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A METHOD FOR SOLVING THE KDV EQUATION AND SOME NUMERICAL EXPERIMENTS

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ABSTRACT

In this paper, by means of difference method for discretization of space partial derivatives of KDV equation, an initial value problem in ordinary differential equations of large dimension is produced. By using this ordinary differential equations the existence and the uniqueness of the solution of the KDV equation and the conservation of scheme are proved. This ordinary differential equation can be solved by using implicit Runge–Kutta methods, so a new method for finding the numerical solution of the KDV equation is presented. Numerical experiments not only describe in detail the procedure of two solitons collision, soliton reflex and soliton produce, but also show that this method is very effective.

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1 Introduction

The Korteweg-de Vries (KDV) equation was established in 1895, and it has the form:

$$u_t + uu_x + \mu u_{xxx} = 0 \tag{1}$$

where the sign of the parameter μ determines the propagation direction and the shape of waves. In this paper we take $\mu = 1$. The KDV equation is the simplest model of the dispersive waves and, under certain simplified conditions, covers the following cases: surface waves of long wavelength in liquids, plasma waves, lattice waves, weakly nonlinear magnetodynamic waves. Why KDV equation has attracted so much attention from mathematicians? The main reason is that this equation can be widely applied.

It is well known that the KDV equation has soliton's solutions [1] :

$$u(x,t) = 3D sech^2 \frac{\sqrt{D}}{2}(x - Dt) \tag{2}$$

We take the initial condition

$$u(x,0) = \phi(x), \qquad for \quad all \quad x \tag{3}$$

and the boundary condition

$$u(a,t) = f(t), \quad u(b,t) = g(t), \quad for \quad all \quad t \tag{4}$$

If a and b are finite, (1), (3) and (4) compose an initial and boundary problem. If $a = -\infty, b = +\infty$, (1) and (3) compose an initial problem.

Nowadays there are three main numerical methods for solving nonlinear wave equations. They are the finite difference method, the finite element method, and the spectral or quasi-spectral methods [2]. In this paper a new differential-difference method is presented as follows: first by using the difference method with second order for discretization of the function and its space partial derivatives, an initial problem in ordinary differential equations of large dimension is obtained, whose Jacobi matrix is real and anti-symmetrical approximately, so all its eigenvalues are pure imaginary approximately. By using this ordinary equations the conservation of the scheme and the convergence of the solution are proved when the step of space tends to 0. This ordinary differential equations can be solved by using the ready-made effective numerical methods, such as implicit Runge-Kutta methods. The discretization of partial derivative of time t can be chosen to achieve a satisfactory solution in a sufficient large interval of time.

The first numerical experiment describes the procedure of collision of two solitons prettily well. It shows that the above presented method is very effective. The second numerical experiment describes the reflex procedure of quasi-soliton, and the third numerical experiment exhibits a procedure in which a quasi-soliton is produced from some vibrations. The third experiment inspires us to find out the condition of producing quasi-soliton in an initial and boundary problem.

2 Semi-discrete scheme

The initial problem can be regarded as an initial and boundary problem numerically or approximately. Taking $x \in [a,b]$, $t \in [0,T]$, where a and b are chosen such that $u(a,t)$ and $u(b,t)$ are sufficiently small absolutely. Hence the following equalities hold approximately when $t \in [0,T]$.

$$u(a,t) = 0, u(b,t) = 0 \tag{5}$$

So we can treat the initial problem in the same way as treat an initial and boundary problem. But in the neighbourhoods of the point a and b they have to equal 0 approximately.

In this paper let $a = 0$, $b = 40$, T is determined according to requirements.

We take $n+1$ points in $[0,b]$. So the step of x is $h = \frac{b-a}{n}$, denote mesh point by $x_i = ih$, $i = 0,1,...,n$, and $u(x_i,t)$, $\frac{\partial u(x,t)}{\partial x}|_{x=x_i}$, $\frac{\partial^3 u(x,t)}{\partial x^3}|_{x=x_i}$, by u_{hi}, u_{hx_i}, u_{hxxx_i}, or briefly by u_i, u_{x_i}, u_{xxx_i}, $i = 0,1,...,n$, respectively.

