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ALGEBRAIC CONFORMAL FIELD THEORY [♣]

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Abstract. Many features of conformal field theory are special versions of structures which are present in arbitrary two-dimensional quantum field theories. Therefore it makes sense to describe two-dimensional conformal field theories in the context of the algebraic theory of superselection sectors. While most of the results of the algebraic theory are rather abstract, conformal field theories offer the possibility to work out many formulæ explicitly. In particular, one can construct the full algebra $\overline{\mathcal{A}}$ of global observables and the endomorphisms of $\overline{\mathcal{A}}$ which represent the superselection sectors. Some explicit results are presented for the level 1 so(N) WZW theories; the algebra $\overline{\mathcal{A}}$ is found to be the enveloping algebra of a Lie algebra $\overline{\mathcal{L}}$ which is an extension of the chiral symmetry algebra of the WZW theory.

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1 Introduction

The qualifications "algebraic" and "conformal" used in the title refer to specific approaches to the description of (classes of) quantum field theories. In recent years much progress has been made in this field, but most of the work has been devoted to particular aspects of either the algebraic description or the conformal field theory framework. Here I review some recent attempts to arrive at a more unified treatment. More precisely, the main issues to be addressed are the following:

- The description of two-dimensional conformal field theories in terms of the algebraic theory of superselection sectors.
- The presentation of explicit examples for various non-trivial structures which are present in general two-dimensional quantum field theory.

Let me summarize very briefly what is meant by the terms "algebraic field theory" and "conformal field theory". The common theme is the idea to define a quantum field theory from first principles so that a completely non-perturbative description is obtained. In particular, the quantum theory is treated in its own right, and not considered as some kind of "quantization" of a classical theory – which may or may not exist. (The quantization principle is e.g. the basis of the usual Lagrangian (path integral) approach to quantum field theory, which, as a consequence, is essentially applicable only to perturbation theory, either around the trivial vacuum or around some other classical solution.) In conformal field theory, this idea is implemented as follows ([1, 2]; for further references, see e.g. [3]):

- The central object is the (maximally extended, chiral) *symmetry algebra* $\mathcal{W}$, an infinite-dimensional Lie algebra over $\mathbb{C}$ which contains the Virasoro algebra Vir as a subalgebra.
- The fields of the theory fall into irreducible highest weight modules V_j of $\mathcal{W}$, with the highest weight vectors corresponding to the *primary fields* $\phi_i(z, \bar{z})$.
- The fields must satisfy a closed *operator product algebra*, which is an associative short-distance expansion (bootstrap). The structure of the operator product algebra is strongly constrained by the $\mathcal{W}$-symmetry.
- The *fusion rules* describe the "invariant content" of the operator product algebra. The fusion rule coefficients $\mathcal{N}_{ij}{}^{k}$ can be viewed as the structure constants of a commutative associative ring with unit (the identity primary field) and conjugation.
- The set of (specialized) characters of the modules V_i is closed under transformations belonging to the modular group PSL(2, $\mathbb{Z}$) of the torus which is generated by the operations S and T. The *Verlinde formula* expresses the fusion rule coefficients $\mathcal{N}_{ij}{}^{k}$ through the entries S_{ij} of the modular transformation matrix S, $\mathcal{N}_{ij}{}^{k} = \sum_l S_{il} S_{jl} S^*_{kl} / S_{0l}$ (the index 0 refers to the identity field).
- There is an underlying *quantum group* structure which shows up in various ways: via the description of both the chiral blocks of the conformal field theory and the representation theory of the quantum group in terms of braided monoidal categories; via the construction of highest weight modules of the quantum group through "screened vertex operators"; and via an isomorphism between the fusion rules and the "truncated Kronecker products" of the quantum group modules.
- The concept of *quantum dimensions* $\mathcal{D}_j$ of the primaries Φ_j is motivated both by the Verlinde formula ($\mathcal{D}_j = S_{j0}/S_{00}$) and by the quantum group structure.

Algebraic field theory, or more precisely the algebraic theory of superselection sectors ([4, 5]; for a review and further references, see e.g. [6]), is a more general approach to quantum field theory, and hence provides in particular another description of two-dimensional conformal field theories (actually this is true only up to questions related to unitarity, see section 2). This approach can be characterized as follows.

- The central object is the *observable algebra* $\mathcal{A}$, a unital C^*-algebra which is realized as an algebra of bounded operators acting on a Hilbert space $\mathcal{H}$.
- The *superselection sectors* $\mathcal{H}_j$ are orthogonal subspaces of $\mathcal{H}$. They are in one to one correspondence with inequivalent irreducible representations π_j of $\mathcal{A}$.
- Complete information on the physical content of the theory is, under suitable assumptions, already contained in the vacuum sector $\mathcal{H}_0$. In particular, the representations π_j correspond to certain endomorphisms $\rho_j : \mathcal{A} \to \mathcal{A}$ of the observable algebra.
- Equivalence classes of composite endomorphisms $\rho_i \circ \rho_j$ of $\mathcal{A}$ can be decomposed into classes of irreducible endomorphisms. The composition is commutative and associative; it possesses a unit *id* and, under appropriate conditions, a conjugation.
- The *field algebra* $\mathcal{F}$ is obtained from $\mathcal{A}$ by adjoining *charged fields* f to $\mathcal{A}$ which realize the endomorphisms. The composition of the classes $[\rho_j]$ provides a natural "cross product" of the representations π_j of $\mathcal{A}$ and, as a consequence, of the elements of the field algebra (more precisely, of the "reduced field bundle").
- The *statistics operators*, giving information about the non-commutativity of the cross product, can be defined entirely in terms of the endomorphisms ρ_j. They can be used to associate *statistical dimensions* D_j to the superselection sectors $\mathcal{H}_j$.
- The *Tomita-Takesaki modular theory* provides operators which are abstract versions of the CPT transformation and of dilation, called the *modular conjugation* J and the *modular operator* $\mathbf{\Delta}$, respectively.

The summary given above suggests a close connection between many of the concepts used in conformal field theory and in algebraic field theory. Indeed, more detailed investigation shows that, very roughly, e.g. the following correspondences hold:

observable algebra $\mathcal{A}$	$\longleftrightarrow$	chiral symmetry algebra $\mathcal{W}$
superselection sectors / endomorphisms ρ_j	$\longleftrightarrow$	primary fields ϕ_j
field algebra $\mathcal{F}$	$\longleftrightarrow$	primary and descendant fields
composition of classes $[\rho_j]$	$\longleftrightarrow$	fusion rules
cross product in the field bundle	$\longleftrightarrow$	operator product algebra
statistical dimension D_j	$\longleftrightarrow$	quantum dimension $\mathcal{D}_j$

From these similarities it is rather clear that one can hope to get more insight into conformal field theory from algebraic field theory and vice versa. This hope can be substantiated by elaborating upon the virtues of the respective approaches. In particular, one should be aware of the following:

- While the formalism and constructions of algebraic field theory are mathematically rigorous as well as extremely powerful, many of the results look – certainly to most physicists not working in the field – very abstract. Therefore it is helpful to analyse concrete examples. Previously, the only accessible examples were just free theories. In contrast, two-dimensional conformal field theories provide a huge number of interesting non-trivial theories where many aspects of the theory can be worked out explicitly.

