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ABSTRACT

The purpose of this paper is to investigate algebraic conditions which give information about the controllability of invariant control systems on nilpotent Lie groups. With the same purpose, the authors in [1] use the co-adjoint representation and define the concept of symplectic vectors. We study the existence of these objects to analyze the controllability. In particular, we obtain a characterization when G is simply connected.

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1. INTRODUCTION

The purpose of this paper is to find algebraic conditions which give information about the controllability of invariant control systems on Lie groups. These systems, of type $\Sigma = (G, D)$ are important because of their theoretical interest and their applications, [3], [6], and are determined by the specification of the following data:

$$\dot{g} = X(g) + \sum_{j=1}^{k} \mu_j Y^j(g)$$

$g \in G$, X, $Y^j \in \tilde{g}$, $1 \leq j \leq k$, where G is a connected Lie group, $\tilde{g}$ is the Lie algebra of G and $X, Y^1, \ldots, Y^k$ are left-invariant vector fields which satisfy the rank condition

$$\mathrm{span}_{\mathcal{L}.A}\{X, Y^1, \ldots, Y^k\} = \tilde{g} .$$

The class of admissible control $\mathcal{U} = \mathcal{U}(k)$, is the piecewise constant functions

$$\mu : [0, \infty) \to \mathbf{R}^k$$

and $\mathcal{D}$ is a family of vector fields associated with Σ, i.e.

$$\mathcal{D} = \{X + \sum_{j=1}^{k} \mu_j Y^j | \mu \in \mathcal{U}\} .$$

For each $Z \in \mathcal{D}$, we denote by $(Z_t)_{t\in\mathbf{R}}$ the 1-parameter group of diffeomorphisms on G generated by the vector field Z.

The rank condition, obviously necessary, is very natural [8] and secures the transitivity of Σ on G, i.e. the action of the group of diffeomorphisms

$$G_\Sigma = \{Z^1_{t_1} \circ Z^2_{t_2} \circ \ldots \circ Z^\nu_{t_\nu} | Z^j \in \mathcal{D}, t_j \in \mathbb{R}, \nu \in \mathbb{N}\}$$

is transitive on some $g \in G$, and consequently on all elements of G. In this context, the concept of controllability of the system Σ from the initial condition $g \in G$ can be established.

Under which circumstances is the semigroup

$$S_\Sigma = \{Z^1_{t_1} \circ Z^2_{t_2} \circ \ldots \circ Z^\nu_{t_\nu} | Z^j \in \mathcal{D}, t_j \geq 0, \nu \in \mathbb{N}\}$$

transitive on G?

That is:

In which cases on g, G and $\mathcal{D}$ do we have

$$S_\Sigma(g) = G \ ?$$

In the first place, the invariance of the vector fields in $\mathcal{D}$ show that if e denotes the neutral element of G, then for each $g \in G$

$$S_\Sigma(g) = g \cdot S_\Sigma(e)$$

In particular, we can concentrate on the positive orbit $S_\Sigma(e)$.

Many people have dealt with this problem under various assumptions on G and $\mathcal{D}$, [2], [4], [5], [6], [7].

In [1], the following idea is give: "If there is a function $f : G \to \mathbb{R}$ strictly increasing on the positive trajectories, i.e., on each $\varphi \in S_\Sigma$, Σ cannot be controllable".

To develop this idea the authors make use of co-adjoint representation of $\tilde{g}$, define the concept of symplectic vector for invariant vector fields and obtain a necessary condition for the controllability of Σ.

In Sec. 2 we focus the study on searching for algebraic conditions on $\tilde{g}$, which warrant the existence of symplectic vectors. An important class of systems adequate for this situation are the so-called nilpotent systems, i.e., invariant systems on nilpotent Lie groups. In Sec. 3 we analize the controllability of these systems. In Sec. 4 we compute an example on the Heisenberg group.

