IC/92/412

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

BOUNDEDNESS FOR A SYSTEM OF REACTION-DUFFUSION EQUATIONS WITH MORE GENERAL ARRHENIUS TERM PART I

S.S. Okoya

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

MIRAMARE-TRIESTE

IC/92/412

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

BOUNDEDNESS FOR A SYSTEM OF REACTION–DIFFUSION EQUATIONS WITH MORE GENERAL ARRHENIUS TERM PART I

S.S. Okoya *
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

In this paper, we consider an extended model of a coupled nonlinear reaction–diffusion equation with Neumann–Neumann boundary conditions. We obtain upper linear growth bound for one of the components. We also find the corresponding bound for the case of Dirichlet–Dirichlet boundary conditions.

MIRAMARE – TRIESTE
November 1992

* Permanent address: Department of Mathematics, Obafemi Awolowo University, Ile Ife, Nigeria.

1 Introduction

In this note, we consider an extended model of a system of reaction-diffusion equations arising in the combustion theory given by

$$U_t - \alpha\triangle U = -Uf(V), \quad \text{in } \Omega\times(0,\infty) \qquad (1.1a)$$

$$V_t - \beta\triangle U - \delta\triangle V = QUf(V), \quad \text{in } \Omega\times(0,\infty) \qquad (1.1b)$$

with the Neumann-Neumann boundary conditions

$$\frac{\partial U}{\partial\eta} = \frac{\partial V}{\partial\eta} = 0, \quad \text{on } \partial\Omega\times(0,\infty) \qquad (1.2)$$

and the non-negative, bounded , uniformly continuous but arbitrary initial data

$$U(x,0) = U_0(x); V(x,0) = V_0(x) \quad \text{in } \Omega \qquad (1.3)$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, η is the outward normal to $\partial\Omega$, $\triangle$ is the Laplacian, Q is a positive constant and α, β, and δ are positive constants such that $\beta^2 \leq 4\alpha\delta$, which assures the parabolicity of the system. We assume that $f(V)$ is given by the more general Arrhenius rate law (see [4] and [6])

$$f(V) = KV^n \exp\left(\frac{-E}{RV}\right) \qquad (1.4)$$

where K is the pre-exponential factor, E is the activation energy, R is the gas constant. Typically, $n \in \{-2, 0, \frac{1}{2}\}$ for most practical reactions but for the first in the series of analyses we assume $n = 0$.

The global existence and unboundedness of the solution for the scalar equations with various source terms has been extensively discussed (see [2], and the literature cited therein). But for systems, the problem of unboundedness is receiving considerable attension and relatively little is known ([1], [3], [7] [8] [9] and [10]).

This model reduces to a particular system in [1] whenever $\beta = 0$. In fact they considered the Arrhenius rate law

$$f(V) = K \exp\left(\frac{-E}{RV}\right) \tag{1.5}$$

with the boundary condition principally taken has Dirichlet-Neumann. They obtain nonboundedness of the temperature V by establishing lower linear growth bounds to complement previously obtained upper linear growth bounds.

The model (1.1-1.3) with $\beta \neq 0$ has been studied in [8] for a general $f(V)$ such that $f(V)$ is a continuously differentiable non-negative function satisfying

$$\lim_{s\to\infty} \frac{1}{s}\log(1 + f(s)) = 0. \tag{1.6}$$

The paper establish the existence of unique strong global solutions for the behaviour of solutions for (1.1-1.3 and 1.6) and also investigated their large time behaviour.

The purpose of this paper is to establish upper growth bound for the component V subject to some boundary conditions for all $t > 0$ and for $n = 0$.

