)$.

The above result suggests that we should define the notion of weak solutions of (22) as follows: Let us say u is a *weak solution* (resp. *weak subsolution, weak supersolution*) if u is a solution (resp. subsolution, supersolution) and $u \in W(A,\theta,\Omega)$.

The two following theorems are our main results of this section. They relate to the boundedness of weak solutions of (22) that are already bounded on the boundary. We are going to establish several estimates of a weak solutions u in term of their boundary estimates and the following global quantities

$$n, r, s, t, \alpha, \gamma, |\Omega| \text{ and } \|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(\Omega)}, \|\theta\, d_\Omega^{-\gamma}\|_{L^s(\Omega)} \tag{27}$$

In the proofs of the theorems of this section. We will usually use $C_1, C_2, \ldots$ to denote the constants that depend on the global quantities stated in (27), and C for the generic constants.

THEOREM 1 *Let (H.1), (H.2) be satisfied and u is a weak subsolution (supersolution) of (22) such that $u \leq 0$ ($u \geq 0$) on $\partial\Omega$. Then we have*

$$\sup_\Omega u\ (-u) \leq C\left\{\|u^+\ (u^-)\|_{L^1(\Omega)} + k(\Omega)\right\} \tag{28}$$

where C is a constant depends on the global quantities as in (27), and

$$k(\Omega) = \left\{\|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(\Omega)}(\||f|^2\lambda^{-1}\|_{L^s(\Omega)} + \||g|^2\theta^{-1}\|_{L^s(\Omega)})\right\}^{1/2} \tag{29}$$

PROOF: We initially assume that u is a weak subsolution and $k = k(\Omega) > 0$. For arbitrary real numbers $\beta \geq 1, N > k$, Let J be the function in $C^1[k, \infty)$ given by

$$J(t) = \begin{cases} t^\beta - k^\beta & \text{for } t \in [k, N] \\ \beta N^{\beta-1}(t-N) + N^\beta - k^\beta & \text{for } t > N \end{cases} \tag{30}$$

We easily see that J' is a bounded increasing function on $[k, \infty)$ and J'' exists for $t \neq N$, and $J'' \in L^\infty[k, \infty)$, and $J''(t)t \leq \beta J'(t)$. Next, we put $w = u^+ + k$ and

$$\varphi = \int_k^w (J'(t))^2 dt \tag{31}$$

Since $D\varphi = (J'(w))^2 Dw = (J'(w))^2 Du^+$ and $0 \leq \varphi \leq u^+(J'(w))^2$, we see that φ (and also $D\varphi$) vanishses wherever u^+ does. This implies $\varphi \in W_0(A, \theta, \Omega)$ and then φ is an admissible test function in (26). Putting φ into that equation and estimating, we easily get

$$\begin{aligned}\int_\Omega a_{ij}(J'(w))^2 D_i w D_j w\, dx \leq & \int_\Omega (|a| + |b|)u^+(J'(w))^2|Dw|\, dx + \int_\Omega |c|(u^+J'(w))^2\, dx \\ & \int_\Omega |f|(J'(w))^2|Dw|\, dx + \int_\Omega |g|u^+(J'(w))^2\, dx\end{aligned} \tag{32}$$

Let I_1, I_2, I_3, I_4 denote the integrals on the right side of (32). Using the Young inequality and (H.2), for any positive $\epsilon > 0$, we see that

$$\begin{aligned} I_1 &\leq \epsilon\int_\Omega \lambda(J'(w))^2|Dw|^2\, dx + C(\epsilon)\int_\Omega \theta v^2\, dx \\ I_2 &\leq \int_\Omega \theta v^2\, dx \end{aligned}$$

where $v = u^+J'(w)$. Because $w \geq k$, we also have

$$\begin{aligned} I_3 &\leq \int_\Omega \frac{|f|}{k} w(J'(w))^2|Dw|\, dx \leq \epsilon\int_\Omega \lambda(J'(w))^2|Dw|^2\, dx + C(\epsilon)\int_\Omega h(wJ'(w))^2\, dx \\ I_4 &\leq \int_\Omega \frac{|g|}{k} wu^+(J'(w))^2\, dx \leq \int_\Omega \theta v^2\, dx + \int_\Omega h(wJ'(w))^2\, dx \end{aligned}$$

where $h = k^{-2}(|f|^2\lambda^{-1} + |g|^2\theta^{-1})$. Using these estimates, (H.1), (32) and choosing ϵ small enough, we obtain

$$\int_\Omega \lambda|D(J(w))|^2\, dx \leq \int_\Omega \lambda(J'(w))^2|D(w)|^2\, dx \leq C\{\int_\Omega \theta v^2\, dx + \int_\Omega h(wJ'(w))^2\, dx \tag{33}$$

By theorem 7.8 in [GT], we know that v is weakly differentiable and

$$|Dv| \leq |Du^+|J'(w) + u^+J''(w)|Dw| \leq 2\beta J'(w)|Dw|$$

and then $v \in W_0(A, \theta, \Omega)$. By remark 1.4 of the previous section, we can find a positive number $q = q(n, t, r, \alpha, \gamma)$ and positive constant C_1 depending only on the global quantities such that for any ϵ

$$\int_\Omega \theta v^2\, dx \leq C_1\{\epsilon\int_\Omega \lambda(J'(w))^2|Dw|^2\, dx + C(\epsilon)\beta^q\int_\Omega v^2\, dx\} \tag{34}$$

From (33),(34) with ϵ small and by replacing v by the larger quantity $wJ'(w)$, we then have

$$\int_\Omega \lambda|D(J(w))|^2\, dx \leq C_2\beta^q\int_\Omega (h+1)(wJ'(w))^2\, dx \tag{35}$$

Since $J(w) \in W_0(A, \theta, \Omega)$, we can apply lemma 4 and the Holder inequality to get

$$\begin{aligned} \|J(w)\|^2_{L^{t_*}(\Omega)} &\leq C(n, t, \alpha)\|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(\Omega)}\int_\Omega \lambda|D(J(w))|^2\, dx \\ &\leq C_3\|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(\Omega)}\beta^q(\|h\|_{L^s(\Omega)} + 1)\|wJ'(w)\|^2_{L^{s_*}(\Omega)} \end{aligned} \tag{36}$$

where $\frac{1}{t_*} = \frac{1}{2}(1 + \frac{1}{t}) - \frac{1}{n} + \frac{\alpha}{2n} < \frac{1}{s_*} = \frac{1}{2}(1 - \frac{1}{s})$. Let us recall the definition of k and note that (36) still holds in case $f, g, k \equiv 0$ in these arguments. Letting $N \to \infty$ in the definition of J, we then have

$$\|w^\beta - k^\beta\|_{L^{t_*}(\Omega)} \leq C_4\beta^\tau\|w^\beta\|_{L^{s_*}(\Omega)} \tag{37}$$

where $\tau = (q+2)/2$ and C_4 does not depend on k, β. Moreover, since $k \leq w$, (37) and the Minkowsky inequality will give

$$\|w^\beta\|_{L^{t_*}(\Omega)} \leq C_5\beta^\tau\|w^\beta\|_{L^{s_*}(\Omega)}$$

or

$$\|w\|_{L^{\beta\chi s_*}(\Omega)} \leq (C_5\beta^\tau)^{1/\beta}\|w\|_{L^{\beta s_*}(\Omega)} \tag{38}$$

where $\chi = t_*/s_*$. Note that the norms in (37) and then in (38) initially are finite only for $\beta = 1$. However, by induction, we see that they will be finite for $\beta = \chi^m$ for $m = 1, 2, \ldots$. Let us infinitely iterate on (38) to get

$$\sup_{\Omega} w \leq C_5^{\sigma_1} \chi^{\sigma_2} \|w\|_{L^{s_*}(\Omega)} \tag{39}$$

where $\sigma_1 = \sum_{m=1}^{\infty} \chi^{-m}, \sigma_2 = \tau \sum_{m=1}^{\infty} m\chi^{-m}$ are finite constants. Finally by the interpolation inequality and the defintion of w, we obtain (28) for subsolutions. In case of supersolutions, we need only replace u by $-u$ in the arguments. Our proof is complete.

Q.E.D.

