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# MÖBIUS INVERSE PROBLEM FOR DISTORTED BLACK HOLES

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ABSTRACT

Hawking "thermal" radiation could be a means to detect black holes of micron sizes, which may be hovering through the universe. We consider these micro-black holes to be distorted by the presence of some distribution of matter representing a convolution factor for their Hawking radiation. One may hope to determine from their Hawking signals the temperature distribution of their material shells by the inverse black body problem. In 1990, Nan-xian Chen has used a so-called modified Möbius transform to solve the inverse black body problem. We discuss and apply this technique to Hawking radiation. Some comments on supersymmetric applications of Möbius function and transform are also added.

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## 1 Introduction

Recently, the remarkable 160 years old inversion/transform formula of Möbius has vigorously entered Physics through a number of important applications to inverse problems, [1]. Its future in mathematical physics and even in more applicative science seems to be extremely fruitful, [2], although Hughes, Frankel, and Ninham, [3], have shown that Möbius function occurs only because the expansion of the reciprocal $\zeta$ function in Mellin transforms produces it. The interest in Möbius inversion was raised by Nan-xian Chen, who proved a modified Möbius transform in order to apply it to such problems like finding out the phonon density of states , the inverse black body radiation problem, and getting the solution for inverse Ewald summation. In an astrophysical context the Möbius transform has been applied to the analysis of *IRAS* data of interstellar dust emission [4].

The present work is prompted by the inverse black body problem in the realm of black hole physics. Hawking heuristic discovery of black body radiation from Schwarzschild horizons is a famous result of quantum field theory in curved space-times, [5]. Consequently, one might be well motivated to look for primordial black holes, and more generally, for small (mini) black holes, from the point of view of remote sensing, i.e., the determination of the "surface" temperature distributions from the spectral measurement of their Hawking radiation. We have in mind the so-called *distorted black holes* discussed by Geroch and Hartle [6]. These are Einstein vacuum solutions obtained by the Weyl technique of generating solutions from axially symmetric potentials of the flat space Laplace equation. Weyl black holes are stationary and carry with them an external distribution of matter; they can have only spherical or toroidal horizons. In this astrophysically interesting class of objects, one may include mini-black holes surrounded by thin matter shells [7]. We call the *horizon temperature distribution* the quantity conjugated to the Hawking power spectrum through the inverse transform but this obviously does not mean we assign the distribution directly to the horizon surface. In the case of *distorted black holes* because of the matter distribution the effective horizon temperature is not constant and therefore it is meaningful to think of a sort of surface temperature distribution. There is no restriction imposed by the zeroth law of black hole thermodynamics [8] which applies only to *isolated* static or stationary black holes. Perhaps it is in order to recall that the generalized black-hole thermodynamics has been settled 20 years ago as a consequence of Hawking radiance, but this in turn was the starting point for still greater puzzles. The interpretation of black hole entropy is unclear and the number of accessible microstates is completely unknown at the present time. The interior (better surface/horizon) microstates are quantum correlated with the outside fields in the right way to produce black body radiation in each quantum mode. The intrinsic black hole entropy is just one fourth of the area of the event horizon, and one may think that each Planck unit of area is one degree of freedom but no more concrete ideas have emerged.



Besides, there are different conceptions regarding the fundamental role of black holes [9]. According to statistical mechanics the entropy is the logarithm of the number of accessible microstates, assuming ergodic behaviour. In Planck units the Schwarzschild entropy is $S = 4\pi M^2$. This would suggest that a black hole of mass $M$ has $\exp(4\pi M^2)$ states and the black hole could be in any of them equally likely. Presumably such a huge number of states may be strongly degenerate. Another remarkable puzzle is the fate of information about the material of the black hole and in what way it is encoded in the Hawking radiation. The last puzzle has to do with the final stages of the evaporation process when the black-hole is Planck-sized and when the semiclassical approximation used by Hawking is no longer valid. Until now no definite progress has been made with any of these puzzles.

## 2 Inverse Problem for Planck Law

In what follows we shall consider the Planck radiation in the radiometric- remote sensing approach.