- On the other hand, while many general results about conformal field theory have been obtained, such as the Verlinde formula, the polynomial equations for the chiral blocks, and various ideas about the classification programme, it is often extremely difficult to apply these results in practice. (Let me name just one problem, namely the explicit calculation of operator product coefficients. Although it is known how these numbers can be computed for any given conformal field theory, they have been fully worked out only for the diagonal $c < 1$ minimal unitary series and for su(2) WZW theories.) One reason for this is that in order to apply the general results, it is in fact usually *not* sufficient to know the chiral algebra and its spectrum of primary fields, but one needs further information specific to the given (class of) theories, such as free field realizations, which have to be found case by case.

  In contrast, in algebraic field theory the analysis proceeds in a completely canonical manner, as the problems are economically separated into two parts:

  – As a starting point, one specifies the observable algebra $\mathcal{A}$. This is quite straightforward for the case of conformal field theories, since as already pointed out, $\mathcal{A}$ corresponds essentially to the chiral algebra $\mathcal{W}$. (Actually the relation between $\mathcal{A}$ and $\mathcal{W}$ is a bit more involved, see below.)

  – Everything else, such as the construction of the endomorphisms of $\mathcal{A}$ which represent the superselection sectors, is then in principle merely a matter of applying canonically defined general procedures.

In conclusion, on one hand conformal field theories provide interesting explicit examples for algebraic field theory, and hence may lead to further insight into its general structure. On the other hand, one may hope to use algebraic field theory techniques to simplify the solution of various problems which arise in the analysis of specific conformal field theories.

2 Open problems

Unfortunately, investigations in the direction indicated in the introduction are still at an early stage, and I think it is fair to say that so far no really surprising or fundamentally new results have been obtained. Therefore it seems appropriate to mention some open questions which should be addressed in the future. (I will not go into any detail; for more precise information on the concepts used here, see below and [7–10].)

- Precise relation between the associative algebra $\mathcal{A}$ and the Lie algebra $\mathcal{W}$:

  In the known examples, the global observable algebra $\overline{\mathcal{A}}$ can be obtained as the universal enveloping algebra of an underlying Lie algebra $\overline{\mathcal{L}}$ which contains $\mathcal{W}$. Is this generic for conformal field theories, or maybe even for a larger class of theories?

- Haag duality:

  The double commutants of the local versions $\mathcal{W}(\mathcal{O})$ of the chiral algebra are known [9] to satisfy Haag duality. In contrast, for the global chiral algebra $\mathcal{W}$ Haag duality requires an extension of $\mathcal{W}$ to the Lie algebra $\overline{\mathcal{L}}$ just mentioned. Does there exist a general prescription for determining the extension $\overline{\mathcal{L}}$, and does it have a geometric interpretation? For example, what is the meaning of the Lie group whose Lie algebra is $\overline{\mathcal{L}}$?

- Non-diagonal modular invariants:

  The algebraic field theory description of conformal field theories uses only the information about one chiral half of the theory. Nevertheless it seems to be possible in principle to determine the operator product coefficients (including all relevant normalizations) completely. How can this be reconciled with the existence of non-diagonal modular invariants and the fact [11] that operator product coefficients generically do depend on the chosen modular invariant?
  (I expect that the resolution of this problem is the following: Assuming that W is maximal, only automorphism modular invariants need to be considered, and the operator product coefficients are invariant under the relevant fusion rule automorphisms.)

- Verlinde formula:

  A version of the Verlinde formula holds for two-dimensional quantum field theory in general ([12]; see also [13]). In the generic case the matrix S appearing in the formula is defined entirely in terms of the endomorphisms ρ_j (the same is true for an analogue of the modular matrix T), and there is no obvious connection with the behavior of characters under modular transformations. Does such a connection nevertheless exist? If this is the case: Are the (generalized) modular transformations related to the modular operator and modular conjugation of the Tomita-Takesaki theory? (For investigations related to this subject, compare [14].) And, can the connection elucidate the origin of the fact that the characters of a conformal field theory span irreducible modules of PSL(2, $\mathbb{Z}$)?

- Unitarity:

  As algebraic field theory starts from the very beginning with an algebra of operators on a Hilbert space, by construction it directly applies only to unitary theories. In contrast, in the conformal field theory approach no such restriction exists. Moreover, for various aspects of conformal field theory, unitary and non-unitary theories can be treated on the same footing. For many unitary conformal field theories there even exist descriptions (e.g. free field realizations, such as the Wakimoto construction for WZW theories) which use non-unitary systems as intermediate steps. Does this imply that the algebraic field theory framework can be extended to describe non-unitary theories as well?

- Classification problem:

  What can one learn about the classification of two-dimensional quantum field theories from the classification of (rational) conformal field theories, or vice versa – not just abstractly, but for the actual characterization of (new) classes of theories? For example, what is the algebraic field theory formulation of various methods for obtaining new conformal field theories from known ones, such as coset constructions, conformal embeddings, or modding out of discrete symmetries?

- Quantum groups:

  The quantum dimensions of conformal field theory and the statistical dimensions of algebraic field theory can be interpreted as the quantum dimensions of modules of certain quantum groups. In both cases such modules have been constructed in terms of objects present in the theory (screened vertex operators in the conformal field theory case), but they play a somewhat artificial role in the theory (e.g. in the conformal case, the non-highest weight states do not correspond to conformal fields). Can one get a deeper understanding of the role of the quantum group, or possibly of more relevant related symmetry objects such as "quasi Hopf algebras" [15] or "braided groups" [16]? Are the fusion rules isomorphic to the tensor products in the category of representations of this symmetry object?

- Quantum dimensions:

  For the theories considered so far, one can express the quantum dimensions $\mathcal{D}_j$ of the primary fields, i.e. the statistical dimensions of the corresponding superselection sectors, as [8] $\mathcal{D}_j = 2^{\mathrm{ind}([\rho_j])/4}$, with $\mathrm{ind}([\rho_j])$ the index of the representative endomorphisms $\rho_j \in [\rho_j]$ of an underlying Majorana algebra. For theories with more complicated fusion rules this simple formula is certainly no longer valid since the quantum dimensions are generically not just fractional powers of 2. Does there exist a generalization of the relation which holds for arbitrary conformal field theories or even for other two-dimensional theories?

- Small quantum dimensions:

  For WZW theories, and presumably for all rational conformal field theories, there exists [17] – for $\mathcal{D} \leq 2$ – a specific relation between the value of the quantum dimension $\mathcal{D}_j$ and the non-integer part of the conformal dimension Δ_j of a primary. In terms of algebraic field theory this constitutes a relation between the statistical dimension and the statistics phase of $[\rho_j]$; is this relation also valid for other two-dimensional quantum field theories?

3 Aspects of the algebraic theory of superselection sectors

As already mentioned, I do not have much to say about the problems listed above. Rather, what one has obtained so far is just a detailed description of the algebraic field theory version of a few rather simple conformal field theories. Of course I hope that such investigations ultimately will shed light on some of the problems. The theories in question will be introduced later on. For now, let me start by reviewing a few facts about the algebraic theory of superselection sectors in some more detail. First let me repeat that in algebraic field theory the central role is played by a C^*-algebra $\mathcal{A}$, called the observable algebra [5–7]. This "quasilocal" observable algebra $\mathcal{A}$ is the closure (in the uniform topology) of the union of the *local observable algebras* $\mathcal{A}(\mathcal{O})$. For every double cone (i.e., intersection of a forward and a backward light-cone) $\mathcal{O}$ of the underlying Minkowski space-time there is a unital C^*-algebra $\mathcal{A}(\mathcal{O})$ of observables localized in this cone. The elements of $\mathcal{A}(\mathcal{O})$ are by definition those bounded operators on the Hilbert space $\mathcal{H}$ of the theory which correspond to observables that can be measured inside $\mathcal{O}$, but not outside $\mathcal{O}$ (only self-adjoint operators of this type are observables in the usual sense, but it is convenient to admit also arbitrary complex linear combinations of these). The physical Hilbert space $\mathcal{H}$ breaks up into a direct sum $\mathcal{H} = \oplus\mathcal{H}_j$ of orthogonal Hilbert spaces $\mathcal{H}_j$ – the superselection sectors, carrying inequivalent irreducible representations π_j of $\mathcal{A}$. The sector containing the vacuum state $|0\rangle$ of the theory is called the vacuum sector $\mathcal{H}_0$, and the associated representation of $\mathcal{A}$ is known as the vacuum representation π_0. The guiding principle of algebraic field theory is that all physical information should be contained in the vacuum representation of the observable algebra.