Let us now discard the assumption $u \leq 0 (u \geq 0)$ on the boundary. To do this, we must slightly weaken (H.2) as follows

(H.2)' Let us assume as in (H.2) but the conditions on a_i, c will now be: $|a|^2\lambda^{-1}$ and $|c|^2\theta^{-1} \in L^s(\Omega)$

Then we have

THEOREM 2 *Let us suppose (H.1), (H.2)' and u is a weak subsolution (supersolution) of (22) and $\sup_{\partial\Omega} |u|$ is finite. Then we have*

$$\sup_{\Omega} u(-u) \leq C\{\|u\|_{L^1(\Omega)} + \sup_{\partial\Omega} |u| + \bar{k}(\Omega)\} \tag{40}$$

where C depends on the global quantities as in theorem 1 and

$$\bar{k}(\Omega) = k(\Omega) + \sup_{\partial\Omega} |u| (\||a|^2\lambda^{-1}\|_{L^s(\Omega)} + \||c|^2\theta^{-1}\|_{L^s(\Omega)})^{1/2} \tag{41}$$

PROOF: For any finite number M, the function $u - M$ will satisfy the corresponding inequality (26) of u with f_i, g are respectively replaced by $\bar{f}_i = f_i + Ma_i, \bar{g} = g - Mc$. So if u is weak subsolution (resp. supersolution), we can consider the function $u - M$ where $M = \sup_{\partial\Omega} |u| (resp. -\sup_{\partial\Omega} |u|)$. This function obviously satisfies the requirement on the boundary of theorem 1. Repeating the argument of theorem 1 (with a little change in estimating the terms involving a_i, c) we easily get (28) with $u, k(\Omega)$ are respectively replaced by $u - M, \bar{k}(\Omega)$. Our desired estimate (40) now follows.

Q.E.D.

If u is a weak solution of (22) then it is both a subsolution and supersolution. Therefore, we get the global boundedness as follows.

THEOREM 3 *Let us suppose (H.1), (H.2)' and u is a weak solution of (22) and $\sup_{\partial\Omega} |u|$ is finite. Then we have*

$$\sup_{\Omega} |u| \leq C\{\|u\|_{L^1(\Omega)} + \sup_{\partial\Omega} |u| + \bar{k}(\Omega)\} \tag{42}$$

where $C, \bar{k}(\Omega)$ are given as in theorem 2.

One important corollary of the above estimates is the general weak maximum principle for subsolution of (22) from which we can derive a uniqueness theorem for the Dirichlet problem for (22). To this end, we consider the following condition on a_i and c.

$$\int_{\Omega} \{a_i D_i \eta + c\eta\}\, dx \geq 0 \tag{43}$$

for all non negative $\eta \in W_0(A, \theta, \Omega)$. We also require that $W_0(A, \theta, \Omega)$ be stable under multiplications. Such a special cas has been considered in proposition 3. Namely, we have

THEOREM 4 *Let (H.1), (H.2) and (43) be satisfied. We also assume that $\Lambda \in L^s(\Omega)$. Then if u is a weak subsolution (supersolution) of (22) we have*

$$\sup_{\Omega} u(-u) \leq \sup_{\partial\Omega} u^+(u^-) + Ck(\Omega) \tag{44}$$

where $C, k(\Omega)$ as in theorem 1.

PROOF: Let u be a weak subsolution and $M_0 = \sup_{\partial\Omega} u^+$. Since (43) holds, we easily see that $u - M_0$ satisfies (26) with the same f and g. So we can assume without loss of generality that $M_0 = 0$, or in other words, $u^+ \in W_0(A, \theta, \Omega)$. Morever, if $\phi \in W_0(A, \theta, \Omega)$ such that $\phi u \geq 0$ then $\phi u \in W_0(A, \theta, \Omega)$ and $D(\phi u) = \phi Du + uD\phi$. Using (43) with $\eta = \phi u$ and eliminating the non negative terms, we have

$$\int_{\Omega} \{a_{ij} D_j u D_i \phi + (a_i + b_i)\phi D_i u\}\, dx \leq \int_{\Omega} \{f_i D_i \phi + g\phi\}\, dx \tag{45}$$

Let us assume $k > 0$. We put $M = \sup_{\Omega} u^+$ and

$$N(u) = M - u^+ + k, \text{ and } w = \log \frac{M + k}{N(u)}$$

then $w \geq 0$ in Ω, $w = 0$ on $\partial\Omega$. Since $Dw = \dfrac{Du^+}{N(u)}$ and $N(u) \geq k$, we see that $w \in W_0(A, \theta, \Omega)$. We propose to show that w is also a subsolution of an equation of the form (22) which, in fact, satisfies the conditions of theorem 1. Let $\eta \in W_0(A, \theta, \Omega)$ satisfying $\eta \geq 0$ and $supp(\eta) \subset supp(w) = supp(u^+)$. We have $\phi = \dfrac{\eta}{N(u)}$ is a valid test function and $\phi u \geq 0$ in Ω. Therefore, (45) and then the Young inequality will give

$$\begin{aligned}
&\int_{\Omega} \{a_{ij} D_j w D_i \eta + \eta a_{ij} D_j w D_i w + (a_i + b_i)\eta D_i w\}\, dx \\
&\quad \leq \int_{\Omega} \{\frac{\eta|g|}{N(u)} + \frac{f_i D_i \eta}{N(u)} + \frac{\eta f_i D_i w}{N(u)}\}\, dx \\
&\quad \leq \int_{\Omega} \left\{\left(|g|k^{-1} + \frac{1}{2}|f|^2\lambda^{-1}k^{-2}\right)\eta + \frac{f_i D_i \eta}{N(u)}\right\}\, dx + \frac{1}{2}\int_{\Omega} \lambda\eta|Dw|^2\, dx
\end{aligned}$$

By (H.1), we then get

$$\int_{\Omega} \{a_{ij} D_j w D_i \eta + (a_i + b_i)\eta D_i w\}\, dx \leq \int_{\Omega} (\hat{g}\eta + \hat{f}_i D_i \eta)\, dx \tag{46}$$

where $\hat{g} = |g|k^{-1} + \frac{1}{2}|f|^2\lambda^{-1}k^{-2}, \hat{f}_i = \dfrac{f_i}{N(u)}$. It is easy to see that the coefficients of (46) satisfy (H.1), (H.2). So we can apply theorem 1 to this equation, bearing in mind the definition of k, to get an estimate for w as follows

$$\log\left(\frac{M+k}{k}\right) \le \sup_\Omega w \le C_1(\|w\|_{L^1(\Omega)} + 1) \tag{47}$$

Now we want to discard the term $\|w\|_{L^1(\Omega)}$ in (47). The function $\phi = \dfrac{u^+}{N(u)}$ is a valid test function in $W_0(A,\theta,\Omega)$ and

$$D\phi = \frac{(M+k)Du^+}{N(u)^2} = \frac{(M+k)Dw}{N(u)}$$

Then (45) again gives

$$\int_\Omega a_{ij}D_jwD_iw\ dx \le \frac{1}{M+k}\int_\Omega \{(|a|+|b|)u^+|Dw| + \frac{|g|u^+}{N(u)} + \frac{(M+k)}{N(u)}|f||Dw|\}\ dx$$

Put $v = u^+/(M+k)$. By the Young inequality and (H.1),(H.2) we get

$$\int_\Omega \lambda|Dw|^2\ dx \le C\int_\Omega (\theta v^2 + h)\ dx + \frac{1}{2}\int_\Omega \lambda|Dw|^2\ dx \tag{48}$$

where $h = (|g|^2\theta^{-1} + |f|^2\lambda^{-1})/N(u)^2 \le \bar{h} = (|g|^2\theta^{-1} + |f|^2\lambda^{-1})/k^2$. By noticing that $|v| \le 1$, $N(0) \ge N(u) \ge 0$ and $|Dv| = \dfrac{N(u)|Dw|}{N(0)} \le |Dw|$, we can use lemma 6 with $\eta \equiv 1, \beta = 1$ and $\epsilon > 0$ to get

$$\int_\Omega \theta v^2\ dx \le C_3\{\epsilon\int_\Omega \lambda|Dw|^2\ dx + C(\epsilon)\}$$

By choosing ϵ small, this estimate, (48), the Holder inequality and the definition of k, we get

$$\int_\Omega \lambda|Dw|^2\ dx \le C_4\int_\Omega (h+1)\ dx \le C_5$$

Using lemma 4 and this result, we have

$$\|w\|^2_{L^1(\Omega)} \le C(|\Omega|)\|w\|^2_{L^{t_*}(\Omega)} \le C_6$$

From (47), we thus obtain $\dfrac{M+k}{k} \le C_7$ and then a bound for M. When $f,g,k \equiv 0$, we can let $k \to 0$ in the foregoing argument to get the result. The proof is complete.

Q.E.D.

We now establish the coercivity property of L. Let us put

$$\kappa = \left\{C\|\theta\ d_\Omega^{-\gamma}\|_{L^s(\Omega)}\|\lambda^{-1}\ d_\Omega^{\alpha}\|_{L^t(\Omega)}\right\}^{1/2} \tag{49}$$

where C is the constant appears in (6) of lemma 3. The following two lemmas can be proved easily by using the Holder inequality and lemma 3

LEMMA 7 *Let (H.1), (H.2) be satisfied. Then the operator (21) satisfy*

$$(Lu,u) \ge (1 - 2\kappa - \kappa^2)\|u\|_*^2$$

for any $u \in W_0(A,\theta,\Omega)$ and $\|.\|_*$ is as in proposition 1.

LEMMA 8 *Let f_i, g satisfy (H.2). The linear functional*

$$T\phi = \int_\Omega \{f_iD_i\phi + g\phi\}\ dx$$

belongs to $W_0^*(A,\theta,\Omega)$, the dual space of $W_0(A,\theta,\Omega)$.