What is known as Planck law is the analytical formula for the power spectrum or spectral brightness of the black body radiation:

$$P(\nu, T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(\frac{h\nu}{kT}) - 1}, \tag{1}$$

In radiometry the power spectrum is called spectral radiance, and characterizes the source spectral properties as a function of position and direction. The total radiated power spectrum ( which in radiometry is called radiant spectral intensity to be used for point i.e., far away sources) will be:

$$W(\nu) = Const \cdot \int_0^\infty A(T) B(\nu, T) dT \ , \tag{2}$$

where the area-temperature distribution is denoted by A(T), and the Boltzmann-Planck factor by B($\nu$,T). The inverse black body problem is to solve the integral equation for A(T) for given total radiated power spectrum, which may be known experimentally or otherwise. This problem was solved by Bojarski, [10], who introduced a thermodynamic coldness parameter, $u = h/kT$, and an area coldness distribution, a(u), as more convenient variables to get an inverse Laplace transform of the total radiated power. The coldness distribution is obtained as an expansion in this Laplace transform. Later, people working in the field have given a simpler approach and tried to improve in various ways the result of Bojarski.

More precisely, the total power spectrum is rewritten as:

$$W(\nu) = \tfrac{2h\nu^3}{c^2} \int_0^\infty \tfrac{a(u)}{\exp(u\nu)-1} du \ , \tag{3}$$

A series expansion of the denominator leads to:

$$W(\nu) = \frac{2h\nu^3}{c^2} \int_0^\infty \exp(-u\nu) \cdot \sum_{n=1}^{\infty} (1/n)\, a(u/n) du \ , \tag{4}$$

and we already see that the sum:

$$f(u) = \sum_{n=1}^{\infty} (1/n)\, a(u/n) \ , \tag{5}$$



is the Laplace transform of:

$$g(\nu) = \frac{c^2}{2h\nu^3} W(\nu) \ , \tag{6}$$

From the Laplace transform f(u), Chen obtained a(u) by his modified Möbius expansion,

$$a(u) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f(u/n) \ , \tag{7}$$

reproducing in a simple way a result already obtained by Kim and Jaggard, [11].

# 3 Möbius Expansions

Let us try to justify the expansion of Chen. The Möbius expansion refers to special sums (which I call d-sums) of any number-theoretic function f(n), running over all the factors of n, 1 and n included. Such a kind of running is indicated by the symbol $d|n$. If:

$$F(n) = \sum_{d|n}^{n} f(d) \ , \tag{8}$$

then the last term of the sum, f(n), could be written as a sum of F functions, which we shall call Möbius d-sum:

$$f(n) = \sum_{d|n}^{n} \mu(d) F(n/d) \ , \tag{9}$$

in which F(n), i.e., the previous sum, becomes the first term. Since each F-term in the second expansion is a d-sum, it is clear that we have an overcounting unless the factors $\mu(d)$ (Möbius functions) are sometimes naught and even negative. The partition of the factors of n implied by the Möbius function is such that $\mu(1)$ is 1, $\mu(n)$ is $(-1)^r$ if n includes r distinct prime factors, and $\mu(n)$ is naught in all the other cases. In particular, all the squares have no contribution to the Möbius d-sums, and all the prime numbers have negative contribution. Another terminology is to call squarefree the integers selected by means of the Möbius function. The trick of Chen is to apply such an inversion of finite sums to ordinary infinite summations just because all the integers could be considered as factors of the infinite set of n. This is tantamount to saying that infinity is the natural number having the infinitely many smaller natural numbers as factors. With this trick, if:

$$f_1(x) = \sum_{n=1}^{\infty} f_2(x/n) , \tag{10}$$

then,

$$f_2(x) = \sum_{n=1}^{\infty} \mu(n) f_1(x/n) , \tag{11}$$

This is the modified Möbius transform (MMT) of Chen. For the inverse blackbody radiation, $f_1(x) = uf(u)$ and $f_2 = ua(u)$. So one gets the coldness distribution simply by multiplying the Laplace transform of the total power spectrum by the coldness parameter and then applying the MMT (Chen trick).