The causal structure of space-time is implemented by the requirement that elements belonging to local algebras $\mathcal{A}(\mathcal{O}_1)$, $\mathcal{A}(\mathcal{O}_2)$ should commute if $\mathcal{O}_1$ and $\mathcal{O}_2$ are relatively space-like. In particular, denoting the space-like complement of $\mathcal{O}$ by $\mathcal{O}^c$, any element of $\pi_0(\mathcal{A}(\mathcal{O}^c))$ must commute with any element of $\pi_0(\mathcal{A}(\mathcal{O}))$, or in other words $\pi_0(\mathcal{A}(\mathcal{O}^c)) \subseteq \pi_0(\mathcal{A}(\mathcal{O}))'$, where the prime denotes the commutant (in the space $\mathcal{B}(\mathcal{H}_0)$ of bounded operators on the vacuum Hilbert space $\mathcal{H}_0$). The stronger property

$$\pi_0(\mathcal{A}(\mathcal{O}^c)) = \pi_0(\mathcal{A}(\mathcal{O}))' \tag{3.1}$$

which states that, in a certain sense, the observable algebra $\mathcal{A}$ is maximal, is known as *Haag duality* (of the vacuum representation [1]). While not every possible local algebra $\mathcal{A}(\mathcal{O})$ is Haag dual, one can always enlarge it such that it acquires this property, namely by forming the double commutant $\mathcal{A}(\mathcal{O})'' \supseteq \mathcal{A}(\mathcal{O})$.

If Haag duality holds for π_0, one usually identifies $\mathcal{A}(\mathcal{O})$ with its vacuum representation $\pi_0(\mathcal{A}(\mathcal{O}))$. This makes sense because in this case all the asymptotically vacuum-like representations π_j of $\mathcal{A}$ are obtainable, up to unitary equivalence, by composing π_0 with appropriate ("transportable") localized endomorphisms ρ_i of the observable algebra,

$$\pi_i \cong \pi_0 \circ \rho_i . \tag{3.2}$$

An endomorphism ρ is called localized in $\mathcal{O}$ iff ρ is the identity on $\mathcal{A}(\mathcal{O}^c)$, and it is called transportable iff it satisfies a somewhat stronger property [7] (which will not be of interest here). Localized irreducible endomorphisms ρ and $\tilde\rho$ are defined to be (unitarily) equivalent, $\rho \cong \tilde\rho$, if there exists a unitary operator U such that

$$\tilde\rho(a) = U\rho(a)U^* \tag{3.3}$$

for all $a \in \mathcal{A}$. It follows from Haag duality and causality that the unitary operator U appearing in (3.3) must in fact belong to some local observable algebra. Thus, defining, for any unitary $U \in \mathcal{A}$, a corresponding inner automorphism σ_U of $\mathcal{A}$ by

$$\sigma_U(a) = U\,a\,U^* \tag{3.4}$$

for all $a \in \mathcal{A}$, one has $\rho \cong \tilde\rho$ iff $\tilde\rho = \sigma_U \circ \rho$ for some unitary $U \in \mathcal{A}$.

The field algebra $\mathcal{F}$ is obtained from $\mathcal{A}$ by adjoining charged fields f which realize the endomorphisms ρ, in the sense that $a\,f = f\,\rho(a)$ for all $a \in \mathcal{A}$. Given the equivalence (3.2), one is naturally led to define a "cross product" of the representations π_i via

$$\pi_i \times \pi_j :\cong \pi_0 \circ \rho_i \circ \rho_j . \tag{3.5}$$

When the endomorphisms ρ_j are localized, this corresponds to an operator product of the corresponding charged fields f_j, and the structure constants of this product can in principle be determined form the intertwiners between the endomorphism $\rho_i \circ \rho_j$ and the irreducible endomorphisms ρ_k into which it can be decomposed. One also defines an associated cross product of "generalized state vectors" Ψ_j which are pairs (ρ_j, ψ_j) with $\psi_j \in \mathcal{H}_0$, via

$$\Psi_i \times \Psi_j = (\rho_j \circ \rho_i\,,\, \rho_j(U_i)U_j|0\rangle) \tag{3.6}$$

with the unitary operator U_i defined by $\rho_i = \sigma_{U_i}$. One then finds [7]

$$\Psi_i \times \Psi_j = \epsilon(\rho_i, \rho_j)\,\Psi_j \times \Psi_i \tag{3.7}$$

with the *statistics operator*

$$\epsilon(\rho_i, \rho_j) = \rho_j(U_i)\,U_j\,U_i^*\,\rho_i(U_j^*). \tag{3.8}$$

Denoting $\epsilon_\rho := \epsilon(\rho, \rho) \in (\rho^2(\mathcal{A}))'$, and defining the linear map η_ρ by $\eta_\rho \circ \rho = id$, it follows for irreducible ρ that $\eta_\rho(\epsilon_\rho) = \lambda_\rho\,\mathbb{1}$. (One can show that the map η_ρ always exists, and is unique except for the pathological case $\lambda_\rho = 0$ which is never realized in conformal field theory). The number λ_ρ is called the *statistics parameter* of ρ_j (or of the superselection sector $\mathcal{H}_j$), its phase

[1] One may also investigate the analogue of (3.1) for other representations of $\mathcal{A}(\mathcal{O})$. Thus Haag duality is to be considered as a property of the representation, not of the algebra $\mathcal{A}(\mathcal{O})$ itself. A necessary condition for π_j to satisfy Haag duality is that its statistical dimension D_j is equal to 1.

is called the *statistics phase*, and its inverse modulus $D_\rho := |\lambda_\rho|^{-1}$ the *statistical dimension* of ρ_j. For conformal field theories, the statistics phase is determined by the conformal dimension Δ_j of the associated primary field ϕ_j, according to

$$\lambda_j \equiv \lambda_{\rho_j} = e^{2\pi i \Delta_j} / D_j, \tag{3.9}$$

and D_j coincides with the quantum dimension $\mathcal{D}_j$ of ϕ_j [7].

Finally I briefly mention a few basics of the Tomita-Takesaki modular theory [18]. By examining the operation $a|0\rangle \mapsto a^*|0\rangle$ on the Hilbert space of a local observable algebra $\mathcal{A}(\mathcal{O})$, one is led to define the modular conjugation J (an anti-unitary involution) and the (positive, selfadjoint) modular operator $\mathbf{\Delta}$. They satisfy $J\mathcal{A}(\mathcal{O})J = \mathcal{A}(\mathcal{O})'$ and, for any real t, $\mathbf{\Delta}^{it}\mathcal{A}(\mathcal{O})\mathbf{\Delta}^{-it} = \mathcal{A}(\mathcal{O})$ (these properties can for example be employed to facilitate the investigation of Haag duality [9]) as well as $J\mathbf{\Delta}^{it}J = \mathbf{\Delta}^{it}$. For a given representation π of $\mathcal{A}(\mathcal{O})$, the maps $\sigma_t : a \mapsto \pi^{-1}(\mathbf{\Delta}^{it}\pi(a)\mathbf{\Delta}^{-it})$ form a group known as the modular automorphism group, which plays an important role in the analysis of the thermodynamic properties of the theory.