Using lemmas (7),(8) and the Lax-Milgram theorem in the Hilbert space $W(A,\theta,\Omega)$ (see, for example, [D0]) we get the following existence theorem for the Dirichlet problem.

THEOREM 5 *Let (H.1), (H.2) be satisfied. Suppose that for κ given by (49) we have*

$$(1 - 2\kappa - \kappa^2) > 0$$

Then for any $\varphi \in W(A,\theta,\Omega)$ there exists a unique weak solution of the following problem.

$$\begin{cases} Lu = -D_if_i + g & in\ \Omega \\ u = \varphi & on\ \partial\Omega \end{cases}$$

However, we have the following compactness result

PROPOSITION 5 *Under the conditions (H.1), (H.2) the space $W_0(A,\theta,\Omega)$ can be compactly imbedded into $L^2(\Omega,\theta)$ where $L^p(\Omega,\theta)$ consists of the measurable functions on Ω with finite norm*

$$\|u\|_{L^p(\Omega,\theta)} = \left(\int_\Omega |u|^p\theta\ dx\right)^{1/p}$$

PROOF: At first, we prove that $W_0(A,\theta,\Omega)$ is compactly imbedded into $L^1(\Omega)$. Let K be a bounded subset of $W_0(A,\theta,\Omega)$, we must show that K is a precompact subset of $L^1(\Omega)$. For every $u \in K$, by the Holder inequality and ii) of lemma 4, we get

$$\int_S |u|\ dx \le |S|^{1-\frac{1}{t_*}}\left(\int_S |u|^{t_*}\ dx\right)^{1/t_*} \le C(K)|S|^{1-1/t_*} \tag{50}$$

where S is any measurable subset of Ω and $|S|$ is its Lebesgue measure. So that, if we put $\Omega_j = \{x \in \Omega : d_\Omega(x) > 2^{-j}\}$ for any $j \in N$, then

(*) $\forall\epsilon > 0, \exists j$ such that $\int_{\Omega\setminus\Omega_j} |u|\ dx < \epsilon$ for every $u \in K$.

Fixing $j \in N$ and taking any $\epsilon > 0$ to be given, for any $\delta > 0$ and $u \in K$, we put

$$G_\delta = \{x \in \Omega_j : d(x, \partial\Omega_j) > \delta\}.$$

$$\hat{u}(x) = \begin{cases} u(x) & \text{for } x \in \Omega_j \\ 0 & \text{for } x \notin \Omega_j \end{cases}$$

Then for any $h \in R^n$ such that $0 < |h| < \delta < 2^{-j}$ and δ is sufficiently small, say $\delta < \delta_1(\epsilon, K)$, we get from (50) that

$$\int_{\Omega\setminus G_\delta} |u|\, dx \text{ and } \int_{\Omega\setminus G_\delta} |\bar{u}(x+h) - \bar{u}(x)|\, dx < \epsilon/2 \tag{51}$$

Moreover, since $\Omega_j \subset\subset \Omega$, $\lambda^{-1} \in L^1(\Omega_j)$, we have

$$\begin{aligned} \textstyle\int_{G_\delta} |\bar{u}(x+h) - \bar{u}(x)|\, dx &\le |h| \textstyle\int_{G_\delta}\int_0^1 |D(u(x+th)|\, dt dx \quad \le \delta \textstyle\int_{\Omega_j} |Du|\, dx \\ &\le \delta \left(\textstyle\int_{\Omega_j} \lambda^{-1}\, dx\right)^{1/2} \left(\textstyle\int_{\Omega_j} |Du|^2 \lambda\, dx\right)^{1/2} \le C(K,j)\delta \\ &< \epsilon/2 \end{aligned} \tag{52}$$

if δ is again small, say $\delta < \delta_2(\epsilon, K, j)$. From (51),(52) and theorem 2.21 of [AD] we conclude that

(**) The natural imbedding $W_0(A, \theta, \Omega) \to L^1(\Omega_j)$ is compact for every $j \in N$

Consequently, (*), (**) and theorem 2.22 of [AD] assert that $W_0(A, \theta, \Omega)$ is compactly imbedded into $L^1(\Omega)$. Because $W_0(A, \theta, \Omega)$ is continuously imbedded into $L^{t_*}(\Omega)$, by interpolation method, we can assert that the natural imbedding $W_0(A, \theta, \Omega) \to L^q(\Omega)$ is compact for every $q \in [1, t_*)$, and a fortiori, $q = 2 \le s_* < t_*$. By lemma 6 and remark 1.4 (without condition on Λ), we see that $W_0(A, \theta, \Omega)$ is compactly imbedded into $L^2(\Omega, \theta)$.

Q.E.D.

Arguing as in [TR] (p.277), we get the following Fredholm alternative for the operator $L_\sigma u = Lu + \sigma\theta u$

THEOREM 6 *Suppose (H.1),(H.2). Then there exists a countable, isolated set of real number Σ such that if $\sigma \in \Sigma$, the operator $L_\sigma u = Lu + \sigma\theta u$ is a bijective mapping from $W_0(A, \theta, \Omega)$ to $W_0^*(A, \theta, \Omega)$. For $\sigma \in \Sigma$, the null spaces of L_σ and its formal adjoint L_σ^* are of positive, finite dimension and the range of L_σ is orthogonal complement of the null space of L_σ^*.*

Moreover, if (43) holds, we have $\Sigma \subset (0, \infty)$.

4 LOCAL ESTIMATES

In this section we study the local boundedness of weak solutions of (22). But for the local estimates, some control on Λ seems to be necessary. It should guarantee that the spaces W will be stable under multiplications with smooth functions to obtain various local test functions of the equation (26). Therefore, together with the conditions (H.1) and (H.2) of section 3, we will add the following one

(H.3) With s as in (24) of (H.2), we assume that $\Lambda \in L^s(\Omega)$.

Then under (H.1), (H.2) and (H.3), we will establish the local estimates of weak solutions in $W(A, \theta, \Omega)$. We remark that our assumptions on the coefficients make the equation (22) be very singular and degenerate near the boundary $\partial\Omega$. We can compare them with those considered in [TR], [GT], [CT] to see that our class of equations is larger than those in the paper cited. However, we will still be able to prove the results derived there. In fact the conditions (H.2), (H.3) may be stated in a more general fashion (see remarks following theorem 3.1)

Hereafter, we will denote by $B_R(x_0)$ the ball of radius R, centered at $x_0 \in R^n$. In dealing with the concentric balls and their center x_0 being assummed to be understood we will simply write $B_R(x_0) = B_R$. We also write $\Omega_R = \Omega \cap B_R$, $\partial\Omega_R = \partial\Omega \cap B_R$.

For any subset S of Ω such that $|S| > 0$, we make use of the denotations

$$\begin{aligned} \lambda(S) &= \|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(S)} \\ \Lambda(S) &= |S|^{-(\frac{1}{s} + \frac{1}{t}) - \frac{\alpha}{n}} \|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(S)} \|\Lambda\|_{L^s(S)} \end{aligned} \tag{53}$$

As before we denote by t_*, s_* the reals given by $\frac{1}{t_*} = \frac{1}{2}(1 + \frac{1}{t}) - \frac{1}{n} + \frac{\alpha}{2n}$ and $\frac{1}{s_*} = \frac{1}{2}(1 - \frac{1}{s})$. For reals $p > 0$, and u is a measurable function on Ω, let us put

$$\Phi(u, p, S) = \left\{ \frac{1}{|S|} \int_S |u|^p\, dx \right\}^{1/p} \tag{54}$$

It is well known that (see [GT])

$$\lim_{p\to\infty} \Phi(u, p, S) = \sup_S |u| \tag{55}$$

Furthermore, in our theorems and the proofs, we will frequently refer to the global quantities given in (27) of the previous section.

Now we have the local version of the theorem 1 as follows

THEOREM 7 *Let u be a weak subsolution (supersolution) of (22). Suppose (H.1), (H.2), (H.3) are satisfied and $u \le 0 (u \ge 0)$ on $\partial\Omega$. Then for any ball $B_{2R}(x_0)$ in R^n such that $x_0 \in \bar{\Omega}$ and $p \ge s_*$, we have*

$$\sup_{\Omega_R} u(-u) \le C\{\Phi(u^+(u^-), p, \Omega_{2R}) + k(\Omega_{2R})\} \tag{56}$$

where

$$k(\Omega_{2R}) = R^{n(\frac{1}{s_*} - \frac{1}{t_*})} \{\lambda(\Omega)(\||f|^2\lambda^{-1}\|_{L^s(\Omega)} + \||g|^2\theta^{-1}\|_{L^s(\Omega)})\}^{1/2} \tag{57}$$

and the constant C depends only on the global quantities of (27) and $\Lambda(\Omega_{2R})$.