1.1 MAIN RESULTS

I. Existence of symplectic vectors

Theorem 2.2 Let $\tilde{h}$ be an ideal de $\tilde{g}$ such that $\tilde{g}/\tilde{h}$ is not an Abelian algebra. If $\pi : \tilde{g} \to \tilde{g}/\tilde{h}$ is the canonic projection and there is $Z \in \tilde{g}$ such that $\pi(Z) \in Z(\tilde{g}/\tilde{h})$ is non-null, then there is a symplectic vector λ for Z. ◇

II. Controllability

Let $\Sigma = (G, D)$ be a nilpotent system

$$\dot{g} = X(\tilde{g}) + \sum_{j=1}^{k} \mu_j Y^j(\tilde{g})$$

we denote by

$$Z(\tilde{g}) = \text{the centre of the Lie algebra } \tilde{g} \text{ of the group } G$$
$$\tilde{h} = \text{span}_{\mathcal{L}.A}\{Y^1, Y^2, \ldots, Y^k\}$$
$$Zt(\tilde{h}) = \bigcap_{j=1}^{k} \ker d\rho(Y^j) = \text{the centralizer of } \tilde{h}$$

and

$$H = \text{the connected Lie subgroup of } G \text{ with Lie algebra } \tilde{h}.$$

Theorem 3.1. If $Z(\tilde{g}) \subsetneq Zt(\tilde{h})$ then Σ cannot be controllable. ◇

Theorem 3.6 If G is a simply connected Lie groups, then Σ is controllable $\Leftrightarrow$ $\tilde{h} = \tilde{g}$ ◇

Moreover, we give in proposition 3.3 an equivalent result with Theorem 7.3 in [5] and Proposition 3.4 asserts this results for nilpotent systems.

2. EXISTENCE OF SYMPLECTIC VECTORS

Let G be a connected Lie group with Lie algebra $\tilde{g}$. The adjoint representation ρ of G is the homomorphism

$$\rho : G \to Aut(\tilde{g})$$

defined via the differential in the neutral element $e \in G$ of the interior automorphisms

$$i_g : G \to G, \quad i_g(h) = g \cdot h \cdot g^{-1} \ .$$

The derivative $d\rho$ in each $W \in g$ is

$$d\rho(\omega)(\cdot) = [w, \cdot] \ .$$

The co-adjoint representation, denoted by ρ^*, is the linear representation of G in the dual space $\tilde{g}^*$ (contragradient of ρ) and may be obtained by the action

$$G \times \tilde{g}^* \to \tilde{g}^*, \quad (g, \lambda) \to \rho^*(g)(\lambda)$$

as follows:

$$\rho^*(g)\lambda)(w) = \lambda(\rho(g^{-1})w), w \in \tilde{g}$$

such that its derivative evolves on elements $\lambda \in \tilde{g}^*$ by bracket evaluation, i.e.

$$d\rho^*(w) * (\lambda)(\xi) = \lambda[w, \xi], \xi \in \tilde{g} \ .$$

The stabilizer of λ by the action of ρ^*

$$E_\lambda = \{g \in G | \rho^*(g)(\lambda) = \lambda\}$$

is a closed Lie subgroup of G with Lie algebra

$$e_\lambda = \{w \in \tilde{g} | d\rho^*(\omega)(\lambda) = 0\}$$

It is well known that, for each $\lambda \in \tilde{g}^*$ the orbit of λ by ρ^*

$$\theta_\lambda = \rho^*(G)(\lambda)$$

is a submanifold of $\tilde{g}^*$, of even dimension, diffeomorphic to the homogeneous space G/E_λ and such that the tangent space in λ is

$$\lambda + d\rho^*(\lambda) \simeq \tilde{g}/e_\lambda, [9] \ .$$

If $\Sigma = (G, \mathcal{D})$ is an invariant system, ρ^* induces a system

$$\rho^*(\Sigma) =: (\tilde{g}^*, d\rho^*(\mathcal{D}))$$

where

$$d\rho^*(\mathcal{D}) = \{d\rho^*(Z)/Z \in \mathcal{D}\}$$

and for each initial condition $\lambda \in \tilde{g}$ the system

1. $\Sigma_\lambda =: (G/E_\lambda, d\pi(\mathcal{D}))$,
   where $\pi : G \to G/E_\lambda$ is the canonic projection, and the system.
2. $\rho^*(\Sigma)_\lambda$, i.e. $\rho^*(\Sigma)$ restricted to the orbit θ_λ, are equivalent by ρ^*, i.e. the dynamics for these systems is the same for each admissible control $u \in \mathcal{U}$.