4 Deviations from the canonical framework

As outlined in the introduction, it should be possible to incorporate concrete models of two-dimensional conformal field theory into the framework of algebraic field theory by heavily exploiting the information provided by the conformal field theory approach, e.g. the knowledge about the chiral algebra $\mathcal{W}$ and its representation theory. In particular, one would like to obtain the endomorphisms ρ_i which represent the superselection sectors by imposing the requirement that their equivalence classes satisfy the fusion rules of the conformal field theory, i.e.

$$[\rho_i \circ \rho_j] = \sum_k \mathcal{N}_{ij}{}^k [\rho_k]. \tag{4.1}$$

It turns out [8] that the realization of this programme requires to relax the canonical framework of algebraic field theory that was described in section 3 in two respects. Namely,

- one allows the observable algebra to contain unbounded operators, and
- one allows the observable algebra to contain global elements.

The former generalization is necessary because the generators of the chiral algebra $\mathcal{W}$ themselves are not bounded (which is a manifestation of the fact that the nontrivial representations of $\mathcal{W}$ are infinite-dimensional). The source of the second generalization is the observation that the relevant "space-time" (which is one-dimensional as a result of the factorization of the symmetry algebra of the conformal field theory into a purely holomorphic and a purely antiholomorphic part) is a compactified light-cone [19] and hence is topologically a circle S^1. In terms of the chiral algebra, this is shows up in the fact that $\mathcal{W}$ contains central generators; these are not multiples of unity, but nevertheless just numbers in any irreducible representation, in particular in π_0, and hence are not local observables (they can be measured in any arbitrary space-time region). Note that the presence of central terms also implies that π_0 is not faithful, and hence the observable algebra $\mathcal{A}$ cannot be identified with the image $\pi_0(\mathcal{A})$.

As a consequence, it is natural to work with a global algebra $\overline{\mathcal{A}}$ of unbounded observables, and to assume that this algebra contains $\mathcal{W}$ as a subalgebra, or more precisely (since $\mathcal{W}$ is a Lie algebra while $\overline{\mathcal{A}}$ should be associative),

$$\overline{\mathcal{A}} \supseteq \mathcal{U}(\mathcal{W}), \tag{4.2}$$

where $\mathcal{U}(\mathcal{W})$ denotes the universal eveloping algebra of $\mathcal{W}$. After having obtained all desired results in terms of $\overline{\mathcal{A}}$, one can recover, in principle, the local algebras $\mathcal{A}(\mathcal{O})$ by building bounded functions of the operators in $\overline{\mathcal{A}}$ and by smearing with test functions with support in $\mathcal{O}$. Nevertheless the introduction of $\overline{\mathcal{A}}$ must be considered as a major deviation form the canonical framework, because in the canonical approach the physics is actually not encoded in the local algebras $\mathcal{A}(\mathcal{O})$ themselves, but in the net structure of inclusions among the local algebras.

Having relaxed the quasilocality of $\mathcal{A}$ it is natural to look for non-localized endomorphisms of the chiral algebra that represent its inequivalent representations. It turns out that in order to obtain these endomorphisms, it is necessary to extend the chiral algebra, so that in particular (4.2) can be refined to

$$\overline{\mathcal{A}} \supset \mathcal{U}(\mathcal{W}). \tag{4.3}$$

This can be understood as a consequence of the fact that the global chiral algebra $\mathcal{W}$ does not satisfy Haag duality. There is a simple argument why $\mathcal{W}$ cannot be Haag dual. For simplicity consider the Virasoro subalgebra Vir $\subseteq \mathcal{W}$. The local algebra Vir$(\mathcal{O})$ consists of diffeomorphisms that are localized in an interval $\mathcal{O} \subset S^1$, in the sense that outside this interval they act as the identity map. The double commutant Vir$(\mathcal{O})''$ of Vir$(\mathcal{O})$ in the space $\mathcal{B}(\mathcal{H}_0)$ can serve as a local observable algebra and satisfies Haag duality [9]. The algebra Vir$(\mathcal{O})''$ certainly contains not only the diffeomorphisms, but also bounded functions of operators which act by multiplication by functions that are constant on $\mathcal{O}^c = S^1\setminus\mathcal{O}$. Therefore a necessary condition for being able to describe the observable algebra $\mathcal{A}(\mathcal{O})$ satisfying Haag duality as bounded functions of a Lie algebra of unbounded observables, is that Vir$(\mathcal{O})$ be extended by the Lie algebra of functions on the circle. (Of course it must checked that the extended observable algebra has the same superselection structure as the chiral algebra one started with. In addition, having obtained the non-localized endomorphisms, one has to show that it is possible to construct localized ones giving rise to the same representations.)

Given the necessity of extending the chiral algebra $\mathcal{W}$, one finds oneself in a rather intricate situation: The task is to construct endomorphisms of an algebra which itself is not yet fully known. Fortunately, the experience from conformal field theory suggests a way out of this dilemma: One assumes that there exists some underlying large associative algebra $\mathcal{M}$ (whose existence is related to a free field realization of the theory) from which both $\mathcal{W}$ and its as yet unknown extension can be constructed, and in addition that the endomorphisms ρ_j are endomorphisms of this underlying algebra $\mathcal{M}$ as well.

In the cases analysed so far [8, 10], such an algebra $\mathcal{M}$ does exist. Moreover, it turns out that for all these theories the extension of $\mathcal{W}$ is again a Lie algebra $\overline{\mathcal{L}}$ so that

$$\overline{\mathcal{A}} = \mathcal{U}(\overline{\mathcal{L}}) \tag{4.4}$$

with

$$\overline{\mathcal{L}} \supset \mathcal{W}. \tag{4.5}$$

Let me stress that it is trivially true that $\overline{\mathcal{A}}$ can be written as the enveloping algebra of some Lie algebra $\overline{\mathcal{L}}$ (namely one whose Lie bracket is defined as the commutator with respect to the product of $\overline{\mathcal{A}}$). It is also certainly plausible that (4.2) holds. What is nontrivial is that the Lie algebra $\overline{\mathcal{L}}$ contains $\mathcal{W}$ as a Lie subalgebra.

In addition, in the cases that have been treated, the Lie algebra $\overline{\mathcal{L}}$ is obtained after a finite number of steps, each step consisting either of application of the endomorphisms ρ or of taking commutators. Again it is clear that, given a Lie algebra $\mathcal{W}$ realized in an associative algebra $\mathcal{M}$ in terms of commutators, and given an endomorphism ρ of $\mathcal{M}$, one may construct a Lie algebraic extension of $\mathcal{W}$ on which the corresponding lifted endomorphisms close, and it is also clear that closure is obtained when adjoining to $\mathcal{W}$ successively the generators obtained by application of ρ and by commutation. What is not at all obvious is that one arrives this way at a closed Lie algebra after a finite number of steps.