PROOF: At first, we suppose that $k = k(\Omega_{2R}) > 0$ and u is a weak subsolution. For any reals $\beta \ge 1$ and $N > k$, let us use again the special function J defined in the proof of theorem 1 and then put

$$w = u^+ + k;\ \phi(w) = \int_k^w (J'(t))^2 dt;\ \zeta = \eta^2\phi$$

where η is a nonnegative function in $C_0^1(B_{2R})$ and $\eta \equiv 1$ in B_R. Since

$$0 \le \phi \le (w - k)(J'(w))^2 = u^+(J'(w))^2, \qquad D\phi = (J'(w))^2 Dw = (J'(w))^2 Du^+$$

we see that ϕ (and $D\phi$) vanishes on $\partial\Omega$ and then $\zeta \in W_0(A, \theta, \Omega)$. We can thus substitute ζ into (26) to get the following estimate

$$\int_\Omega a_{ij}\eta^2(J'(w))^2 D_j w D_i w\, dx$$
$$\le -2\int_\Omega a_{ij}\eta\phi D_j w D_i\eta\, dx + 2\int_\Omega |a|\eta|D\eta|(u^+J'(w))^2\, dx$$
$$+\int_\Omega (|a|+|b|)\eta^2 u^+(J'(w))^2|Dw|\, dx + \int_\Omega |c|\eta^2(u^+J'(w))^2\, dx$$
$$+\int_\Omega |f|\eta|D\eta|u^+(J'(w))^2\, dx + \int_\Omega |f|\eta^2(J'(w))^2|Dw|\, dx + \int_\Omega |g|\eta^2 u^+(J'(w))^2\, dx \tag{58}$$

Let us denote by I_0 the integral on the left side and by $I_1, \dots, I_7$ the integrals on the right side. We denote

$$U = u^+J'(w); \quad and \quad W = wJ'(w).$$

By (H.2), the Young inequality and the fact that $\phi \le w(J'(w))^2$, we can find a constant C such that

$$\begin{aligned} |I_1| &\le \left|\int_\Omega \phi(wJ'(w))^2)^{-1}2a_{ij}(\eta J'(w)D_j w)(wJ'(w)D_i\eta)\, dx\right| \\ &\le \tfrac{1}{2}\int_\Omega a_{ij}\eta^2(J'(w))^2 D_i w D_j w\, dx + C\int_\Omega a_{ij}W^2 D_i\eta D_j\eta\, dx \end{aligned}$$

Because $U \le W$ and $k \le w$ and $\lambda \le \Lambda$, for any $\epsilon > 0$, we can apply the Young inequality again to estimate the integral I_2, I_3, I_5, I_6, I_7 in a standard way and then use the condition (H.2) to obtain the following estimate from (58)

$$\begin{aligned} \tfrac{1}{2}\int_\Omega a_{ij}\eta^2(J'(w))^2 D_i w D_j w\, dx &\le 2\epsilon\int_\Omega \lambda\eta^2(J'(w))^2|Dw|^2\, dx + C\int_\Omega \Lambda|D\eta|^2 W^2\, dx + \\ &\quad +C(\epsilon)\{\int_\Omega \theta\eta^2 U^2\, dx + k^{-2}\int_\Omega h\eta^2 W^2\, dx\} \end{aligned}$$

where $h = |f|^2\lambda^{-1} + |g|^2\theta^{-1} \in L^s(\Omega)$ and C is still a general constant. Using the ellipticity of (H.1) and choose small enough, we get

$$\int_\Omega \lambda(\eta J'(w))^2|Dw|^2\, dx \le C\{\int_\Omega \Lambda|D\eta|^2 W^2\, dx + \int_\Omega \theta\eta^2 U^2\, dx + \int_\Omega k^{-2}h\eta^2 W^2\, dx\} \tag{59}$$

We now consider the second term on the right side of (59). Since $U \in W_0(A, \theta, \Omega)$ and $|DU| \le 2\beta J'(w)|Dw|$, for any positive ϵ, we can apply lemma 6 to obtain

$$\int_\Omega \theta\eta^2 U^2\, dx \le C_1\{\epsilon\int_\Omega \lambda\eta^2(J'(w))^2|Dw|^2\, dx + C(\epsilon)\beta^\tau\int_\Omega \eta^2 U^2\, dx + \int_\Omega \lambda|D\eta|^2 U^2\, dx\} \tag{60}$$

for some τ, C_1 depend only on the global quantitites of (27). Substituting (60) into (59) and choose ϵ small, we then get

$$\int_\Omega \lambda\eta^2(J'(w))^2|Dw|^2\, dx \le C_2\beta^\tau\int_\Omega (\Lambda|D\eta|^2 + k^{-2}h\eta^2 + \eta^2)W^2\, dx \tag{61}$$

Let us write Ω_η for the support of η in Ω and apply lemma 4 for the function $\eta J(w)$, which readily belongs to $W_0(A, \theta, \Omega)$ (even in the case B_{2R} is not contained in Ω), and then we get

$$\begin{aligned} \Phi^2(\eta J(w), t_*, \Omega_\eta) &\le C|\Omega_\eta|^{-2/t_*}\lambda(\Omega_\eta)\int_\Omega (\lambda\eta^2(J'(w))^2|Dw|^2 + \lambda|D\eta|^2(J(w))^2)\, dx \\ &\le C_3|\Omega_\eta|^{-2/t_*}\lambda(\Omega_\eta)\beta^\tau\int_\Omega (\Lambda|D\eta|^2 + k^{-2}h\eta^2 + \eta^2)(wJ'(w))^2\, dx \end{aligned} \tag{62}$$

Letting $N \to \infty$ and using the Holder inequality to estimate the terms on the right side of (62) (in recalling the definitions of $t_*, s_*, \Lambda(\Omega_R), k$) and the fact that $|\Omega_R| \simeq R^n$ (see remark 3.3 next to this proof), we thus obtain

$$\Phi^2(\eta(w^\beta - k^\beta), t_*, \Omega_\eta) \le C_3C_4\beta^{\tau+2}\Phi^2(w^\beta, s_*, \Omega_\eta) \tag{63}$$

we see that C_3 depends only on the global quantities. While for C_4, we have

$$C_4 \le R^2\sup|D\eta|^2\Lambda(\Omega_{2R}) + 1 + |\Omega_\eta|^{2(\frac{1}{s_*}-\frac{1}{t_*})} \le R^2\sup|D\eta|^2\Lambda(\Omega_{2R}) + C(|\Omega|) \tag{64}$$

because $t_* > s_*$. Now we specify the cut-off function η as follows. Let m_1, m_2 be such that $1 \le m_1 < m_2 \le 2$ and set $\eta \equiv 1$ in B_{m_1R}, $\eta \equiv 0$ in $\Omega\backslash B_{m_2R}$ with $|D\eta| \le 1/(m_2 - m_1)R$. Then from (63),(64) we get

$$\Phi(\eta(w^\beta - k^\beta), t_*, \Omega_{m_1R}) \le \frac{C_5\beta^{\bar\tau}}{(m_2 - m_1)}\Phi(w^\beta, s_*, \Omega_{m_2R})$$

where $\tau = (\tau+2)/2$ and C_5 are the constants depending only on the quantities stated in the theorem. Since $k \le w$, we also have

$$\Phi(w^\beta, t_*, \Omega_{m_1R}) \le \Phi(\eta(w^\beta - k^\beta), t_*, \Omega_{m_1R}) + k^\beta \le \frac{C_6\beta^{\bar\tau}}{m_2 - m_1}\Phi(w^\beta, s_*, \Omega_{m_2R})$$

Put $\chi = t_*/s_* > 1$. For $q \ge s_*$, we can write the above estimate as

$$\Phi(w, \chi q, \Omega_{m_1R}) \le \left(\frac{C_7q^{\bar\tau}}{m_2 - m_1}\right)^{s_*/q}\Phi(w, q, \Omega_{m_2R}) \tag{65}$$

Taking any $p \ge s_*$, we set $q_i = \chi^i p$, $m_i = 1 + 2^{-i}$ for $i = 0, 1, 2, \dots$ and inductively, we have (65) holds for every i. So that we can infinitely it rate on (65) to get

$$\Phi(w, \infty, \Omega_R) \le C_8^{\sigma_1}(2\chi^{\bar\tau})^{\sigma_2}\Phi(w, p, \Omega_{2R})$$

where, as before, $\sigma_1 = \sum\chi^{-i}$, $\sigma_2 = \sum i\chi^{-i}$. By (55), we get the theorem when $k > 0$. The case $k = 0$ can be proved similarly by letting $k \to 0$. Finally, replacing u by $-u$ in the argument, we get the result for supersolutions.

Q.E.D.