$$\begin{cases} u_{hi} = \frac{1}{3}(u_{hi-1} + u_{hi} + u_{hi+1}) \\ u_{hx_i} = \frac{1}{2h}(u_{hi+1} - u_{hi-1}) \\ u_{hxxx_i} = \frac{1}{2h^3}(u_{hi+2} - 2u_{hi+1} + 2u_{hi-1} - u_{hi-2}) \end{cases} \tag{6}$$

They all have second accuracy. Put (6) in (1), we have

$$\frac{du_i}{dt} = -\frac{(u_{hi+1} + u_{hi} + u_{hi-1})(u_{hi+1} - u_{hi-1})}{6h} - \frac{u_{hi+2} - 2u_{hi+1} + 2u_{hi-1} - u_{hi-2}}{2h^3} \tag{7}$$

$$i = 1,2,...,n-1$$

Denote $\mathbf{u}_h = (u_{h1}, u_{h2}, ..., u_{hn-1})^T$, $\mathbf{f}_h = (f_{h1}, f_{h2}, ..., f_{hn-1})^T$, then the equations (7) can be written as

$$\frac{d\mathbf{u}_h}{dt} = \mathbf{f}_h$$

By using notation in [3] the formula (7) can be rewritten as

$$\frac{\partial \mathbf{u}_h}{\partial t} = \frac{1}{3}(\mathbf{u}_h D_0 \mathbf{u}_h + D_0 \mathbf{u}_h^2) + D_+ D_- D_0 \mathbf{u}_h \tag{8}$$

Now we can prove

THEOREM 1. *The (7) or (8) is a conservational scheme, that is*

$$||\mathbf{u}_h(.,t)||_h^2 = c \tag{9}$$

where constant c independs on t.

PROOF: It is easily verified that

$$D_+ D_- D_0 u_i = \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2h^3} = D_0 D_+ D_- u_i$$

hence

$$D_+D_-D_0\mathbf{u}_h = D_0D_+D_-\mathbf{u}_h$$

$$\begin{aligned}(\mathbf{u}_h, \frac{\partial \mathbf{u}_h}{\partial t}) &= \frac{1}{2}\frac{\partial ||\mathbf{u}_h||_h^2}{\partial t} \\ &= \frac{1}{3}[(\mathbf{u}_h, \mathbf{u}_h D_0 \mathbf{u}_h)_h + (\mathbf{u}_h, D_0\mathbf{u}_h^2)_h] + (\mathbf{u}_h, D_+D_-D_0\mathbf{u}_h)_h \\ &= \frac{1}{3}[(\mathbf{u}_h^2, D_0\mathbf{u}_h)_h - (D_0\mathbf{u}_h, \mathbf{u}_h^2)_h] + (\mathbf{u}_h, D_+D_-D_0\mathbf{u}_h)_h \\ &= (\mathbf{u}_h, D_+D_-D_0\mathbf{u}_h)_h = -(D_0D_+D_-\mathbf{u}_h, \mathbf{u}_h)_h \\ &= \frac{1}{2}(\mathbf{u}_h, (D_+D_-D_0 - D_0D_+D_-)\mathbf{u}_h)_h = 0\end{aligned}$$

We get

$$||\mathbf{u}_h(.,t)||_h^2 = ||\mathbf{u}(.,0)||_h^2 = ||\phi(.)||_h^2 = c$$

where constant c independs on t.

Let $\mathbf{v}_h(x,t) = \dfrac{\partial \mathbf{u}_h(x,t)}{\partial t}$. The existence and the uniqueness of the solution for (1), (3), (4) or (1), (3) and the convergence of solutions for (7) when step of space tends to 0 can be proved by [3], if we have proved the following Theorem

THEOREM 2. *There exist constants T_1 and $k_i, i = 0,1,2,$ which independ on h but depend on $\phi(x)$ and its derivatives of order three and lower, such that*

$$||\mathbf{u}(.,t)||_h \le k_0 \qquad for \quad all \quad t \tag{10}$$

$$|\mathbf{u}(x_i,t)|_h \le k_1 \qquad for \quad all \quad i, \quad 0 \le t \le T_1 \tag{11}$$

$$||\mathbf{v}(.,t)||_h \le k_2 \qquad for \quad 0 \le t \le T_1 \tag{12}$$

PROOF: By Theorem 1. the equality $||\mathbf{u}_h(.,t)||_h^2 = ||\phi(.)||_h^2$ holds for any h. There exists h_0 such that if $h \le h_0$ (this condition is assumed to be fulfilled from now on), then

$$||\phi(.)||_h^2 \le 2\int_a^b \phi^2(x)dx = k_0^2$$

hence $||\mathbf{u}_h(.,t)||_h \le k_0$, for all t.