In [1] the authors give the following definition "$\lambda \in \tilde{g}^*$ is a symplectic vector for $\omega \in \tilde{g}$ if the co-adjoint orbit θ_λ is not trivial and $\beta(W) > 0$, $\forall \beta \in \theta_\lambda$.

The authors make use of this definition on invariant systems $\Sigma = (G, D)$

$$\dot{g} = X(\tilde{g}) + \sum_{j=1}^{k} \mu_j Y^j(\tilde{g})$$

and obtain a necessary condition for the controllability of this type of systems via the following result:

"If there is a vector field ξ belonging to the centralizer of the subalgebra $\tilde{h}$ such that the non-null vector field $Z = [X, \xi]$ has a symplectic vector, then Σ cannot be controllable".

In fact, the existence of a symplectic vector λ f. z permits us to construct the function

$$f_\xi : \theta_\lambda \to \mathbb{R}$$

$$\beta \to f_\xi(\beta) = \beta(\xi)$$

such that for each $j = 1, 2, \ldots, k$ the directional derivatives of f_ξ in relationship with the vector fields generating $d\rho^*(\mathcal{D})$ satisfy

$$d\rho^*(Y^j) \cdot f_\xi(\beta) = \beta[Y^j, \xi] = 0 \quad \text{and}$$

$$d\rho^*(X) \cdot f_\xi(\beta) = \beta(z) > 0 \ .$$

In particular, f_ξ is strictly increasing on each $\varphi \in S_\Sigma$. Therefore the systems $\rho^*(\Sigma)_\lambda$ cannot be controllable in θ_λ and consequently Σ is not controllable in G.

The last argument follows like this: it is possible to project Σ over any homogeneous space G/H in a system Σ_H. It is clear that the controllability of Σ in G implies the controllability of Σ_H in G/H.

Now we analyze the existence of symplectic vectors. Let $\tilde{g}$ be the Lie algebra of the group G.

Proposition 2.1. If $\tilde{g}$ is not Abelian and $Z \in Z(\tilde{g})$ is not a null vector field, then there is a symplectic vector for Z.

Proof. By definition, for each $\omega \in \tilde{g}$ and $g \in G$, we have

$$\rho(g)(\omega) = \frac{d}{dt}\Big|_{t=0} g \cdot \exp(t\omega) g^{-1}$$

In particular, every $\lambda \in \tilde{g}^*$ is constant on the adjoint orbit of elements ω which belong at the centre of $\tilde{g}$. We denote by $\mathcal{A}$ the union of the family of non trivial co-adjoint orbits θ_λ with $\lambda \in \tilde{g}^*$.

We claim:

$\mathcal{A}$ is an open non void subset of $\tilde{g}^*$. In fact, if $\Delta \in End(\tilde{g}^*)$ and $\Delta(\lambda) \neq 0$, then for a continuity argument there is a neighbourhood $V = V(\lambda)$ such that $\Delta(\beta) \neq 0$, $\forall \beta \in V$.

The hypothesis: $\tilde{g}$ is not Abelian secures the existence of an element $\omega \in \tilde{g}$ such that the endomorphism $d\rho(\omega)$ is not trivial and since the algebra $\tilde{g}^*$ separates points in $\tilde{g}$, there is $\lambda \in \tilde{g}^*$ such that

$$\beta(\cdot) = \lambda[w, \cdot] \in \tilde{g}^*$$

is not the null application. In particular, the orbit θ_λ is not trivial, and this property is valid in a neighbourhood of λ. This proves our claim.