REMARK 3.1: In checking the estimates of the proof, we see that the function $a_i, b_i, c, f_i, g, \Lambda$ may need not to share the same control numbers r, s, γ and the function θ. In fact, we can state (H.2) and (H.3) in a more general form as follows

(H.3)' There exists reals $\alpha, t, s_i, r_j, \gamma_j$ and the nonnegative measurable functions θ_j for $i = 1,2,3$ and $j = 1,2,3,4$ such that $r_j, s_i, t \in (1,\infty]$, $\alpha \in [0,2)\setminus\{1\}$ and

$$\alpha - 2 + n\left(\frac{1}{s_i} + \frac{1}{t}\right) < 0$$

$$\alpha - 2 + n\left(\frac{1}{r_j} + \frac{1}{t}\right) < \gamma_j \le 0$$

and

i/ $\lambda^{-1}\, d_\Omega^\alpha \in L^t(\Omega)$, $\theta_j d_\Omega^{-\gamma_j} \in L^{r_j}(\Omega)$ and $\Lambda \in L^{s_1}(\Omega)$.

ii/ $|a|^2\lambda^{-1}, |b|^2\lambda^{-1}, |c|$ are respectively bounded by $\theta_1, \theta_2, \theta_3$.

iii/ $|f|^2\lambda^{-1} \in L^{s_2}(\Omega)$ and $g^2\theta_4^{-1} \in L^{s_3}(\Omega)$.

Actually, we need only consiuer a candidate θ in (60) and for s_* in (63) we should take

$$\frac{1}{s_*} = \frac{1}{2}(1 - \frac{1}{\min s_i})$$

REMARK 3.2: If $\gamma_4 = 0$, we can choose $\theta_4 = g$ to see that the condition on g becomes $g \in L^{s_3}(\Omega)$, which is the usual condition considered in [LU],[TR]. The equations studied in these papers are just the special case of (H.3)' when $\alpha = \gamma_j = 0$. In general, our assumptions obviously violate those of [LU],[TR]. For example, we can take $\alpha = 0$, $r, s, t = \infty$ and $\gamma \in (-2,-1]$, then $a_i, b_i, c \in L^\infty_{loc}(\Omega)$ and $|a|^2, |b|^2, |c| = O(d_\Omega^\gamma)$ near $\partial\Omega$. The methods used in these papers can not work in this situation.

REMARK 3.3: The result of theorem 7 gives both local estimates inside Ω and near $\partial\Omega$. We have just required that the ball $B_{2R}(x_0)$ must have its center lying in $\bar{\Omega}$). In deriving (63) and the estimate of the constant there, we have implicitly used the fact that Ω_R can be replaced by R^n. But we know this is the case, since $\partial\Omega$ is of class C^2 and then Ω satisfies a *uniform interior cone condition* on $\partial\Omega$(see for example [Kuf]). If $B_{2R} \subset \Omega$, we do not need the assumption $u \le (\ge) 0$ on the boundary. When $B_{2R} \not\subset \Omega$, we only require that $u^+(u^-)$ be zero on $\partial\Omega_{2R} = \partial\Omega \cap B_{2R}$, the part of $\partial\Omega$ lying inside B_{2R}.

As in section 3, to discard the assumption of u on the boundary, we should replace (H.2) by (H.2)' and then by similar argument as in theorem 2 we get

THEOREM 8 *Suppose (H.1), (H.2)' and (H.3). Let u be a weak solution(supersolution) of (22). Assume that for some portion Γ of $\partial\Omega$, we have $\sup_\Gamma |u|$ is finite. Then for any ball $B_{2R}(x_0)$ having its center lies in $\bar{\Omega}$ and not intersecting $\partial\Omega\setminus\Gamma$, we have*

$$\sup_{\Omega_R} u(-u) \le C\{\Phi(u^+(u^-), p, \Omega_{2R}) + \sup_\Gamma |u| + \bar{k}(\Omega_{2R})\} \tag{66}$$

where $p \ge s_*$, C depends only on the global quantities and now

$$\bar{k}(\Omega_{2R}) = k(\Omega_{2R}) + R^{n(\frac{1}{s_*} - \frac{1}{s_*})} \sup_\Gamma |u| \{\lambda(\Omega)(\||a|^2\lambda^{-1}\|_{L^s(\Omega)} + \||c|^2\theta^{-1}\|_{L^s(\Omega)})\}^{1/2} \tag{67}$$

Chaining together a finite sequence of balls that cover an open subset of Ω, we get the local boundedness of weak subsolutions, solutions of (22) as follows

THEOREM 9 *Suppose as in theorem 8. Let u be a weak subsolution (solution) of (22). Then for any open subset Ω' of Ω that lies at a positive distance from $\partial\Omega\setminus\Gamma$, we can majorize $\sup_{\Omega'} u$ ($\sup_{\Omega'} |u|$) by some positive constant that depends only on the global quantities of (27), $\sup_\Gamma |u|$, $\|u\|_{L^{s_*}(\Omega)}$, $\Lambda(\Omega)$, $\bar{k}(\Omega)$, Ω' and the distance $dist(\Omega', \partial\Omega\setminus\Gamma)$.*

5 HOLDER REGULARITY

In this last section we consider the Holder continuity of weak bounded solutions of (22). Let U be such a solution and assume that we already know the quantity $M_0 = \sup_\Omega |u|$. As we have seen in theorem 8, in order to get such a bound for $|u|$ on Ω, we must consider a somewhat weaker condition (H.2)' instead of (H.2). Moreover, the condtion (H.3) is also required so that we can use the local test functions. Nevertheless, in considering the Holderness of solutions, we could not hope to have just such general assumptions. A new condition seems necessary to weaken the non-uniformity of the equation (22). Therefore, besides the ellipticity as in (H.1) we will modified (H.2), (H.3) and hereafter assume that

(H.4) There exists real numbers r, s, t, α, γ and a nonnegative measurable function θ such that $r, s, t \in (1,\infty]$, $\alpha \in [0,2)\setminus\{1\}$ and

$$\alpha - 2 + n\left(\frac{1}{s} + \frac{1}{t}\right) < 0 \tag{68}$$

$$\alpha - 2 + n\left(\frac{1}{r} + \frac{1}{t}\right) < \gamma \le 0 \tag{69}$$

and

i/ $\lambda^{-1}\, d_\Omega^\alpha \in L^t(\Omega)$, $\Lambda d_\Omega^{-\alpha} \in L^s(\Omega)$ and $\theta\, d_\Omega^{-\gamma} \in L^s(\Omega)$.

ii/ $|a|^2\lambda^{-1}, |f|^2\lambda^{-1}, |g|^2\theta^{-1}$ and $|c|^2\theta^{-1}$ belong to $L^s(\Omega)$

iii/ $|b|^2\lambda^{-1} \le \theta$

For any measurable subset S of Ω such that $|S| > 0$, we put

$$\Lambda^*(S) = |S|^{-(\frac{1}{s} + \frac{1}{t})} \|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(S)} \|\Lambda d_\Omega^{-\alpha}\|_{L^s(S)}$$

and assume that

$$\Lambda^* = \sup_{|S|>0} \Lambda^*(S) < \infty \tag{70}$$

For each equation of the form (22) and satisfied (H.4), we will associate the following quantities

$$\begin{aligned} k(L) &= \left\{\|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(\Omega)}(\||a|^2\lambda^{-1}\|_s + \||f|^2\lambda^{-1}\|_s + \||c|^2\theta^{-1}\|_s + \||g|^2\theta^{-1}\|_s)\right\}^{1/2} \\ k(L, \Omega_R) &= R^{\tau_0} k(L) \end{aligned} \tag{71}$$

where $\|.\|_s$ denotes the norm in $L^s(\Omega)$, $\Omega_R \neq \emptyset$ and $\tau_0 = n(\frac{1}{s_*} - \frac{1}{t_*}) > 0$. We again refer repeatedly to the global quantities stated in (27) of section 3.

At first, let us prove a weaker version of theorem 7 but it is appropriate for the purpose of this section

PROPOSITION 6 *The estimate (56) of theorem 7 still holds with $k(\Omega)$ now given by $k(L)$ of (71) and the constant C now depending on the global quantities of (27) and $\Lambda^*(\Omega_{2R})$.*

PROOF: We need only adjust the proof of theorem 7 by considering the following two cases according to the position of the center of the ball $B_{2R}(x_0)$ as follows

i/ $d_\Omega(x_0) \leq 4R$: We then have $d_\Omega(x) < 6R$ for any point $x \in \Omega_{2R}$. Since $\alpha \geq 0$,

$$|\Omega_{2R}|^{-\alpha/n}\|\Lambda\|_{L^t(\Omega_{2R})} \leq CR^{-\alpha}\|\Lambda\|_{L^t(\Omega_{2R})} \leq C6^{-\alpha}\|\Lambda d_\Omega^{-\alpha}\|_{L^t(\Omega_{2R})}$$

and hence $\Lambda(\Omega_{2R}) \leq C(\Omega)\Lambda^*(\Omega_{2R})$.

ii/ $d_\Omega(x_0) > 4R$: We can find a number $m \geq 2$ such that $mR \leq d_\Omega(x) \leq (m+4)R$ for any x in Ω_{2R}. Repeating the proof of theorem 7 until (61), then we can apply lemma 4 with $\alpha = 0$ and t'_* given by $\frac{1}{t'_*} = \frac{1}{2}(1 + \frac{1}{t}) - \frac{1}{n}$. So that

$$\Phi^2(\eta J(w), t'_*, \Omega_\eta) \leq C|\Omega_\eta|^{1+\frac{1}{t}-\frac{2}{n}}\|\lambda^{-1}\|_{L^t(\Omega_\eta)}\int_\Omega \lambda|D(\eta J(w))|^2\, dx$$