# 4 Distorted Black Holes and their Coldness Distribution

We come now to some well-known black hole results. In his note to *Nature* in 1974, Hawking argued that for Schwarzschild black holes (SBHs) the number of particles emitted in wave packet modes propagating along null u-lines is $[\exp(2\pi\omega/\kappa) - 1]^{-1}$ times the number of particles that would have been absorbed from similar wave packets incident on the black hole from $I^-$ along null v-lines, a result that holds for a perfect black body with a thermodynamic temperature in geometric units of $\kappa/2\pi$. According to Hawking, the rate of emission for a given bosonic mode is $\Gamma_n B(\nu, T_H)$, where $\Gamma_n$ is the absorption coefficient for the mode, and $B(\nu, T_H)$ is the Boltzmann-Planck factor. Calculations of absorption coefficients were provided by Page, [12], and in the case of massless scalar fields by Matzner [13] and more recently by Sanchez [14]. She was able to show that s-wave contribution predominates in Hawking radiation of the massless scalar fields. The contributions from all the higher partial waves are supressed by the rapid decrease of the Boltzmann-Planck factor in the range $kr_S \geq 1$. At the same time a strong increase of the maxima of the effective potential was found for partial waves with $l \geq 1$. The Hawking emission regime in the range $kr_S \leq 1$, corresponds to only phase shifts in the phase of the wave packets passing through the Schwarzschild sphere. We intend to study in a future publication the more realistic case of the modified Debye potentials introduced by Mo and Papas, [15], and to update in this way the results of Fabbri, [16].

Consider now micron-sized SBHs ($M \sim 10^{24}g$) for which no known massive particles are emitted, and suppose it radiates in the Hawking regime. According to Page, about 16% of its Hawking flux goes into the photon flux, the rest being neutrino emission. Suppose now that such SBHs are of Weyl type. From the remote sensing point of view we may introduce a horizon coldness parameter $u_S = h/kT_{DH} = 4\pi/\kappa_D$, where $T_{DH}$ is the effective horizon temperature of the distorted black holes, i.e.,

$$T_{DH} = (8\pi M)^{-1} e^{2\mathcal{U}} \tag{12}$$

The variable $\mathcal{U}$ is related to the exponents that characterize the Weyl metrics, which one can put in a general form as follows

$$g^{ab} = e^{2U-2V} h^{ab} + r^{-2} e^{2U}(1 - e^{-2V})\phi^a \phi^b - e^{-2U} t^a t^b \tag{13}$$

where $h^{ab}$ is a flat, positive-definite, three-dimensional metric, $\phi^a$ is the rotational Killing field of the metric, and $t^a$ are surface-orthogonal Killing fields. With this notation, $\mathcal{U}$ will be

$$\mathcal{U} = U - \frac{1}{2}V - \ln(\frac{1}{2}\sqrt{r/m}) \tag{14}$$

In the case of Weyl distorted-black holes a simple application of the MMT will give for the horizon coldness distribution the expression:

$$a(4\pi/\kappa_D) = \frac{c^2}{2h\nu^3} \sum_{n=1}^{n=\infty} \frac{\mu(n)}{n} f(\frac{4\pi}{n}\kappa_D^{-1}) \tag{15}$$

where f is the inverse Laplace transform of the total photon power spectrum.



# 5 Möbius Transform and Supersymmetry

We would like to mention that almost simultaneously with Chen, Donald Spector, [17], pointed out another application of the Möbius function. In his paper he showed the equivalence between the Möbius function and the Witten topological index in supersymmetric theories with discrete spectra. The Witten index operator $W = (-1)^F$ distinguishes fermionic from bosonic states and operators in supersymmetric frameworks. It is just the number of fermionic zero modes minus the bosonic zero modes (the spectral asymmetry). In order to obtain the number theoretic interpretation of the Witten index, Spector has made use of Gödel numbering in associating the natural numbers with the states of a quantum system . Since W is ill- defined, Witten, [18], suggested to use a regularized generalized partition function $\Delta = trWe^{-\beta H}$ as a better order parameter for studying SUSY breaking. Spector developed his arguments based only on examples of systems with purely discrete spectra for which $\Delta$ is independent of $\beta$ as claimed by Witten. However, it is known that the continuous part of spectra causes beta dependence of $\Delta$, [19]. Another proof of Spector, again for systems with discrete spectra, concerns the Möbius inverse transform, which shows up whenever one cancels some of the bosonic degrees of freedom of a bosonic Hamiltonian by means of a corresponding fermionic Hamiltonian, supersymmetrizing the bosonic degrees of freedom to be deleted.

It would be also of interest to consider in the future the supersymmetric black holes recently investigated by Renata Kallosh, [20], in the number theoretic perspective. The extremal solutions of the supergravity equations of motion preserve half of the supersymmetries. An interesting future problem will be to establish a supersymmetric Planck law. For that it is necessary to obtain further insight in the underlying probability distributions of Planck law [21].

In conclusion, detailed studies are required to exploit further the many possibilities that number theoretic methods promise to open up in many fields of mathematical physics [22].

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