Since

$$(\mathbf{v}_h, D_+D_-D_0\mathbf{v}_h)_h = -(D_0D_+D_-\mathbf{v}_h, \mathbf{v}_h)_h = 0$$

$$\mathbf{v}_h = \frac{\partial \mathbf{u}_h}{\partial t} = \frac{1}{3}(\mathbf{u}_h D_0\mathbf{u}_h + D_0\mathbf{u}_h^2) + D_+D_-D_0\mathbf{u}_h$$

then

$$\frac{\partial \mathbf{v}_h}{\partial t} = \frac{1}{3}(\mathbf{v}_h D_0\mathbf{u}_h + \mathbf{u}_h D_0\mathbf{v}_h + 2D_0\mathbf{u}_h\mathbf{v}_h) + D_+D_-D_0\mathbf{v}_h$$

hence

$$\begin{aligned}\frac{\partial(||\mathbf{v}_h||_h^2)}{2\partial t} &= (\mathbf{v}_h, \frac{\partial \mathbf{v}_h}{\partial t})_h \\ &= \frac{1}{3}[(\mathbf{v}_h^2, D_0\mathbf{u}_h)_h + (\mathbf{v}_h, \mathbf{u}_h D_0\mathbf{v}_h)_h + 2(\mathbf{v}_h, D_0\mathbf{u}_h\mathbf{v}_h)_h] + (\mathbf{v}_h, D_+D_-D_0\mathbf{v}_h)_h \\ &= \frac{1}{3}[(\mathbf{v}_h^2, D_0\mathbf{u}_h)_h - (D_0\mathbf{u}_h, \mathbf{u}_h\mathbf{v}_h)_h] \\ &= \frac{1}{3}[(\mathbf{v}_h^2, D_0\mathbf{u}_h)_h + (\mathbf{u}_h, D_0\mathbf{u}_h\mathbf{v}_h)_h]\end{aligned}$$

The remaining part of the proof is same as the proof of lemma 3 in [3], we omit it here.

Equations (7) or (8) is an ordinary differential equations with $n-1$ dimension. The discrete form of the initial condition (3) are $u_i(0)=\phi(x_i)$,i=1,...,n-1. Denote

$$\mathbf{u}(0)=(u_1(0),u_2(0),...,u_{n-1}(0))^T$$

We have

$$\mathbf{u}|_{t=0}=\mathbf{u}(0) \tag{13}$$

Let $u_{-1}=u_0$, $u_{n+1}=u_n$, when the initial problem was reseached. In the experiments of soliton reflex and produce, the boundary condition $u_0'=0$, $u_n'=0$ have to append to (13), so we also let $u_{-1}=u_0$, $u_{n+1}=u_n$. (7), (13) compose an initial value problem in ordinary differential equations.

The Jacobi matrix of right side functions in (7) is denoted by

$$J=\frac{1}{h^3}A+\frac{1}{6h}B \tag{14}$$

where

$$A=-\begin{pmatrix} 0 & 1 & -0.5 & & & & & & & \\ -1 & 0 & 1 & -0.5 & & & & & & \\ 0.5 & -1 & 0 & 1 & -0.5 & & & & & \\ & 0.5 & -1 & 0 & 1 & -0.5 & & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & & 0.5 & -1 & 0 & 1 & -0.5 \\ & & & & & & 0.5 & -1 & 0 & 1 \\ & & & & & & & 0.5 & -1 & 0 \end{pmatrix}$$

$$B=\begin{pmatrix} u_0-u_2 & -u_1-2u_2 & & & & \\ 2u_1+u_2 & u_1-u_3 & -u_2-2u_3 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & 2u_{n-3}+u_{n_x-2} & u_{n-3}-u_{n-1} & -u_{n-2}-2u_{n-1} \\ & & & & 2u_{n-2}+u_{n-1} & u_{n-2}-u_n \end{pmatrix}$$

Note that

$$\begin{aligned} u_{i-1}-u_{i+1}&=O(h) \\ u_{i-1}&=u_i+O(h) \end{aligned} \tag{15}$$

and denote

$$C=\begin{pmatrix} 0 & -u_1 & & & & \\ u_1 & 0 & -u_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & u_{n-2} & 0 & -u_{n-1} \\ & & & & u_{n-1} & 0 \end{pmatrix} \tag{16}$$