2 Main result

2.1 Existence of solutions

We can convert equations (1.1)-(1.4) to an abstract first order system in a Banach space $X = C(\overline{\Omega}) \times C(\overline{\Omega})$ of the form

$$\mathbf{U}'(t) = A\mathbf{U}(t) + B(\mathbf{U}(t)), \quad t > 0 \tag{2.1}$$

$$\mathbf{U}(0) = \mathbf{U}_0 \in X,$$

with

$$A\mathbf{U}(t) = (\alpha\Delta U(t), \beta\Delta U(t) + \delta\Delta V(t))^T, \tag{2.2}$$

$$B\mathbf{U}(t) = (-KQUV^n \exp(-E/RT), KQUV^n \exp(-E/RT))^T \tag{2.3}$$

In (2.1), $\mathbf{U} : R^+ \to X$

$$A : D_\infty(\Delta) \times D_\infty(\Delta) \to X \quad \text{and}$$

$$D_\infty(\Delta) = \{u \in W^{2,p}(\Omega) \ \text{ for all p > } \ n\ , \Delta u \in C(\overline{\Omega}), \frac{\partial u}{\partial \eta} = 0.\}$$

It is clear that the map B is locally Lipschitz in X. Therefore, from the general theory of semigroup we obtain the existence of a local solution which is regular. For the global existence we need the fact that the solutions are positive.

2.2 Positivity of U and V

lemma 1: Let (U,V) be the solution of (1.1)-(1.4). If $\phi_0 = V_0 - \frac{\beta}{\alpha-\delta}U_0$ is non-negative for $\alpha > \delta$ and $\frac{1}{|\Omega|}\int_\Omega \phi_0(\xi)d\xi = \rho^2 > 0$, then

$$V \geq \rho^2 \tag{2.4}$$

for all $t \geq 0$ and all $x \in \Omega$ where $|\Omega|$ denotes the measure of Ω.

First of all, note that the positivity and boundedness of U follows from equations (1.1a) and (1.3a) together with the maximum principle for parabolic equations i.e.

$$0 \leq U \leq \|U_0\|_\infty \tag{2.5}$$

where we denote by $\|.\|_\infty$ the supremum norm in $C(\bar{\Omega})$

Proof: let $\phi(x,t) = V - \epsilon U$ where ϵ is a positive constant to be chosen later. Multiplying (1.1a) by ϵ and subtracting the resulting equation from (1.1b) gives

$$\phi_t - \delta\Delta\phi = (Q+\epsilon)Uf(V), \quad \text{in} \quad \Omega \times (0,\infty) \tag{2.6}$$

with

$$\frac{\partial \phi}{\partial \eta} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty),$$

$$\phi(x,0) = V(x,0) - \epsilon U(x,0) = \phi_0(x) \geq 0 \quad \text{on} \quad \Omega \times (0,\infty),$$

where $\epsilon = \frac{\beta}{\alpha-\delta}$. The integral equation corresponding to (2.6) is

$$\phi = \int_\Omega N(x,t;\xi,0)\phi_0(\xi)d\xi + (Q+\epsilon)\int_0^t \int_\Omega N(x,t;\xi,\tau)Uf(\xi,\tau)d\xi d\tau \tag{2.7}$$

where N is the fundamental solution with the Neumann condition. Since $U \geq 0$ and $f(\xi,\tau) \geq 0$ then it is clear that

$$\begin{aligned} \phi &\geq \int_\Omega N(x,t,\xi,0)\phi_0(\xi)d\xi \\ &\geq \tfrac{1}{|\Omega|}\int_\Omega \phi_0(\xi)d\xi = \rho^2 > 0. \end{aligned} \tag{2.8}$$

To obtain (2.8b) we simply consider the expansion of the right hand side of (2.8a) in terms of the eigenfunctions and eigenvalues. We observed that the first term in the expansion is ρ^2 corresponding to the first eigenfunction with

zero eigenvalue and the sum of the subsequent terms is positive. Hence

$$V - \frac{\beta}{\alpha-\delta}U \geq \rho^2$$

and since $U \geq 0$ with $\alpha > \delta$ it follows that

$$V \geq \rho^2. \tag{2.9}$$

We shall now assert that U decays uniformly to zero in the Lemma that follows.