Now we suppose that the thesis of the proposition is not true, i.e., for each $\lambda \in \tilde{g}^*$:

$$\{\lambda\} \subsetneqq \theta_\lambda \to \beta(Z) = 0, \ \forall \beta \in \theta_\lambda .$$

We consider Z a linear application defined on the dual ot $\tilde{g}$ for evaluation, i.e.

$$Z : \tilde{g}^* \to \mathbb{R}, Z(\lambda) = \lambda(Z) .$$

Then, it is clear that $\mathcal{A} \subset \ker(Z)$, but this is a contradiction since $Z \neq 0$ and therefore $\mathrm{Ker}(Z)$ is a hyperplane in $\tilde{g}^*$. So $\exists \lambda \in \tilde{g}^*$ with $\lambda(Z) \neq 0$. Now, Z belongs at the centre of $\tilde{g}$, and thus we obtain

$$\lambda(\rho(G)Z) = \lambda(Z) .$$

Consequently it is necessary that the orbit θ_λ be constant on the vector field Z. Therefore λ (or $-\lambda$) is a symplectic vector for Z. $\diamond$

It is possible to generalize the previous results.

Theorem 2.2 Let $\tilde{h}$ be an ideal of $\tilde{g}$ such that $\tilde{g}/\tilde{h}$ is not an Abelian algebra. If $\pi : \tilde{g} \to \tilde{g}/\tilde{h}$ is the canonic projection and there is $Z \in \tilde{g}$ such that $\pi(Z) \in Z(\tilde{g}/\tilde{h})$ is non-null, then there is a symplectic vector λ for Z.

Proof. Via the proposition 2.1 we obtain a symplectic vector for $\pi(Z)$, since $\tilde{h}$ is an ideal, there is a normal subgroup H of G with Lie algebra $\tilde{h}$. We consider the following commutative diagrams:

$$\begin{array}{ccc} \tilde{g} & \xrightarrow{\lambda} & \mathbb{R} \\ \pi \searrow & & \nearrow \tilde{\lambda} \\ & \tilde{g}/\tilde{h} & \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{g} & \xrightarrow{\rho(g)} & \tilde{g} \\ \pi \downarrow & & \downarrow \pi \\ \tilde{g}/\tilde{h} & \xrightarrow{\tilde{\rho}(gH)} & \tilde{g}/\tilde{h} \end{array}$$

where $g \in G$ and $\tilde{\rho}$ is naturally the adjoint representation of the quotient Lie group G/H with Lie algebra $\tilde{g}/\tilde{h}$. For each $g \in G$ we have the following relationships:

$$\lambda(\rho(g)Z) = \tilde{\lambda}(\tilde{\rho}(gH)\pi(Z)) = \tilde{\lambda} \circ \pi(Z) .$$

The projection π is surjective and consequently the application

$$(\tilde{g}/\tilde{h})^* \xrightarrow{\phi} \tilde{g}^*$$

defined by $\psi(\tilde{\lambda}) = \tilde{\lambda} \circ \pi$ is injective. Moreover, there is $g \in G$ such that

$$\tilde{\lambda} \circ \tilde{\rho}(gH) \neq \tilde{\lambda}$$

and therefore necessarily for this g we have

$$\lambda \circ \rho(g) \neq \lambda .$$

So, θ_λ is a nontrivial orbit and since $\tilde{\lambda}$ is a symplectic vector for $\pi(Z)$ we obtain

$$\lambda(\rho(\sigma)Z) = \tilde{\lambda}(\pi(Z)) > 0$$

and therefore the proof is complete. $\diamond$

In the next section we utilize this result for nilpotent groups.

3. NILPOTENT SYSTEMS

Let $\Sigma = (G, \mathcal{D})$ be a nilpotent system, i.e. Σ is an invariant system

$$\dot{g} = X(g) + \sum_{j=1}^{k} \mu_j Y^j(g)$$

such that the Lie algebra $\tilde{g}$ of G is nilpotent. This class of systems is adequate for implementing theorem 2.2 .We have

Theorem 3.1. If $Z(\tilde{g}) \subsetneqq Zt(\tilde{h})$ then Σ cannot be controllable.