Put (15), (16) in (14) and omit the relative term of $O(h^3)$, we have

$$J \approx \frac{1}{h^3}(A + \frac{h^2}{2}C) = \tilde{J} \tag{17}$$

Then (7) can be rewritten approximately as

$$\frac{d\mathbf{u}}{dt} = \tilde{J}\mathbf{u} \tag{18}$$

Since matrix A and C are real and antisymmetric, the $\tilde{J}$ is also real and antisymmetric. Therefore we have following result by [4]:

THEOREM 3. *All eigenvalues of ordinary differential equations (6), which are obtained from the KDV equation by using semi-discrete scheme (7), are pure imaginary approximately.*

3 Numerical solution of the ordinary equations

From Theorem 1, 3 the steady condition of ordinary differential equations (7) isn't too bad. Therefore the implicit Runge-Kutta methods can be used for finding the numerical solution of the initial value problem in ordinary differential equations (7), (13). We take $n_t + 1$ points in $[0, T]$, the mesh step of t is $h_t = \frac{T}{n_t}$, the mesh point is $t_j = jh_t, j = 0, 1, ..., n_t$. We use two-stage implicit Runge-Kutta methods of order four [5], namely

$$\begin{cases} \mathbf{k}_1^{j+1} = h_t\mathbf{f}(t^j + \frac{3-\sqrt{3}}{6}h_t, \mathbf{u}^j + \frac{1}{4}\mathbf{k}_1^{j+1} + \frac{3-2\sqrt{3}}{12}\mathbf{k}_2^{j+1}) \\ \mathbf{k}_2^{j+1} = h_t\mathbf{f}(t^j + \frac{3+\sqrt{3}}{6}h_t, \mathbf{u}^j + \frac{3+2\sqrt{3}}{12}\mathbf{k}_1^{j+1} + \frac{1}{4}\mathbf{k}_2^{j+1}) \end{cases} \tag{19}$$

$$\mathbf{u}^{j+1} = \mathbf{u}^j + \frac{1}{2}(\mathbf{k}_1^{j+1} + \mathbf{k}_2^{j+1}) \tag{20}$$

for numerical solution of (7), (13). We will omit the superscript $j+1$ on $\mathbf{k_1}$, and $\mathbf{k_2}$ if it does not cause confusion.

(19) is a nonlinear equations of $\mathbf{k}_1$, $\mathbf{k}_2$. Quasi-Nowton-Gauss-Seidel iterative method is used for solving it. Denote

$$\begin{cases} \hat{\mathbf{k}}_{1i}^{(p+1)} = (k_{11}^{(p+1)}, ..., k_{1i}^{(p+1)}, k_{1i+1}^{(p)}, ..., k_{1n-1}^{(p)})^T \\ \hat{\mathbf{k}}_{2i}^{(p+1)} = (k_{21}^{(p+1)}, ..., k_{2i}^{(p+1)}, k_{2i+1}^{(p)}, ..., k_{2n-1}^{(p)})^T \\ \qquad i = 1, 2, ..., n-1, \qquad p = 0, 1, ..., l-1 \end{cases} \tag{21}$$

where $\mathbf{k}_1^{(p+1)} = \hat{\mathbf{k}}_{1n-1}^{(p+1)}, \mathbf{k}_2^{(p+1)} = \hat{\mathbf{k}}_{2n-1}^{(p+1)}$, let

$$\begin{cases} k_{1i}^{(p+1)} = h_t f_i(t^j + \frac{3-\sqrt{3}}{6}h_t, \mathbf{u}^j + \frac{\hat{\mathbf{k}}_{1i}^{(p+1)}}{4} + \frac{3-2\sqrt{3}}{12}\hat{\mathbf{k}}_{2i}^{(p+1)}) \\ k_{2i}^{(p+1)} = h_t f_i(t^j + \frac{3+\sqrt{3}}{6}h_t, \mathbf{u}^j + \frac{3+2\sqrt{3}}{12}\hat{\mathbf{k}}_{1i}^{(p+1)} + \frac{\hat{\mathbf{k}}_{2i}^{(p+1)}}{4}) \\ \qquad i = 1, 2, ..., n-1 \end{cases} \tag{22}$$