5 The level one so($\mathcal{N}$) WZW theories

As indicated above, the extension of $\mathcal{W}$ on which the endomorphisms representing the superselection sectors close, is constructed with the help of an underlying associative algebra $\mathcal{M}$. For many conformal field theories the presence of such an algebra is guaranteed by the existence of a free field realization of the theory: $\mathcal{M}$ is essentially the algebra generated by the Fourier-Laurent modes of the free fields. So far only the simplest free field realizations have been analysed in the algebraic context, namely the cases where the theory can be described by free fermions. These are the following conformal field theories:

- the two-dimensional Ising model, treated in [8], for which $\mathcal{W} = \mathrm{Vir}_{1/2}$;
- the level 1 so($\mathcal{N}$) WZW [2] theories, treated in [10], for which $\mathcal{W} = \mathrm{Vir}_{\mathcal{N}/2} \dot{+} \widehat{so}(\mathcal{N})_1$.

In the following I will discuss the latter theories. They can be realized as the theory of $\mathcal{N}$ free massless Majorana fermions (here $\mathcal{N} = 3, 4, ...$; by formally setting $\mathcal{N} = 1$, one recovers the Ising model). Before going into technical details, let me repeat that all that follows is to a large extent a warm-up exercise, whereas the more interesting problems are those listed in section 2.

The representations of $\widehat{so}(\mathcal{N})_1$ are well known [20]; they extend to the semidirect sum $\mathrm{Vir}_{\mathcal{N}/2} \dot{+} \widehat{so}(\mathcal{N})_1$. For even $\mathcal{N}$ there are four physical representations – the basic representation, the vector, the spinor and the conjugate spinor; they will be denoted by 0, v, s, and c. In this symbolic notation, the nontrivial part of the WZW fusion rules reads

$$\mathrm{v} * \mathrm{v} = \mathrm{s} * \mathrm{s} = \mathrm{c} * \mathrm{c} = 0, \quad \mathrm{s} * \mathrm{c} = \mathrm{v} \qquad \text{for } \mathcal{N} \in 4\mathbb{Z}, \tag{5.1}$$

respectively

$$\mathrm{v} * \mathrm{v} = \mathrm{s} * \mathrm{c} = 0, \quad \mathrm{s} * \mathrm{s} = \mathrm{c} * \mathrm{c} = \mathrm{v} \qquad \text{for } \mathcal{N} \in 4\mathbb{Z} + 2. \tag{5.2}$$

For odd $\mathcal{N}$ the fusion rules are identical to those of the Ising model: There are three physical representations – the basic, vector and spinor representations, denoted by 0, v, and σ – and the nontrivial fusion rules read

$$\mathrm{v} * \mathrm{v} = 0, \quad \sigma * \sigma = 0 + \mathrm{v}, \quad \sigma * \mathrm{v} = \sigma\,. \tag{5.3}$$

The Fourier-Laurent modes of the fermions give rise to the *universal Majorana algebra* Maj. This is the associative *-algebra with identity **1** that is generated freely by the fermion modes b^i_q, $j = 1, \dots, \mathcal{N}$, $p \in \frac{1}{2}\mathbb{Z}$, and central elements **1**, Y, subject to the hermiticity condition $(b^j_p)^* = b^j_{-p}$, the anticommutation relations

$$\{b^i_p, b^j_q\} = \tfrac{1}{2}\,\delta^{ij}\delta_{p+q,0}\,[\mathbf{1} + (-1)^{2q}\,Y], \tag{5.4}$$

and $[b^i_p, Y] = 0 = [b^i_p, \mathbf{1}]$. Note that both integer and half integer mode numbers q are allowed, i.e. both the Neveu-Schwarz (NS) sector ($q \in \mathbb{Z} + \frac{1}{2}$) and the Ramond (R) sector ($q \in \mathbb{Z}$) of the fermion theory are described simultaneously. Correspondingly there is, in addition to **1**, a second central generator Y which distinguishes between the two sectors according to

$$\begin{aligned} \pi_{\mathrm{NS}}(b^i_p) &= 0 \quad \text{for} \quad p \in \mathbb{Z}, & \pi_{\mathrm{NS}}(Y) &= -1, \\ \pi_{\mathrm{R}}(b^i_p) &= 0 \quad \text{for} \quad p \in \mathbb{Z} + \tfrac{1}{2}, & \pi_{\mathrm{R}}(Y) &= 1 \end{aligned} \tag{5.5}$$

(thus $-Y$ corresponds to a rotation by 2π of the space-time S^1).

The global observable algebra $\overline{\mathcal{A}}$ will be obtained as a subalgebra of an appropriate completion of Maj by infinite power series. Moreover, $\overline{\mathcal{A}}$ will be the enveloping algebra of a Lie algebra $\overline{\mathcal{L}}$, which in accordance with the ideas outlined in section 4 contains the chiral algebra $\mathcal{W}$. For WZW theories, $\mathcal{W}$ = Vir $\oplus$ g is the semidirect sum of the Virasoro algebra Vir with an untwisted affine Kac-Moody algebra g; in the case under consideration, $g = \widehat{so}(N)$, and the generators of $\mathcal{W}$ are expressible as infinite series of bilinears of the generators b^i_p of Maj.

$$\begin{aligned} J^a_m &= \tfrac{1}{2}\sum_{i,j}(T^a)_{ij} \sum_{q\in\mathbb{Z}/2} :b^i_q b^j_{m-q}: , \\ L_m &= -\tfrac{1}{2}\sum_{i} \sum_{q\in\mathbb{Z}/2} (q - \tfrac{m}{2}) :b^i_q b^i_{m-q}: + \tfrac{N}{32}(1 + Y)\,\delta_{m,0} \end{aligned} \tag{5.6}$$

for $m \in \mathbb{Z}$. Here T^a, $a = 1,2,\ldots,N(N-1)/2$, are the matrix generators of the simple Lie algebra $so(N)$ in the vector representation, and the colons denote normal ordering. By direct computation, one verifies that the combinations (5.6) satisfy the commutation relations

$$\begin{aligned} [J^a_m, J^b_n] &= m\,\kappa^{ab}\,\delta_{m+n,0} + f^{ab}{}_c\, J^c_{m+n}, \\ [L_m, L_n] &= \tfrac{N}{24}(m^3 - m)\,\delta_{m+n,0} + (m - n)\,L_{m+n}, \\ [L_m, J^a_n] &= -n\, J^a_{m+n}, \end{aligned} \tag{5.7}$$

where $f^{ab}{}_c$ and κ^{ab} are the structure constants and Cartan-Killing form of $so(N)$, respectively, and the summation convention is used for the adjoint indices $a, b, \ldots$. Thus indeed the operators (5.6) generate the semidirect sum of the Virasoro algebra Vir (generated by the L_m) and the untwisted affine Lie algebra $\widehat{so}(N)$ (generated by the J^a_m), with the respective central generators fixed to the values $c = N/2$ and $k = 1$, respectively.