Since $(m+4)/m \leq 3$ and

$$\|\lambda^{-1}\|_{L^t(\Omega_\eta)}\int_\Omega \Lambda|D\eta|^2(wJ'(w))^2\, dx \leq (\frac{m+4}{m})^\alpha\|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(\Omega_\eta)}\int_\Omega \Lambda d_\Omega^{-\alpha}|D\eta|^2(wJ'(w))^2\, dx$$

$$\|\lambda^{-1}\|_{L^t(\Omega_\eta)} \leq R^{-\alpha}\|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(\Omega_\eta)} \leq C|\Omega_\eta|^{-\alpha/n}\|\lambda^{-1}\, d_\Omega^\alpha\|_{L^t(\Omega_\eta)}$$

we then see that the estimate (64) of the constant of (63) still holds with $\Lambda(\Omega_{2R})$ now is replaced by $\Lambda^*(\Omega_{2R})$. Moreover, because the new t'_* is still greater than s_*, the proof can go on as before.

From these considerations we get the proposition.

Q.E.D.

Now we can state the main results of this section as follows

THEOREM 10 *If u is a weak solution of (22) and (H.1), (H.4) hold then u is locally Holder continuous in Ω. For any ball $B_{R_0} \subset \Omega$ and $R \leq R_0$, we have*

$$osc_{B_R} u \leq CR^\sigma(R_0^{-\sigma}M_0 + k(L)) \tag{72}$$

where C, σ are positive constants which depend only on the global quantities and Λ^*, M_0.

THEOREM 11 *If u is a weak solution of (22) and (H.1), (H.4) hold then u is locally Holder continuous up to the boundary. For any $x \in \partial\Omega$ and for any $R_0 > 0$ and $R \leq R_0$, we have*

$$osc_{\Omega_R} u \leq C\{R^\sigma(R_0^{-\sigma}M_0 + k(L)) + \hat{\omega}(R^\mu R_0^{1-\mu})\} \tag{73}$$

where $\hat{\omega}(R) = osc_{\partial\Omega_R} u$ and C, σ, μ are positive constants depending on the quantities stated in theorem 68.

Having theorems 10,11 and arguing as in theorem 8.29 of [GT] we can assert the Holderness of weak solution as follows

THEOREM 12 *Let the operator L satisfy (H.1) and (H.4) and u be a bounded weak solution of (22). Suppose that there exist positive constants K, σ_0 such that for any x in a boundary portion Γ and $R > 0$ such that for $\partial\Omega_R = B_R \cap \partial\Omega \subset \Gamma$, we have*

$$osc_{\partial\Omega_R} u \leq KR^{\sigma_0}$$

It follows that for any subdomain Ω' strictly contained in $\Omega \cup \Gamma$, we have the estimate

$$\|u\|_{C^\sigma(\Omega')} \leq C\{M_0 + K + k(L)\} \tag{74}$$

where C, σ are positive constants depending on K, σ_0 and $d' = dist(\Omega', \partial\Omega\backslash\Gamma)$, the global quantities and Λ^*, M_0.

Moreover if $\Gamma = \partial\Omega$, we can take $\Omega' = \Omega$ and $d' = diam(\Omega)$

To prove theorems 10,11 we will use the following well known result (see for example [LU], [GT]).

LEMMA 9 *Let ω, π be non decreasing functions on an interval $(0, R_0]$ satisfying, for all $R \leq R_0$, the inequality*

$$\omega(\tau R) \leq \beta\omega(R) + \pi(R) \tag{75}$$

for some β, τ in $(0,1)$. Then for any $\mu \in (0,1)$ and $R \leq R_0$, we can find positive constants $C = C(\beta, \tau)$, $\sigma = (1-\mu)\log\beta/\log\tau$ such that

$$\omega(R) \leq C\{R^\sigma R_0^{-\sigma}\omega(R_0) + \pi(R^\mu R_0^{1-\mu})\} \tag{76}$$

Moreover, in proving (72),(73) we can assume without loss of generality that $R \leq R_0/4$ (otherwise, we can take $C = 2.4^\sigma$). Thus for the proof of (72) we will also assume that B_R and B_{4R} are both contained in Ω.

We put $M_1 = \sup_{\Omega_R} u$, $m_1 = \inf_{\Omega_R} u$, $M_4 = \sup_{\Omega_{4R}} u$, $m_4 = \inf_{\Omega_{4R}} u$, $M_\partial = \sup_{\partial\Omega_{4R}} u$, $m_\partial = \inf_{\partial\Omega_{4R}} u$, and

$$\omega(\Omega_R) = \sup_{\Omega_R} u - \inf_{\Omega_R} u, \qquad \omega(\partial\Omega_R) = \sup_{\partial\Omega_R} u - \inf_{\partial\Omega_R} u$$

It is easy to see that (72),(73) are respectively the results of applications of lemma 9 if we are able to prove that

$$\omega(B_{4R}) \le C\{\omega(B_{4R}) - \omega(B_R) + k(L, B_{4R})\} \tag{77}$$

$$\omega(\Omega_{4R}) \le C\{\omega(\Omega_{4R}) - \omega(\Omega_R) + k(L, \Omega_{4R}) + \omega(\partial\Omega_{4R})\} \tag{78}$$

where C is a positive constant depends only on the quantities stated in our theorems but not on R. We need only apply lemma 9 with $\beta = 1 - C^{-1}$, $\tau = 1/4$ and $\sigma = (1-\mu)\log\beta/\log\tau < \mu\tau_0$, where τ_0 is given after (71).

In proving (77), let us put $k = k(L, \Omega_{4R})$ and consider the two special functions w_1, w_2 defined on B_{4R} as

$$\begin{aligned} w_1 &= \log\left(\frac{M_4 - m_4 + k}{2(M_4 - u) + k}\right) \\ w_2 &= \log\left(\frac{M_4 - m_4 + k}{2(u - m_4) + k}\right) \end{aligned}$$

and for (78),

$$\begin{aligned} w_3 &= \log\left(\frac{M_4 - m_4 + k}{M_4 - m_4 + k - (u - M_\partial)^+}\right) \\ w_4 &= \log\left(\frac{M_4 - m_4 + k}{M_4 - m_4 + k - (u - M_\partial)}\right) \end{aligned}$$

which are defined on Ω_{4R}. From now on, we shall assume that $k > 0$. The case $k = 0$ can be treated by the same arguments and then let $k \to 0$.

If we can find a positive constant C_1 not depending on R such that

$$\sup_{B_R} w_1 \qquad or \qquad \sup_{B_R} w_2 \le C_1 \tag{79}$$

$$\sup_{\Omega_R} w_3 \le C_1 \tag{80}$$

then since $\sup_{\Omega_R} w_4 \le \sup_{\Omega_R} w_3$, $M_1 \le M_4$, $m_1 \ge m_4$, $m_1 \ge m_\partial$, we easily see that (79),(80) respectively imply (77) and (78).

Therefore, in the rest of this section, our goal is to find an *uniform bound from above* (not depending on R) of w_1 (or w_2) and w_3 in Ω_R. This may be derived through proposition 6 as follows: At first, we will prove that the functions w_i are subsolutions of some equations that satisfy the same structure conditions as (22). So that, we can apply the results of section 4 to bound the supremum norms of w_i by the L_p norms. Finally, we give the uniform esmates of L_p norms for some p big enough.

Thus, we first prove

PROPOSITION 7 *The function w_1, w_2, w_3 are weak subsolutions of the equations $\tilde{L}(w) = 0$ in Ω_{4R}, which satisfy the conditions (H.1) and (H.4). Namely, we have*

$$\int_\Omega a_{ij} D_j w D_i\eta \, dx + \int_\Omega b_i \eta D_i w \, dx \le \int_\Omega \tilde{f}_i D_i \eta \, dx + \int_\Omega \tilde{g}\eta \, dx \tag{81}$$

for every η in $W_0(A, \theta, \Omega_{4R})$.

Moreover, $\tilde{f}, \tilde{g}$ satisfy (H.4) and for the correponding quantity $k(\tilde{L}, \Omega_R)$, we have

$$k(\tilde{L}, \Omega_{2R}) \le C(M_0) \tag{82}$$

PROOF: We consider first the functions w_1, w_2. According to their definitions, we can write w_1, w_2 in the form

$$w = \log\frac{M'}{N(u)} \tag{83}$$

where M' is a positive constant, $N(u) = 2\beta(M - u) + k$ with $M - u \ge 0$, and β is respectively ± 1. Since $N(u) \ge k > 0$, w is bounded. Moreover, $Dw = 2\beta Du/N(u)$ implies that $w \in W(A, \theta, \Omega_{4R})$.