$k_{1i}^{(p+1)}$, $k_{2i}^{(p+1)}$ can be solved from (22), since $k_{10} = k_{20} = 0$, $k_{1-1} = k_{2-1} = 0$, $k_{1n+1} = k_{2n+1} = 0$, $k_{1n} = k_{2n} = 0$. We take $\mathbf{k}_1^{j(l)}$, $\mathbf{k}_2^{j(l)}$ as the approximate initial value of $\mathbf{k}_1^{j+1}$,

k_2^{j+1} that are $k_1^{j+1(0)} = k_1^{j(l)}$, $k_2^{j+1(0)} = k_2^{j(l)}$, and let $l = 4$ for the experiments in this paper. When k_1^{j+1}, k_2^{j+1} are obtained, the u^{j+1} can be obtained from (20) too.

The equations (19) will be a linear system when put J= $\frac{1}{h^3}A$ approximately. The above iterative method coincides with the Gauss-Seidel iterative method. If take

$$h_t < \frac{2}{3+\sqrt{3}}h^3 \approx 0.42265h^3 \tag{23}$$

then the coefficient matrix of the linear equations (19) becomes a diagonal dominate matrix. Thus Gauss-Seidel iterative method is convergent. Condition (23) is very similar to steady conditions in [6], [7] . Let $h = 0.1$, $h_t = 0.4 \times 10^{-3}$ all numerical experiments mentioned in this paper are successful. This shows that our choice of h_t is proper.

4 Numerical experiments

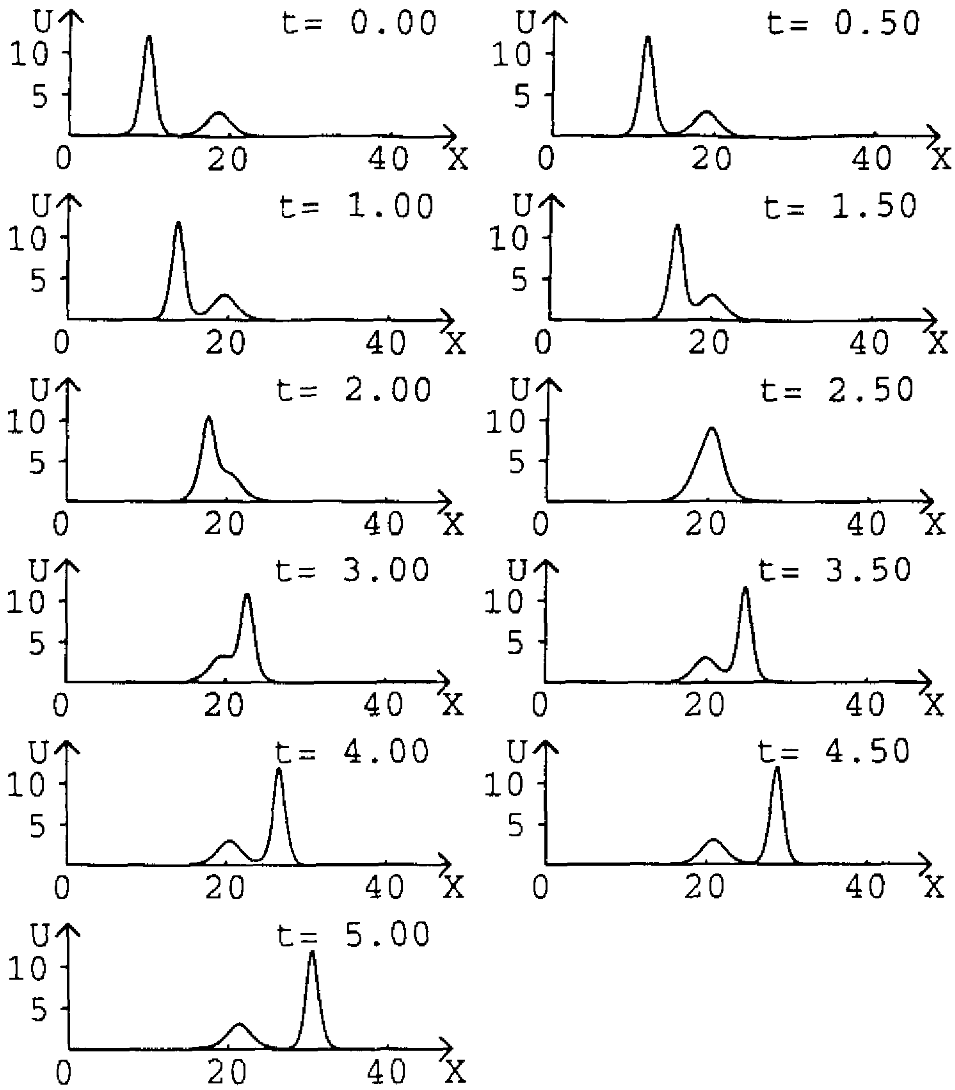

FIG. 1

1). Collision of two solitons.