Proof. Let ξ belong to $Z(\tilde{g}) - Zt(h)$ and let us define the non-null vector field $Z = [X, \xi]$. Since $\tilde{g}$ is nilpotent, the descendent central series $\tilde{g}^{(0)} = \tilde{g}$ and $\tilde{g}^{(i+1)} = [\tilde{g}, \tilde{g}^{(i)}]$, $i \in \mathbb{N}$.

Therefore, there is $n \in \mathbb{N}$ such that

$$\hat{g} = \tilde{g}^{(0)} \supsetneqq \tilde{g}^{(1)} \supsetneqq \ldots \supsetneqq \tilde{g}^{(n-1)} \supsetneqq \tilde{g}^{(n)} = 0$$

Let

$$i_0 = \min\{i | Z \in \tilde{g}^{(i)} \backslash \tilde{g}^{(i+1)}\} .$$

Since $Z \in g^{(1)} \backslash g^{(n)}$, we have $1 \leq i_0 \leq n-1$. Moreover,

$$\pi : \tilde{g} \to \tilde{g}/\tilde{g}^{(i_0+1)}$$

satisfies

$$\tilde{g}^{(i_0)}/\tilde{g}^{(i_0+1)} = z(\tilde{g}/\tilde{g}^{(i_0+1)})$$

consequently, $\tilde{g}^{(i_0)}/\tilde{g}^{(i_0+1)}$ is a non Abelian algebra. Now $\pi(Z) \neq \tilde{g}^{(i_0+1)}$, so the Theorem 2.2 secures the existence of a symplectic vector for Z and Σ is not contollable. $\diamond$

Remark 3.2. Now we study the case when $\tilde{h}$ is of co-dimension 1 in the nilpotent algebra $\tilde{g}$. For the sake of completeness we explicity a general result closely related to theorem 7.3 in [5].

Proposition 3.3. Let $\Sigma = (G, \mathcal{D})$ be an invariant control system and H the connected subgroup of G whose Lie algebra $\tilde{h}$ is an ideal of co-dimension 1 in $\tilde{g}$. If,

1. H is closed, then Σ is controllable $\Leftrightarrow G/H \simeq S^1$
2. H is not closed, then Σ is controllable.

Proof. The subalgebra $\tilde{h}$ is an ideal, therefore H is a normal subgroup of G.

1. If H is closed, the homogeneous space G/H is a Lie group and we can project Σ in an invariant system

$$\pi(\Sigma) = (G/H, \{X + \tilde{h}\})$$

on the 1-dimensional manifold G/H.

It is evident that we can suppose $X \in \tilde{h}$. If not, since Σ is transitive, it will also be controllable on the grou, $G = H$, (each $h \in H$ is a finite product of elements of the form

$$\exp(t, Y^j), t \in \mathbb{R})$$

We separate the analysis in two cases

a) Compact case: $G/H = S^1$

In this condition Σ is controllable [5].

b) Non compact case: $G/H = \mathbb{R}^+$

Here

$$S_{\pi(\Sigma)}(1) = [1, +\infty)$$

and hence Σ cannot be controllable.

2. We assume that the co-dimension 1 subgroup H is not closed. Therefore, the closure $\bar{H}$ of H is a closed Lie group containing H with Lie algebra $\tilde{g}$. In particular, the connected component of the neutral element $e \in G$ of $\bar{H}$ and G coincide. Therefore, H is a dense subgroup. By the rank condition we have:

$$\tilde{g} = \text{span}_{C.A.}\{X, Y^1, \ldots, Y^k\}$$

and hence the positive orbit of e satisfies

$$\text{int} S_\Sigma(e) \neq \emptyset \ .$$

Since

$$H \subset S_\Sigma(e)$$

it is necessary that

$$\overline{S_\Sigma(e))} = G \ .$$

In particular, [5]

$$S_\Sigma(e) = G$$

and Σ is controllable. ◇

Proposition 3.4. Let $\Sigma = (G, D)$ be a nilpotent system. If the subalgebra $\tilde{h}$ has co-dimension 1 in $\tilde{g}$ then the thesis of the proposition 3.3 is valid.