Let $\eta \in W_0(A, \theta, \Omega_{4R})$ and $\eta \ge 0$, we put $\phi = \dfrac{\beta\eta}{N(u)}$ then

$$D\phi = \left(\frac{D\eta}{N(u)} + \frac{2\beta\eta Du}{N(u)^2}\right)\beta = \frac{D\eta + \eta Dw}{N(u)}\beta$$

Since $\Lambda \in L^s(\Omega)$, we see that $\phi \in W_0(A, \theta, \Omega)$ and it is a valid test function of (26). On substitution into that integral form, we get

$$\begin{aligned} &\tfrac{1}{2}\left(\int_\Omega a_{ij} D_j w D_i \eta \, dx + \int_\Omega a_{ij}\eta D_j w D_i w \, dx + \int_\Omega b_i \eta D_i w \, dx\right) \\ &\qquad \le \beta\left(\int_\Omega \frac{(f_i - a_i u)}{N(u)} D_i\eta \, dx + \int_\Omega \frac{(f_i - a_i u)}{N(u)}\eta D_i w \, dx + \int_\Omega \frac{(g - cu)}{N(u)}\eta \, dx\right) \end{aligned}$$

By (H.1), the Young inequality and the fact that $|u| \le M_0$ and $N(u) \ge k > 0$, we obtain

$$\int_\Omega (a_{ij} D_j w D_i \eta + b_i \eta D_i w) \, dx + \int_\Omega \lambda\eta|Dw|^2 \, dx \le \int_\Omega (\tilde{f}_i D_i \eta + \tilde{g}\eta) \, dx + \frac{1}{2}\int_\Omega \lambda\eta|Dw|^2 \, dx \tag{84}$$

where

$$\tilde{f}_i = \beta\frac{(f_i - a_i u)}{N(u)}, \qquad \tilde{g} = C(M_0)\left\{\frac{|a|^2 + |f|^2}{\lambda k^2} + \frac{|c| + |g|}{k}\right\}$$

Because $|\tilde{f}| \le C(M_0)(|a| + |f|)/k$, recalling the definition of $k(L, \Omega_{4R})$ we get (82).

For w_3, we notice that $u - M_\partial$ satisfies an equation of the form (22) with f_i, g respectively replaced by $f_i + M_\partial c$, $g - M_\partial c$, and $(u - M_\partial)^+$ vanishes on $\partial\Omega_{4R}$. On the other hand, it is not difficult to see that the above arguments still holds if we replace u by u^+, provided that $u^+ = 0$ on $\partial\Omega_{4R}$. Thus w_3 satisfies (81). Taking into account the considerations at the end of section 2, we also have $w_3 \in W_0(A, \theta, \Omega_{4R})$ and then w_3 is a weak subsolution. Finally, since $|M_\partial| \le M_0$, we also get (82). The proposition has been proved.

Q.E.D.

PROPOSITION 8 *For $i = 1, 2, 3$ and any $p \ge s_*$, there exist positive constants C_1, C_2 that depend only on the global quantities and M_0, Λ^* such that*

$$\sup_{\Omega_R} w_i \le C_1\Phi(w_i^+, p, \Omega_{2R}) + C_2 \tag{85}$$

PROOF: The operator $\tilde{L}$ of proposition 7 shares with the original operator L the same quantities $r, s, t, \alpha, \gamma, \lambda, \Lambda$ and θ. In addition, $\tilde{L}$ also satisfies (H.4). By remark 3.3, since w_3 vanishes on $\partial\Omega_{4R}$, the part of $\partial\Omega$ lying in B_{4R}, we see that the arguments in the proof of theorem 7 can be repeated here for w_i and the operator $\tilde{L}$ to get again the estimate

$$\sup_{\Omega_R} w_i \leq C_1\{\Phi(w_i^+, p, \Omega_{2R}) + k(\tilde{L}, \Omega_{2R})\} \tag{86}$$

where C_1, in fact, depends on the same global quantities of (27). Now (85) comes from (86) and (82).

Q.E.D.

In deriving the uniform bounds for $\Phi(w_i^+, p, \Omega_{4R})$, we need the following

PROPOSITION 9 *Let B_R be a ball in R^n and η be a non negative function in $C_0^1(B_R)$ we can find a constant C depends only on the global quantities and M_0 such that*

i) *If $B_R \subset \Omega$, $w = w_1$ or w_2 then*

$$\int_\Omega \lambda\eta^2|Dw|^2\,dx \leq C\left\{\int_\Omega \Lambda|D\eta|^2\,dx + \int_\Omega \theta\eta^2\,dx + \int_\Omega h\eta^2\,dx\right\} \tag{87}$$

ii) *If $\Omega_R \neq \emptyset$, $w = w_3$ then*

$$\int_\Omega \lambda\eta^2|Dw|^2\,dx \leq C\left\{\int_\Omega \Lambda|D\eta|^2\,dx + \int_\Omega h\eta^2\,dx + \int_\Omega \eta^2\,dx\right\} \tag{88}$$

where in both cases

$$h = \frac{|a|^2 + |f|^2}{k^2\lambda} + \frac{c^2 + g^2}{k^2\theta} \tag{89}$$

PROOF:

i) We use again the notations given at the beginning of the proof of proposition 7. Put $\phi = \eta^2/N(u)$, we see that ϕ is a valid test function in $W_0(A, \theta, \Omega)$ and

$$D\phi = \frac{2\eta D\eta}{N(u)} + \frac{2\beta\eta^2 Du}{N(u)^2} = \frac{2\eta D\eta + \eta^2 Dw}{N(u)}$$

then on substitution and noting the following

$$\begin{aligned}
\int_\Omega a_{ij}D_j u D_i\phi\,dx &= \tfrac{1}{2\beta}\left(\int_\Omega a_{ij}\eta^2 D_j w D_i w\,dx + \int_\Omega a_{ij}2\eta D_i\eta D_j w\,dx\right)\\
\left|\int_\Omega a_{ij}2\eta D_i\eta D_j w\,dx\right| &\leq \tfrac{1}{2}\int_\Omega a_{ij}\eta^2 D_i w D_j w\,dx + 2\int_\Omega \Lambda|D\eta|^2\,dx
\end{aligned}$$

and using the Young inequality, the condition (H.4) and the facts that $|u| \leq M_0$, $N(u) \geq k$ to estimate the other terms, we obtain

$$\int_\Omega \lambda\eta^2|Dw|^2\,dx \leq C(M_0)\{\int_\Omega \Lambda|D\eta|^2\,dx + \int_\Omega \theta\eta^2\,dx + \int_\Omega h\eta^2\,dx\}$$

where $h \in L^s(\Omega)$ given as in (89).

ii) Put $V = (u - M_\beta)^+$, $w = w_3$ and $\phi = \eta^2 V/N(V)$ where $N(V) = M_4 - m_4 - V + k$. Since $V = 0$ on $\partial\Omega_{4R}$, ϕ belongs to $W_0(A, \theta, \Omega)$ and

$$D\phi = \frac{2\eta V D\eta + N(0)\eta^2 Dw}{N(V)}, \qquad DV = N(V)Dw$$

where $N(0) = M_4 - m_\beta + k \geq k$. We know that $u - M_\beta$ satisfies (26) with f_i, g accordingly replaced by $\hat{f}_i = f_i + M_\beta a_i$, $\hat{g} = g - M_\beta c$. Then by substituting ϕ into the corresponding integral form and dividing it by $N(0)$ we get

$$\begin{aligned}
&\int_\Omega a_{ij}\eta^2 D_i w D_j w\,dx\\
&\quad\leq -2\int_\Omega a_{ij}\eta\frac{V}{N(0)}D_j w D_i\eta\,dx + \int_\Omega |a|\frac{V^2}{N(0)N(V)}\eta|D\eta|\,dx\\
&\quad+\int_\Omega |a|\eta^2\frac{V}{N(V)}|Dw|\,dx + \int_\Omega |b|\eta^2\frac{V}{N(0)}|Dw|\,dx + \int_\Omega |c|\eta^2\frac{V^2}{N(0)N(V)}\,dx\\
&\quad+\int_\Omega |\hat{f}|\frac{V}{N(0)N(V)}\eta|D\eta|\,dx + \int_\Omega |\hat{f}|\frac{\eta^2}{N(V)}|Dw|\,dx + \int_\Omega |\hat{g}|\eta^2\frac{V^2}{N(0)N(V)}\,dx
\end{aligned} \tag{90}$$

Let us denote the terms on the right side of the above estimate by $I_1, \ldots, I_8$. Because $N(0) \geq V$, we have

$$|I_1| \leq \frac{1}{2}\int_\Omega a_{ij}\eta^2 D_i w D_j w\,dx + 2\int_\Omega \Lambda|D\eta|^2\,dx$$