Let

$$\begin{cases} u_1(x,t) = 12sech^2(x-9.7-4t) \\ u_2(x,t) = 3sech^2\frac{1}{2}(x-18.6-t) \end{cases} \tag{24}$$

be two solitons. When $t = 0$, the distance between their crests is large enough, so the sum of these two solitons satisfies the KDV equation (1) from numerical point of view. The equation (1) with the initial condition

$$u(x,0) = \phi(x) = u_1(x,0) + u_2(x,0) \tag{25}$$

compose an initial problem. We use presented method to solve this problem and obtain Figure 1, which contains 11 graphs corresponding to every $\Delta t = 0.5$ minute. Figure 1 describes vividly the detail of the procedure of the collision. The soliton with large amplitude pursues, annexes, splits out and separates the soliton with smaller one. The shapes of two solitons almost don't change after their collision. It verifies that this numerical method is very good.

2). Reflex of quasi-soliton.

Let

$$u(x,t) = 12sech^2(x-34-4t)$$

This is a quasi-soliton propagating right.

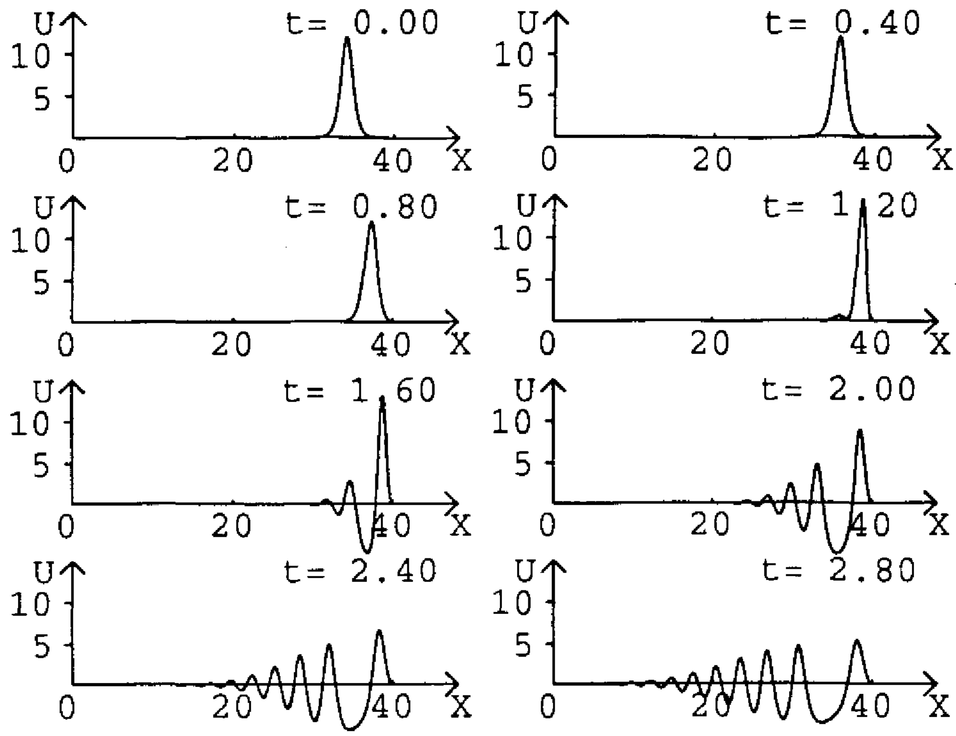

FIG. 2

Taking

$$\begin{cases} \phi(x) = u(x,0) = 12sech^2(x-34) \\ u(0,t) = \phi(0), \quad u(40,t) = \phi(40) \\ u'_x(0,t), \quad u'_x(40,t) = 0 \end{cases} \tag{26}$$

When $x \in [0, 40]$ (1) and (26) compose an initial and boundary problem. We use presented method for solving this problem. The results are described in Figure 2, which is formed by 8 graphs corresponding to every $\Delta t = 0.4$ minute. Along with the increase of time the quasi-soliton propagates to right and holds back, what will happen? This numerical experiments of reflex of quasi-soliton exihibit the whole procedure, where at first quasi-soliton's amplitude becomes higher and narrower and then some vibrations, propagating to opposite direction, appear and increase progressively.