Proof. In fact, in this condition, it is easy to check that

$$[\tilde{g}, \tilde{g}] \subset \tilde{h}$$

and so, $\tilde{h}$ is an ideal of co-dimension 1 in $\tilde{g}$ ◇

Remark 3.5. If the drift vector field X belongs to the centre of $\tilde{g}$ then $\tilde{h}$ satisfies the hypothesis of the proposition 3.4.

Now we characterize the controllability of nilpotent systems when G is a (connected) simply connected Lie group.

Theorem 3.6. Let Σ be a nilpotent system on the simply connected Lie group G. Then

$$\Sigma \text{ is controllable } \Leftrightarrow \tilde{h} = \tilde{g} \ .$$

Proof. It is evident that $\tilde{h} = \tilde{g}$ is a sufficient condition for the controllability of Σ. Now, $\tilde{g}$ is nilpotent and consequently also $\tilde{h}$. We consider the ascendent central series $(\tilde{h}_i)_{i=0}^k$ of $\tilde{h}$ defined by:

$$\tilde{g}_{(0)} = 0, \quad \text{and for} \quad j = 1, 2, 3 \ldots$$

$$\tilde{h}_{(j)} = \{w \in \tilde{h} | [w, \tilde{h}] \subset \tilde{h}_{(j-1)}\}$$

Therefore, there is $n \in \mathbb{N}$ such that

$$0 = \tilde{h}_{(0)} \subsetneq \tilde{h}_{(1)} \subsetneq \ldots \subsetneq \tilde{h}_{(n-1)} = \tilde{h}$$

For each $j \in \{1, 2, \ldots n\}$, $\tilde{h}_{(j)}$ is an ideal for $\tilde{h}$ but not necessarily for $\tilde{g}$. We claim: "If any $\tilde{h}_{(j)}$ is not an ideal of $\tilde{g}$, then Σ cannot be controllable". In fact, if we denote

$$j_0 = \min\{j | \tilde{h}_{(j)} \text{ is not an ideal for } \tilde{g}\}$$

therefore, $\tilde{h}_{(j_0-1)}$ is an ideal of $\tilde{g}$ and since $\tilde{h}_{(1)}$ is the centre of $\tilde{g}$, we obtain $j_0 \geq 1$. Let H_0 be the connected normal subgroup of G whose Lie algebra is $\tilde{h}_{(j_0-1)}$. By hypothesis G is simply connected and therefore

$$P =: G/H_0$$

is a simply connected Lie group with Lie algebra

$$\tilde{p} = \tilde{g}/\tilde{h}_{(j_0-1)} \ .$$

Via the differential of the canonic application

$$\pi : G \to P$$

we can to project Σ in an invariant system $\pi(\Sigma)$ over P. Therefore, the family of vector fields $d\pi(\mathcal{D})$ generates $\tilde{p}$ and the subalgebra of the control vectors of $\pi(\Sigma)$ is $\tilde{h}/\tilde{h}_{(j_0-1)}$.

By the construction of the ascendent central series, we have

$$Z(\tilde{h}) = \tilde{h}_{(j_0)/\tilde{h}(j_0-1)}$$

which is not an ideal of $\tilde{p}$. This shows that

$$Zt(\tilde{h}) \not\supseteq Z(\tilde{p})$$

In fact, the relationship

$$Z(\tilde{h}) \subset Z(\tilde{p})$$

is not possible by the construction of j_0. In this condition the proposition 3.1 secures that $\pi(\Sigma)$ is not controllable on P an hence Σ cannot be controllable on G.

Therefore, for each $j = 0, 1, \ldots, n$, $\tilde{h}(j)$ is an ideal for $\tilde{g}$, in particular for $j = n$. Since Σ is a transitive system, there are two possible cases:

1. $\tilde{h} = \tilde{g}$
2. $\tilde{h}$ is an ideal of co-dimension 1.