For any positive ϵ given, using the fact that $|V| \leq 2M_0$, $N(V) \geq k$, $N(0) \geq V$ and the Young inequality in a standard fashion by now, the term I_2, I_3, I_6, I_7 can be majorized by

$$\epsilon\int_\Omega \lambda\eta^2|Dw|^2\,dx + C(\epsilon, M_0)\int_\Omega \eta^2(|a|^2 + |f|^2)k^{-2}\lambda^{-1}\,dx + C\int_\Omega \Lambda|D\eta|^2\,dx$$

Putting $U = V/N(0)$, by assumption (H.4), we have

$$\begin{aligned}
I_4 &\leq \epsilon\int_\Omega \lambda\eta^2|Dw|^2\,dx + C(\epsilon)\int_\Omega \theta\eta^2 U^2\,dx\\
I_5 \text{ and } I_8 &\leq \tfrac{M_0^2}{k^2}\int_\Omega \eta^2(c^2 + g^2)\theta^{-1}\,dx + \int_\Omega \theta\eta^2 U^2\,dx
\end{aligned}$$

Choosing ϵ small in these estimates, (90) yields

$$\int_\Omega \lambda\eta^2|Dw|^2\,dx \leq C(M_0)\{\int_\Omega \Lambda|D\eta|^2\,dx + \int_\Omega h\eta^2\,dx\} + C\int_\Omega \theta\eta^2 U^2\,dx \tag{91}$$

where C is a general constant and h is as in part i). However, since $U = 0$ on $\partial\Omega_{4R}$, ηU belongs to $W_0(A, \theta, \Omega)$, we can apply lemma 6, and simultaneously notice that $U \leq 1$, $0 \leq N(V) \leq N(0)$, $|DU| = \dfrac{N(V)}{N(0)}|Dw| \leq |Dw|$, to get

$$\int_\Omega \theta(\eta U)^2\,dx \leq C_1\{\epsilon\int_\Omega \lambda\eta^2|Dw|^2\,dx + C(\epsilon)\int_\Omega \eta^2\,dx + \int_\Omega \lambda|D\eta|^2\,dx\} \tag{92}$$

Together with this estimate, (91) gives part ii) of the proposition.

Q.E.D.

Finnally, we are about to get the uniform bounds that are independent of R for the Lebesgue norms $\Phi(w_i^+, p, \Omega_{2R})$. Let η be a function in $C_0^1(B_{4R})$ having its value in $[0,1]$ such that $\eta \equiv 1$ in B_{2R}, $\eta \equiv 0$ outside B_{3R} and $|D\eta| \leq 1/R$.

At first, we consider w_1, w_2 in the ball $B_{4R} \subset \Omega$. It is easy to see that

$$w_1 \geq 0 \Leftrightarrow u \geq \frac{M_4 + m_4}{2} \Leftrightarrow w_2 \leq 0$$

So that, one of the function w_1^+, w_2^+ must be zero on a subset S of B_{2R} such that $|S| \geq \frac{1}{2}|B_{2R}|$. Let w be such function. By lemma 5 we have

$$\begin{aligned} \Phi^2(w, p, B_{2R}) &\leq C(n,t)R^{-2n/p}\|\lambda^{-1}\|_{L^t(B_{2R})}\int_{B_{2R}} \lambda|Dw|^2\, dx \\ &\leq C(n,t)R^{2-n-\frac{n}{t}}\|\lambda^{-1}\|_{L^t(B_{2R})}\int_{\Omega} \lambda\eta^2|Dw|^2\, dx \end{aligned}$$

where $\frac{1}{p} = \frac{1}{2}(1+\frac{1}{t}) - \frac{1}{n} < \frac{1}{s_*}$. Since $B_{4R} \subset \Omega$, we can find a real $m \geq 1$ such that $mR \leq d_\Omega(x) \leq (m+6)R$ for $x \in supp(\eta)$. Applying (87) and noting that

$$\begin{aligned} \|\lambda^{-1}\|_{L^t(B_{2R})}\int_{\Omega} \Lambda|D\eta|^2\, dx &\leq \left(\tfrac{m+6}{m}\right)^{\alpha} R^{-2}\|\lambda^{-1}\, d_\Omega^{\alpha}\|_{L^t(B_{3R})}\int_{B_{3R}} \Lambda d_\Omega^{-\alpha}\, dx \\ \|\lambda^{-1}\|_{L^t(B_{2R})}\int_{\Omega} \theta\eta^2\, dx &\leq R^{\gamma-\alpha}\|\lambda^{-1}\, d_\Omega^{\alpha}\|_{L^t(B_{3R})}\int_{B_{3R}} \theta\, d_\Omega^{-\gamma}\, dx \\ \|\lambda^{-1}\|_{L^t(B_{2R})}\int_{\Omega} h\eta^2\, dx &\leq R^{-\alpha}\|\lambda^{-1}\, d_\Omega^{\alpha}\|_{L^t(B_{3R})}\int_{B_{3R}} h\, dx \end{aligned}$$

we can then apply the Holder inequality to get (recalling the defintion of h, k)

$$\Phi^2(w, p, B_{2R}) \leq C_3(\Lambda^*(B_{3R}) + R^{\sigma_1} + R^{2 - n(\frac{1}{s} + \frac{1}{t}) - \alpha}) \tag{93}$$

where C_3 depends only on the global quantities and $\sigma_1 = \gamma + 2 - \alpha - n(\frac{1}{r} + \frac{1}{t}) > 0$. Thus, we obtain an uniform bound for $\Phi(w, p, B_{2R})$. Since $p > s_*$, theorem 10 follows from the proposition 8.

Now we consider $w = w_3$ defined on Ω_{4R}. Because $\partial\Omega_{4R}$ is of class C^2 and $w = 0$ there, we see that w belongs to $W_T(A, \theta, \Omega_{4R})$ with $T = \partial\Omega_{4R}$ and d_Ω now coincides with d_T given in the definition 1.2 (with some uniform fixed number τ in $(0,1)$ depending on the smoothness of $\partial\Omega$). Hence lemma 4 gives

$$\Phi^2(w, t_*, \Omega_{2R}) \leq C(n, \alpha, t, \tau)R^{-2n/t_*}\|\lambda^{-1}\, d_\Omega^{\alpha}\|_{L^t(\Omega_{2R})}\int_{\Omega_{2R}} \lambda|Dw|^2\, dx \tag{94}$$

where now $\frac{1}{t_*} = \frac{1}{2}(1+\frac{1}{t}) - \frac{1}{n} + \frac{\alpha}{2n} < \frac{1}{s_*}$.

Using (88) to majorized the right side of (94) we obtain

$$\Phi^2(w, t_*, \Omega_{2R}) \leq C_4 R^{2 - n - \frac{n}{t} - \alpha}\|\lambda^{-1}\, d_\Omega^{\alpha}\|_{L^t(\Omega_{2R})}\int_{\Omega_{3R}} (\Lambda|D\eta|^2 + h\eta^2 + \eta^2)\, dx \tag{95}$$

Since $d_\Omega(x) \leq 3R$ for $x \in \Omega_{3R}$, $|D\eta| \leq 1/R$, we have

$$\|\lambda^{-1}\, d_\Omega^{\alpha}\|_{L^t(\Omega_{3R})}\int_{\Omega_{3R}} \Lambda|D\eta|^2\, dx \leq 3^{\alpha} R^{\alpha-2}\|\lambda^{-1}\, d_\Omega^{\alpha}\|_{L^t(\Omega_{3R})}\int_{\Omega_{3R}} \Lambda d_\Omega^{-\alpha}\, dx$$

Using the Holder inequality for the other terms on the right side of (95) and taking into account the definitions of Λ^*, h, k and the conditions on α, γ in (H.4) we finally get a bound for $\Phi(w, t_*, \Omega_{2R})$ which depends only the global quantitites stated (but not on R). Theorem 11 now follows.

6 Ackrowledgement

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References

[AD] Adams,R.A : Sobolev spaces. NewYork, academic press 1975.

[D0] D.M.Duc : A class of strongly degenerate elliptic operators. Bull.Austral.Math.Soc. 39,1989, pp.177-200.

[D1] D.M.Duc : Nonlinear singular elliptic equations. J.London.Math.Soc. 2(40)1989, pp.420-440.

[Kuf] A.Kufner Weighted Sobolev spaces. John Wiley and Son, Chichester(1985)

[LU] O.A.Ladyzeuskaya and N.N.Ural'tseva: Linear and quasilinear elliptic equations. Academic press, NewYork (1968)

[GT] D.Gilbarg and N.S.Trudinger: Elliptic Partial differential equations of second order. Springer Verlag (1983)

[MS] J.K.Moser: On Harnack's theorem for elliptic differential equations. Comm.Pure.Appl.Math. Vol.14 (1961). 577-591.

[ST] G.Stampacchia: Le probleme de Dirichlet pour les equations elliptiques du second ordre a coefficient discontinus. Ann.Inst.Fourier, Vol.15, 1965, 189-258.

[TR] N.S.Trudinger: Linear elliptic operators with measurable coefficients. Ann. Scuola.Norm.Sup.Pisa (3) 27, 265-308(1973)