3). Quasi-soliton produce in initial and boundary problem.

What condition can produce a quasi-soliton is the problem many scientists try to find out recently [6], [7]. This experiment, based on the reflex of soliton, gives a procedure, in which the quasi-soliton is produced from some vibrations. It inspires us to solve this problem.

The quasi-soliton of (1), (26) is denoted by u(x,t). Take a transformation: let $x' = 40 - x$, $t' = T - t$, $u(x,t) = \tilde{u}(x', t')$. It is easy to verify that $\tilde{u}(x', t')$ satisfies the KDV equation also, where x', t' are regarded arguments. Let

$$\begin{cases} \tilde{u}(x', 0) = u(40 - x, T) = \phi_1(x') \\ \tilde{u}(0, t') = \phi_1(0), \quad \tilde{u}(40, t') = \phi_1(40) \end{cases} \tag{27}$$

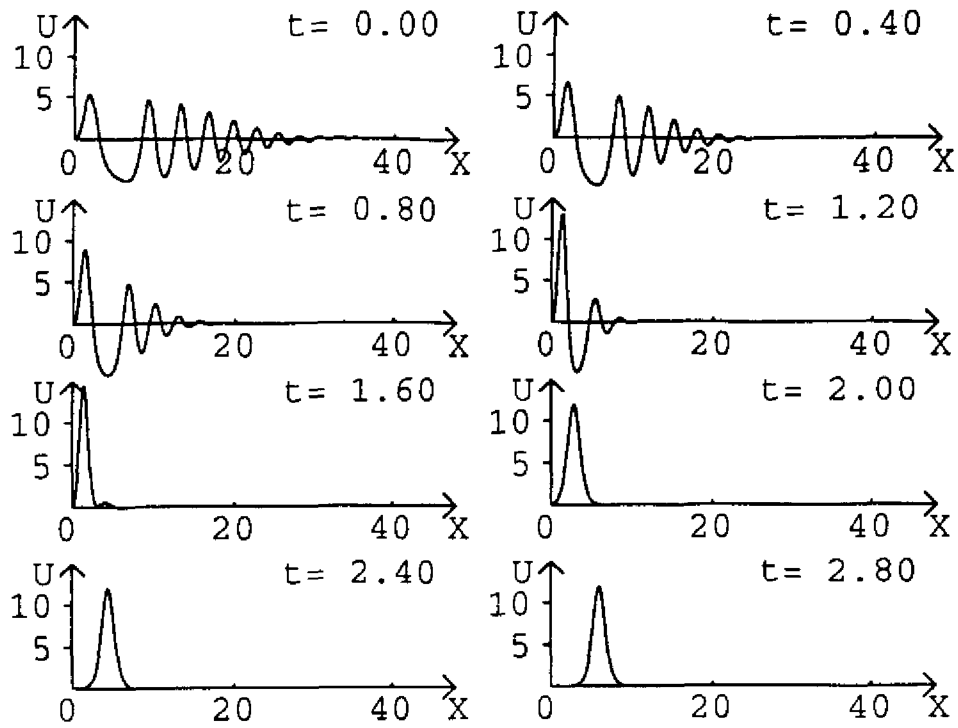

FIG. 3

u(x,T) is obtained from the numerical result of second experiment. So $\tilde{u}(x', 0)$, $\tilde{u}(0, t')$, $\tilde{u}(40, t')$, are defined. Below we denote briefly $\tilde{u}$, x', t', by u, x, t . When $x \in [0, 40]$ (1), (27) form an initial and boundary problem, the Figure 3, which is formed by 8 graphs corresponding to every $\Delta t = 0.4$ minute, is drawn according to the numerical results. The procedure, in which quasi-soliton is produced, can be regarded as the converse procedure of the reflex of quasi-soliton. By comparing the Figure 2 with Figure 3, you can find out that they are symmetric each other about the line $x = 20$ in corresponding time. This experiment gives us an evidence again that our method for solving the KDV equation is successful and efficient.

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