If H is the connected closed Lie group with Lie algebra $\tilde{h}$, the corollary 3.4 secures that:

Σ is controllable $\Leftrightarrow G/H \simeq S^1$

But G/H is a simply connected Lie group and hence the second possibility is in contradiction with our hypothesis. Therefore $\tilde{h} = \tilde{g}$ ◇

Remark 3.7. If $x_0 \in \mathbb{R}^n$ and $A \in M_n(\mathbb{R})$, the solution of $\dot{x} = Ax$, $x(0) = x_0$ is the action of the solution of the matricial equation $\dot{X} = AX$, $X(0) = Id$, on X_0. Therefore, it is possible to study controllability of bilinear systems of type

$$B = \begin{cases} \dot{x} = Ax + \sum_{j=1}^{k} \mu_j A_j x \\ \quad x \in \mathbb{R}^n - \{0\} \end{cases}$$

via the invariant system

$$\Sigma = \begin{cases} \dot{g} = Ag + \sum_{j=1}^{k} \mu_j A_j g \\ \quad g \in G \end{cases}$$

where G is the connected subgroup the $GL_n(\mathbb{R})$, the group of non-singular real matrices, with Lie algebra

$$\tilde{g} = \text{span}_{\mathcal{L.A.}}\{A, A_1, \ldots, A_k\} .$$

In fact,

$$S_B(x_0) = S_\Sigma(Id) \cdot x_0 .$$

Therefore, the results of controllability for invariant systems can be used for bilinear systems.

4. AN EXAMPLE

Let G be the heisenberg group of dimension $2p+1$. The Lie algebra $\tilde{g}$ of G is generated by the elements

$$X_1, \ldots, X_p, Y_1, \ldots, Y_p, Z$$

with the following rules in the non-null brackets

$$[X_i, Y_i] = Z, \quad 1 \le i \le p .$$

It is well known that this algebra has a realization over the vector space of strictly superior matrix of order $p+2$ with the commutator

$$[A, B] = AB - BA .$$

If $\delta(i,j)$ is the matrix of order $p+2$ with one in the (i,j)–coordinate and zeros in the other positions, we can identify for $i, j \in \{1, 2, \ldots, p+2\}$:

$$X_i = \delta(1, i+1), Y_j = \delta(j+1, p+2) \text{ and } Z = \delta(1, p+2) .$$

This way we can identify the elements of G by linear combinations of X_i, Y_j and Z but with one in every diagonal element.

So, $g \in G$ have coordinates

$$g = (x, y, z), \quad x, y \in \mathbb{R}^P, z \in \mathbb{R} .$$

We now consider the dual of $\tilde{g}$

$$\tilde{g}^* = \text{span}_{\mathcal{L.A.}}\{X_1^*, X_2^*, \ldots, X_p^*, Y_1^*, Y_2^*, \ldots, Y_p^*, Z^*\}$$

Each $\lambda \in \tilde{g}^*$ has coordinates

$$\lambda = (a, b, c), \; a, b \in \mathbb{R}^n, \; c \in \mathbb{R}$$

A straightforward calculation shows that the orbit of λ by the co-adjoint representation is

$$\theta_\lambda = \{(a + cy, b - cx, c) | x, y \in \mathbb{R}^n\}$$

In particular

1. $c = 0 \Rightarrow \theta_{(a,b,0)}$ is trivial.
2. $c \neq 0 \Rightarrow \theta(a, b, c) = \{\beta \in \tilde{g}^* | \beta(Z) = 0\} \oplus c \cdot Z^*$.

Therefore, each invariant system Σ of type

$$\Sigma : \begin{cases} \dot{g} = X_{i_0}(g) + \sum_{i \neq i_0} \mu_i X_i(g) + \sum_{j=1}^{p} \mu_j Y_j(g) \\ u \in \mathcal{U} = \mathcal{U}(2p-1) \end{cases}$$

cannot be controllable in G. In fact, $Y_{i_0} \in Zt(\tilde{h})$ and $Z = [X_{i_0}, Y_{i_0}]$ is the centre of $\tilde{g}$ and therefore theorem 3.1 is applicable. We have

1. Each vector $\lambda = (a, b, c)$ with $c > 0$ is a symplectic vector for Z.

2. Since G is a simply connected Lie group then Theorem 3.6 is applicable directly.

Remark 4.1. Let G be a connected and simply connected Lie groups, then Theorem 3.6 permit to construct all the controllable systems on G.

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