When looking for endomorphisms of $\mathrm{Vir}_{N/2} \oplus \widehat{so}(N)_1$, one finds immediately that arbitrary finite sums of the fermion bilinears must be included into the extension $\overline{\mathcal{L}}$. These bilinears form an infinite-dimensional orthogonal Lie algebra

$$O_\infty := \mathrm{span}\left\{ 1,\, b^i_p b^j_q \mid i,j \in \{1,2,\ldots,N\},\ p,q \in \tfrac{1}{2}\mathbb{Z},\ p - q \in \mathbb{Z} \right\} \tag{5.8}$$

(this includes the central generator Y). For example, the action of distinct endomorphisms belonging to the same class will differ by elements of O_∞,

$$\rho(a) - \tilde\rho(a) \in O_\infty \quad \text{for } \tilde\rho \in [\rho], \tag{5.9}$$

for a any even power series in the b^i_p. The commutator of any element of O_∞ with an element of the Virasoro or the affine algebra is again in O_∞; thus $\overline{\mathcal{L}}$ contains the Lie algebra

$$\mathcal{L} = O_\infty \oplus \mathcal{W} = O_\infty \oplus \mathrm{Vir}_{N/2} \oplus \widehat{so}(N)_1 \tag{5.10}$$

(semidirect sum) as a subalgebra. The irreducible highest weight modules of $\mathcal{L}$ are the same as those of $\mathcal{W}$. The NS- and R-sector Fock spaces $\mathcal{H}_{\mathrm{NS}}$, $\mathcal{H}_{\mathrm{R}}$ of Maj split into these irreducible modules as $\mathcal{H}_{\mathrm{NS}} = \mathcal{H}_0 \oplus \mathcal{H}_v$ and as $\mathcal{H}_{\mathrm{R}} = \bigoplus_{i=1}^{2^{N/2}} (\mathcal{H}_s^{(i)} \oplus \mathcal{H}_c^{(i)})$ for N even, respectively $\mathcal{H}_{\mathrm{R}} = \bigoplus_{i=1}^{2^{(N+1)/2}} \mathcal{H}_\sigma^{(i)}$ for N odd.

6 Endomorphisms of the Majorana algebra

The task is now to construct *-endomorphisms ρ_j ($j = 0, v, s, c$ for N even, $j = 0, v, \sigma$ for N odd) of Maj which obey the fusion rules (4.1) as well as the equivalence relations (3.2). For $j = 0$, of course $\rho_0 = id$, the identity map, does the job; more generally, any map of the form

$$\tilde\rho_0 = \sigma_U, \tag{6.1}$$

with the automorphism σ_{U} of Maj defined in analogy with (3.4), and with the unitary U appearing in this definition being an even polynomial in the fermion modes, is a representative of $[\rho_0]$. It is also easy to find the vector endomorphisms ρ_v of Maj by imposing the fusion rule $[\rho_v] \circ [\rho_v] = [id]$: One has $\bar\rho_v \in [\rho_v]$ iff

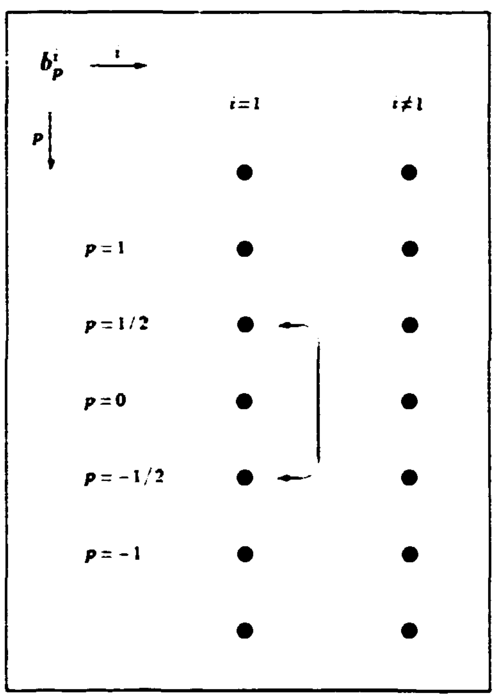
$$\bar\rho_v = \sigma_{U_v} \tag{6.2}$$

with the unitary U_v an odd polynomial in the fermion modes. One can in fact immediately identify also those representatives which directly give $\rho_v \circ \rho_v = id$, namely those σ_{U_v} for which in addition $U_v = U_v^*$; an example is

$$U_v = \begin{cases} b^j_{1/2} + b^j_{-1/2} & \text{on } \mathsf{Maj}_{NS}, \\ \sqrt{2}\, b^j_0 & \text{on } \mathsf{Maj}_{R} \end{cases} \tag{6.3}$$

for fixed $j \in \{1,2,\dots,N\}$.

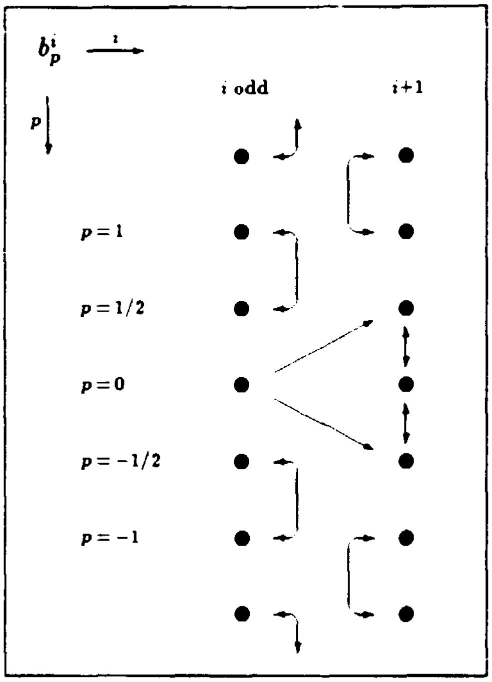
For the spinor endomorphisms, the results look more complicated. For explicit formulæ, I refer to [10]; here I will instead display the endomorphisms in a pictorial manner. Let the modes b^i_p be represented by small disks, with the index $i \in \{1,2,\dots,N\}$ running horizontally and the index $p \in \mathbb{Z}/2$ running vertically, i.e. ${}_p\overset{i}{\bullet}$, and let the action of the maps ρ be depicted by arrows pointing from ${}_p\overset{i}{\bullet}$ to ${}_q\overset{j}{\bullet}$ if $\rho(b^i_p) \propto b^j_q$, respectively by several arrows if b^i_p is mapped into a linear combination of several modes b^j_q. In this notation, the vector endomorphism defined by (6.3) with $j = 1$ acts as shown in the figure (6.4).



(6.4)

ρ_v

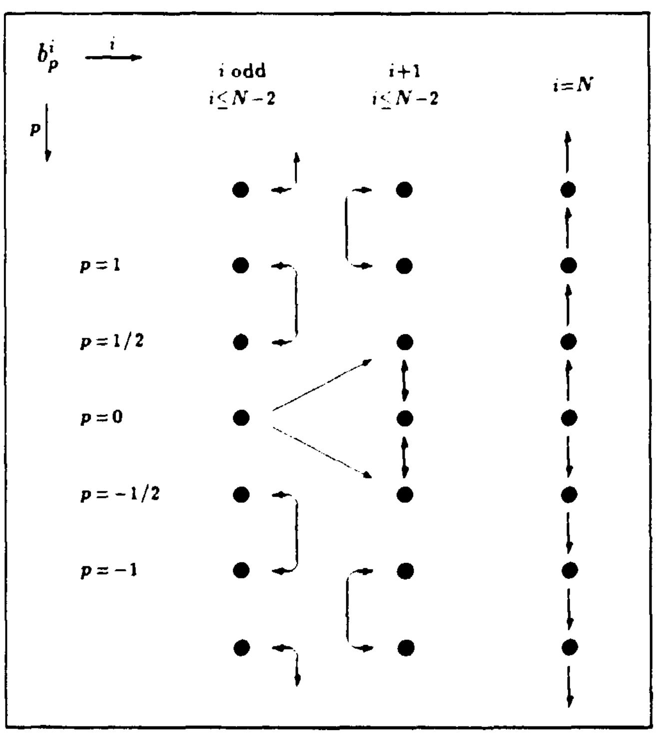
For a typical spinor endomorphism ρ_s the corresponding picture is basically given by



(6.5)

ρ_s, ρ_c

The important difference from (6.4) is that transitions are made between the NS- and R-sectors, and that the map acts nontrivially on an infinite number of modes b_p^i. The action of ρ_c is essentially given by (6.5) as well (up to sign, ρ_c differs from ρ_s only when acting on a finite number of modes b_p^i). In contrast, for ρ_σ an additional new feature arises. Namely, for, say, $i = N$ the arrows point only in one direction, while for $i < N$, ρ_σ coincides with the endomorphism ρ_s of the so($N - 1$) WZW theory. This is displayed in figure (6.6).