<u>lemma 2</u> :Let U and V satisfy (1.1) - (1.4) and Suppose that $\kappa = f(\rho^2)$. Then for $U_0 \in C(\bar{\Omega})$ and $U_0 \geq 0$ U eventually decays to zero exponentially, that is

$$\|U\|_\infty \leq \|U_0\|_\infty \exp(-\kappa t).$$

Proof: Since $V(x,t) \geq \rho^2$ for all $t > 0$ and all $x \in \Omega$ and f(V) is strictly increasing in V for $V > 0$, we thus have that

$$f(V(x,t)) \geq f(\rho^2) = \kappa \tag{2.10}$$

for all $t > 0$ and $x \in \Omega$.

Then from (1.1a) and (2.10) U satisfies the differential inequality

$$U_t - \alpha\Delta U \leq -\kappa U \quad \text{in} \quad \Omega \times (0,\infty), \tag{2.11}$$

$$\frac{\partial U}{\partial \eta} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty),$$

$$U(x,0) = U_0(x) \quad \text{in} \quad \Omega.$$

Define $\overline{U} = U\exp(\kappa t)$,then $\overline{U}$ satisfies

$$\overline{U_t} - \alpha\Delta\overline{U} \leq 0 \quad \text{in} \quad \Omega \times (0,\infty),$$

$$\frac{\partial \overline{U}}{\partial \eta} = 0 \text{ on } \Omega,$$

$$\overline{U}(x,0) = U_0(x) \text{ in } \Omega.$$

From the maximum principle for parabolic equations $\overline{U} \leq U_0$ and it follows that

$$\|U(x,t)\|_\infty \leq \|U_0\|_\infty \exp(-\kappa t). \tag{2.12}$$

The Theorem to be stated below shows that V is uniformly bounded for all t.

2.3 Uniform Boundedness of V

Theorem 1: Consider solutions (U,V) of (1.1)-(1.4) and let $U_0, V_0 \in C(\overline{\Omega})$. Then there exists a constant $G > 0$ such that

$$\|V\|_\infty \leq G$$

for all $t > 0$.

Proof: Consider equation (2.7) in the form

$$V - \epsilon U = \int_\Omega N(x,t;\xi,0)(V_0 - \epsilon U_0)d\xi + (Q+\epsilon)\int_0^t \int_\Omega N(x,t;\xi,\tau)Uf(\xi,\tau)d\xi d\tau.$$

It follows that

$$\|V - \epsilon U\|_\infty \leq \int_\Omega \|N\|_\infty \|V_0 - \epsilon U_0\|_\infty d\xi + (Q+\epsilon)\int_0^t \int_\Omega \|N\|_\infty \|Uf(\xi,\tau)\|_\infty d\xi d\tau. \tag{2.13}$$

From equation (2.12) together with the relation

$$\|f(V)\|_\infty \leq 1$$

we obtain

$$\|Uf(V)\|_\infty \leq \|U_0\|_\infty \exp(-\kappa t). \tag{2.14}$$

Combining (2.13) with (2.14) we obtain

$$\begin{aligned} \|V - \epsilon U\|_\infty &\leq \|V_0 - \epsilon U_0\|_\infty + \kappa^{-1}(Q+\epsilon)\|U_0\|_\infty(1 - \exp(-\kappa t)), \\ &\leq \|V_0 - \epsilon U_0\|_\infty + \kappa^{-1}(Q+\epsilon)\|U_0\|_\infty. \end{aligned} \tag{2.15}$$

Hence

$$\|V\|_\infty \leq \|V_0 - \epsilon U_0\| + \epsilon\|U\| + \kappa^{-1}(Q+\epsilon)\|U_0\|_\infty.$$

Thus the result is established with G taken as the right hand side.

Remarks

(a) The above result can be suitably modified when the boundary conditions are Dirichlet-Dirichlet. Briefly, for this case, let $\varphi(x)$ be the solution of the problem

$$-\Delta\varphi = \mu\varphi \text{ in } \Omega$$

$$\varphi = 0 \text{ on } \partial\Omega.$$

Let μ_1 and φ_1 be the first nonzero eigenvalue and the correponding eigenvector, then φ_1 does not change sign in Ω and so it can be chosen to be positive. Furthermore it is bounded in Ω. Since U satisfies the differential inequality $U_t - \alpha\Delta U = -Uf(V) \leq 0$ then the function $\hat{U} = C_0 \exp(-\mu_1 t)\varphi_1$ satisfy

$$\hat{U}_t - \alpha\Delta\hat{U} = 0, \text{ in } \Omega$$

$$U_0(x) \leq \hat{U}(0,x) = C_0, \text{ in } \Omega$$

and therefore dominate U for C_0 big enough such that $\|U_0(x)\|_\infty \leq C_0\|\varphi_1(x)\|_\infty$ by the comparison theorem for parabolic equations. Hence

$$\|U\|_\infty \leq \|\hat{U}(t)\|_\infty \leq C_1 \exp(-\mu_1 t) \to 0$$

as $t \to \infty$ where $C_1 = C_0 \|\varphi_1\|_\infty$. Hence the main theorem holds true when equation (1.2) is replaced by Dirichlet-Dirichlet boundary conditions with G given as

$$G = \|V_0 - \epsilon U\|_\infty + \epsilon \|U\|_\infty + \mu^{-1} C_1 (Q + \epsilon),$$

while in the body of Theorem 1 the appropriate fundamental solution is equipped with the Dirichlet condition.

(b) The condition $\beta^2 < 4\alpha\delta$ arises from the parabolicity of the system (1.1) and since this results are true for the case $\beta = 0$, our model is an extension of earlier works. Notice that with $\beta = 0$, α and δ represent the mass and the thermal conductivities respectively.

(c) If we replace the coefficient of $f(V)$ by a more general form U^m (where m is a positive integer) our results hold true.

(d) The case of $n \neq 0$ is the subject of another paper.

Acknowledgement

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

References

(1) J. D. Avrin and M. Kirane. *Temperature growth and temperature bounds in special cases of combustion models.* To appear in Applicable Analysis.

(2) J. Bebernes and D. Eberly. *Mathematical problems from combustion.* Applied Math. Sci. Series. Vol. 83, (Springer-Verlag, New York, 1989).

(3) J. Bebernes and A. Lacey. *Finite time blow up for semilinear reaction-diffusion systems.* Journal of differential equations. Vol. 95, No.1 (1992),

(4) T. Boddington, C. Feng and P. Gray. *Thermal explosions, criticality and the disappearance of criticality in systems with distributed temperatures 1. Arbitrary Biot number and general reaction rate laws.* Proc. R. Soc. London. A390, (1983), 247-264.

(5) J. Buckmaster and G. S. S. Ludford. Theory of Laminar flames, (Cambridge University Press, 1982).

(6) F. S. Dainton. Chain reactions: An introduction. (Wiley Press, New York, 1966).

(7) S. L. Hollis, R. H. Martin and M. Pierre. *Global existence and boundedness in reaction-diffusion systems.* SIAM Journal of Math. Analysis. 18, (1987), 744-761.

(8) M. Kirane. *Global bounds and Asymptotics for a system of reaction-diffusion equations.* Journal of Mathematical Analysis and Applications. 138, (1989), 328-342.

(9) M. Kirane. Unbounded solutions for the Brusselator. ICTP Preprint IC/ 92 /30, (1992).

(10) S. S. Okoya. A mathematical model for explosions with chain branching and chain breaking kinetics. Ph.D. thesis, Obafemi Awolowo University, Nigeria (1989)

(11) M. H. Protter and H. F. Weinberger. Maximum principles in differential equations. (Prentice-Hall Englewood Cliffs, N. J. 1967).

(12) F. Williams. Combustion theory, (Second Edition, Adison-Wesley, 1985).

35K57 Partial differential equations;

Parabolic equations and systems;

Reaction-diffusion equations.