(6.6)

ρ_σ

Already from this pictorial representation one can read off that the endomorphisms ρ_j for $j = 0, v, s, c$ are *auto*morphisms, while ρ_σ is a genuine endomorphism, with $\dim(\mathrm{coker}(\rho_\sigma)) = 2$ (the cokernel is spanned by b_0^N and a linear combination of $b_{1/2}^N$ and $b_{-1/2}^N$). It is also easy to check that $\rho_j(b^i_{-q}) = (\rho_j(b^i_q))^*$ and hence the ρ_j are *-endomorphisms. With a bit more effort one verifies that the ρ_j satisfy the requirement (3.2) (actually one checks that (3.2) is valid on $\mathcal{W}$; the result then trivially extends to the whole of $\mathcal{L} = \mathcal{W} \oplus O_\infty$; later on one must check that it also extends to the rest of $\overline{\mathcal{L}}$, which is however again rather simple).

Finally one can show that the endomorphisms ρ_j reproduce the correct fusion rules. If at least one of the maps involved is an automorphism, one merely has to identify a unitary $U \in O_\infty$ such that $\rho_i \circ \rho_j = \sigma_U \circ \rho_k$. This is straightforward, if lengthy. For N even one finds

$$\begin{aligned} &[\rho_v \circ \rho_v] = [id], \\ &[\rho_s \circ \rho_v] = [\rho_v \circ \rho_s] = [\rho_c], \quad [\rho_c \circ \rho_v] = [\rho_v \circ \rho_c] = [\rho_s], \\ &[\rho_s \circ \rho_s] = [\rho_c \circ \rho_c] = [id], \quad [\rho_s \circ \rho_c] = [\rho_c \circ \rho_s] = [\rho_v] \qquad \text{for} \quad N \in 4\mathbb{Z}, \\ &[\rho_s \circ \rho_c] = [\rho_c \circ \rho_s] = [id], \quad [\rho_s \circ \rho_s] = [\rho_c \circ \rho_c] = [\rho_v] \qquad \text{for} \quad N \in 4\mathbb{Z} + 2. \end{aligned} \tag{6.7}$$

Thus the fusion rules (5.1) and (5.2) are reproduced as required. Analogously, one verifies

$$[\rho_\sigma \circ \rho_v] = [\rho_v \circ \rho_\sigma] = [\rho_\sigma] \tag{6.8}$$

for N odd. It remains to analyse the composite endomorphism $\rho_\sigma \circ \rho_\sigma$. This is done in two steps. First, maps $\sigma_U \in [id]$, $\tilde\rho_v \in [\rho_v]$, and operators $\mathcal{I}_0$ and $\mathcal{I}_v$ satisfying the identities

$$\sigma_U(a)\mathcal{I}_0 = \mathcal{I}_0\, \rho_\sigma \circ \rho_\sigma(a), \qquad \tilde\rho_v(a)\mathcal{I}_v = \mathcal{I}_v\, \rho_\sigma \circ \rho_\sigma(a), \tag{6.9}$$

are identified. Thus $\mathcal{I}_0$ and $\mathcal{I}_v$ are intertwiners between $[\rho_\sigma \circ \rho_\sigma]$ and $[id]$ and $[\rho_v]$, respectively, and hence $[\rho_\sigma \circ \rho_\sigma] \supseteq [id] + [\rho_v]$. Next one observes that

$$\mathcal{I}_0\mathcal{I}_0^* = \mathcal{I}_v\mathcal{I}_v^* = \mathbf{1}, \qquad \mathcal{I}_0^*\mathcal{I}_0 = P_0, \qquad \mathcal{I}_v^*\mathcal{I}_v = P_v, \tag{6.10}$$

with P_0, P_v orthogonal projectors ($P_0 + P_v = \mathbf{1}$, $P_0P_0 = P_0$) so that the maps $\mathcal{I}_0 : P_0\mathcal{H}_0 \to \mathcal{H}_0$ and $\mathcal{I}_v : P_v\mathcal{H}_0 \to \mathcal{H}_0$ are bijective. Together it follows that

$$[\rho_\sigma \circ \rho_\sigma] = [id] + [\rho_v]. \tag{6.11}$$

Thus also for odd N the composition of classes $[\rho_j]$ gives the relevant fusion rules, namely (5.3).

7 Construction of $\overline{\mathcal{L}}$

Having obtained the endomorphisms ρ_j, it is straightforward to compute their action on $\mathcal{L} = \mathcal{W} \oplus O_\infty$. Again some of the calculations are lengthy, and I only give rough overview. Motivated by relations such as (5.9), one introduces the notation

$$a \simeq b \;:\Longleftrightarrow\; a - b \in O_\infty, \tag{7.1}$$

which makes the results more readable. For example, in this notation the formulæ

$$\rho_v(L_m) \simeq L_m, \qquad \rho_v(J^a_m) \simeq J^a_m \tag{7.2}$$

describe the action of the endomorphism ρ_v on the chiral algebra $\mathcal{W}$. (7.2) implies $\rho_v(\mathcal{L}) \subsetneq \mathcal{L}$.

For the spinor endomorphisms, the situation is again more involved. First, as already mentioned, ρ_σ for N odd is closely related to ρ_s at $\tilde{N} = N - 1$; moreover, for N even one has

$$\rho_c(a) \simeq \rho_s(a) \tag{7.3}$$

for all $a \in \mathcal{L}$. Therefore I can restrict attention to, say, ρ_s. On the Virasoro algebra, ρ_s acts as

$$\rho_s(L_m) \simeq L_m + F_m, \tag{7.4}$$

where

$$F_m \simeq \tfrac{1}{2}\sum_i \sum_{q\geq m/2} (-1)^{i+2q}\, b^i_{m-q} b^i_q, \tag{7.5}$$

(the operators F_m essentially generate the algebra of functions on S^1). Also, $\rho_s(F_m) \simeq -F_m$ so that in particular $\rho_s \circ \rho_s(L_m) \simeq L_m$. Furthermore,

$$[L_m, F_n] \simeq -n\, F_{m+n}, \qquad [F_m, F_n] \simeq 0. \tag{7.6}$$

For $\rho_s(J^a_m)$, the formulæ are more complicated. The results are best described by introducing the infinite sums

$$D^{ij}_\pm(m) := \sum_{\substack{q>m/2 \\ q\in\mathbb{Z}+(1\pm1)/4}} b^i_{m-q} b^j_q, \qquad m \in \mathbb{Z}, \tag{7.7}$$

of fermion bilinears. In this notation, one has e.g.

$$F_m \simeq \tfrac{1}{2} \sum_i (-1)^i \left(D^{ii}_-(m) - D^{ii}_+(m)\right), \tag{7.8}$$

and also

$$J^a_m \simeq \tfrac{1}{2}\sum_{i,j} (T^a)_{ij} \left[D^{ij}_+(m) + D^{ij}_-(m) - D^{ji}_+(m) - D^{ji}_-(m)\right]. \tag{7.9}$$

The operators $D^{ij}_\pm(m)$ satisfy the commutation relations $[D^{ij}_+(m), D^{kl}_-(n)] \simeq 0$ and

$$[D^{ij}_\pm(m), D^{kl}_\pm(n)] \simeq \delta^{jk} D^{il}_\pm(m+n) - \delta^{il} D^{kj}_\pm(m+n). \tag{7.10}$$

Neglecting finite sums in the bilinears $b^i_p b^j_q$, this is, for fixed subscript $\pm$, the loop algebra associated to sl(N). In fact, the algebra of the zero modes $D^{ij}_+(0)$ is fully isomorphic to sl(N). [2] Including also the generators F_m, one obtains a structure similar to the loop algebra of gl(N). Consequently, the notation $\widetilde{\mathrm{gl}}(N)$ has been chosen in [10] for the vector space spanned by the generators $D^{ij}_\pm(m)$.

The relevance of the operators $D^{ij}_\pm(m)$ is most directly seen by acting on them with a spinor endomorphism; this induces a change in the sign $\pm$ and a shift in the mode number m:

$$\rho_s(D^{ij}_\pm(m)) \simeq D^{ij}_\mp(m \mp \tfrac{1}{2}[(-1)^i - (-1)^j]). \tag{7.11}$$

The shift is not symmetric in i and j; as a consequence, one must consider $D^{ij}_\pm$ and $D^{ji}_\pm$ independently, not just their difference as in J^a_m. Similarly, the shift depends nontrivially on

[2] In contrast, the algebra generated by the zero modes $D^{ij}_-(0)$ is isomorphic to sl(N) only up to finite sums in the bilinears $b^i_0 b^j_0$. This corresponds to the fact that under the embedding sl(N) $\supset$ so(N) none of the modules of sl(N) branches to the irreducible spinor modules of so(N) which are relevant to the R-sector (whereas the singlet and vector of so(N) which are relevant to the NS-sector are just the image of the singlet and defining module of sl(N), respectively).

the $\pm$ subscript so that one must consider D^{ij}_{+} and D^{ij}_{-} independently, not just their sum as in J^a_m. Thus ρ_s can close only if all of the operators $D^{ij}_{\pm}(m)$ are included into $\overline{\mathcal{L}}$.

It also follows that further commutation and application of ρ_s does not introduce any new generators. For example, one has $[D^{ij}_{\pm}(m), \rho_s(D^{kl}_{\pm}(n))] \simeq 0$ and

$$[D^{ij}_{\pm}(m), \rho_s(D^{jk}_{\mp}(n))] \simeq D^{ik}_{\pm}(m + n \pm \tfrac{1}{2}[(-1)^j - (-1)^k]) \tag{7.12}$$

(and an analogous formula for $[D^{ij}_{\pm}(m), \rho_s(D^{kl}_{\mp}(n))]$ for generic values of i, j, k, l). Finally,

$$[L_m, D^{ij}_{\pm}(n)] \simeq -n\, D^{ij}_{\pm}(m+n). \tag{7.13}$$

Thus one can summarize: The Lie algebra $\overline{\mathcal{L}}$ of global observables is the algebra

$$\overline{\mathcal{L}} = O_\infty \oplus \mathrm{Vir}_{N/2} \oplus \tilde{\mathrm{gl}}(N), \tag{7.14}$$

where O_∞ is spanned by the bilinears $b^i_p b^j_q$ (and by $\mathbf{1}$), $\mathrm{Vir}_{N/2}$ is spanned by the infinite sums L_m defined in (5.6), and (for N even, and similarly for N odd) $\tilde{\mathrm{gl}}(N)$ is spanned by the infinite sums $D^{ij}_{\pm}(m)$ defined in (7.7). Here $i,j \in \{1,2,\dots,N\}$, $p,q \in \frac{1}{2}\mathbb{Z}$, $p-q \in \mathbb{Z}$, and $m \in \mathbb{Z}$.

Let me come back to the finding that the endomorphisms ρ_j do not close on the algebra $\mathcal{L}$ introduced in (5.10), but rather on the Lie algebra $\overline{\mathcal{L}}$ which is given by (7.14). It should be noted that the algebra $\mathcal{L}$ is the natural first guess for the Lie algebra of global observables as it corresponds to the chiral symmetry algebra of the WZW theory, supplemented by the fermion bilinears. In contrast, the method described above by which the extension $\overline{\mathcal{L}}$ is found [8, 10] is purely constructive; in particular, a deeper (e.g. geometrical) understanding of the $\tilde{\mathrm{gl}}(N)$ part of $\overline{\mathcal{L}}$ is not yet available.

As already mentioned, the generators of the vector space $\tilde{\mathrm{gl}}(N) \subset \overline{\mathcal{L}}$ satisfy commutation relations which up to finite sums of fermion bilinears coincide with those of the loop algebra of $\mathrm{gl}(N)$, and hence also with the $\widehat{\mathrm{gl}}(N)$ Kac-Moody algebra. Note that $\tilde{\mathrm{gl}}(N)$ is only a sub-vector space of $\overline{\mathcal{L}}$, not a subalgebra: The generators $D^{ij}_{\pm}(m)$ of $\tilde{\mathrm{gl}}(N)$ close under commutation only up to finite sums of fermion bilinears, i.e. form a closed Lie algebra only when combined with the orthogonal Lie algebra O_∞ defined in (5.8).[3] Note that there is no way to obtain a closed algebra by adjusting the finite contributions to the non-antisymmetric generators $D^{ij}_{\pm}(m)$. It would in fact be rather disturbing if this were possible, because then the operators $D^{ij}_{\pm}(m)$ could be interpreted as the modes of a primary conformal field of conformal dimension 1. Such a field would necessarily be a Kac-Moody current, and as a consequence the chiral symmetry algebra of the theory would have to be larger than the semidirect sum of $\widehat{\mathrm{so}}(N)_1$ with $\mathrm{Vir}_{N/2}$, which is, however, just the correct maximal chiral algebra of the WZW theory.

To conclude, let me stress again that there are many questions yet to be answered, see section 2 above. One line of future research is certainly to extend the results to more complicated conformal field theory models. The WZW theories described here already provide an infinite number of models, but all of them are rather simple, manifested by the fact that the level of the relevant affine algebra is equal to one. As WZW theories are believed to be the building blocks of all rational conformal field theories, one should next try to extend the results to higher level WZW theories. Since the progress reported here relies on the realization of the affine algebra in terms of free fermions, one may first look for higher level theories for which

[3] Nevertheless $\tilde{\mathrm{gl}}(N)$ contains the $\widehat{\mathrm{so}}(N)_1$ Kac-Moody Lie algebra as a sub-vector space. This is possible because, upon taking the combinations (7.9) of the $\tilde{\mathrm{gl}}(N)$-generators, one is left with a closed algebra generated only by infinite sums, provided that a finite number of terms in the sums is adjusted properly. The operators obtained this way are precisely the generators J^a_m of the affine algebra $\widehat{\mathrm{so}}(N)_1$.

such a realization is still available. Such theories exist; in fact, they are in one to one correspondence with the conformal embeddings in the classical affine algebras $\widehat{so}(N)_1$ and $\widehat{sl}(N)_1$ [21], so that the present results should apply rather directly to these more complicated models. (Note, however, that for conformal embeddings the maximal chiral symmetry algebra contains the full large level one classical affine algebra rather than only its higher level subalgebra.)

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