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A Zeta Function Approach to the Semiclassical Quantization of Maps

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Abstract

The quantum analogue of an area preserving map on a compact phase space is a unitary (evolution) operator which can be represented by a matrix of dimension $L \propto \hbar^{-1}$. The semiclassical theory for the spectrum of the evolution operator will be reviewed with special emphasize on developing a dynamical zeta function approach, similar to the one introduced recently for the semiclassical quantization of hamiltonian systems.

1 Introduction

Area preserving maps accompany the study of non integrable hamiltonian systems since its infancy. Poincaré introduced the map of the section in phase-space as a means to visualize the essential features of the dynamics in a plane. Since then, the interest and the study of maps developed both in the physics and mathematics communities. Maps appear naturally in the study of physical systems in a large variety of contexts. The most common examples are systems which are driven by external time periodic forces. They can be described stroboscopically at integer multiples of the driving periods. The evolution in time is written as a mapping which expresses the present state of the system in terms of its state at the previous inspection time. The "standard map" (known also as the "kicked rotor") [1] is a paradigm for this class of systems. The motion of a particle in a billiard is another example, where the dynamics is naturally expressed as a map [2,3]. Another important example is the Poincaré scattering map [4,5] which is an important tool in the classical analysis of scattering. However, Poincaré maps of realistic systems are not always convenient or easy to construct. It was found useful to introduce synthetic models, in which structural simplicity is achieved at the loss of contact with physical reality. The "cat map" and the "baker map" are two well known examples of such maps [6]. (Recently Hannay and his co-workers have shown that the baker map can describe the propagation of light in a system of thin lenses and prisms [7]. Thus, even this mathematical toy becomes physically "relevant").

The maps which will be considered here are those defined over a two dimensional compact phase-space with an area A. This includes most of the maps considered to date, but excludes systems whose phase space is a cylinder.

The quantization of chaotic maps played an important rôle in "Quantum Chaos". The quantum counterpart of a classical area-preserving map is a unitary (evolution) operator. An important problem in quantum chaology is to construct the semiclassical approximation of the quantum evolution operator in terms of classical information. This problem was addressed by Miller [8] and later by Tabor [20]. Other approaches were developed for the semiclassical quantization of specific models like the cat map [9,10] and of the baker map [11,12].

In the present paper I shall deal with a particular aspect of the semiclassical theory of maps, namely, the semiclassical evaluation of the spectrum of the evolution operator. The spectra of quantum evolution operators of classically chaotic systems were discussed in many applications and contexts. Consider e.g, the evolution of a system driven by an external periodic force. The stroboscopic description results in a unitary operator whose eigenphases (the quasi-energies) determine the evolution. The distribution of the eigenphases on the interval $[0, 2\pi]$ and their correlations provide important evidence about the nature of the corresponding eigenvectors, especially in the context of dynamical localization [13]. This question was of major importance in studying the quantum kicked rotor and the ionization of Rydberg atoms in a microwave cavity [14]. Recently, it was shown [15,16] that semiclassical quantization of chaotic billiards can be obtained in a way which is based on semiclassically quantized maps. The quantum analogue of the Poincaré scattering map plays here a crucial rôle. As a matter of fact, it was shown that the spectral properties of the hamiltonian to be quantized are intimately connected with the spectral properties of the mapping which is used for the quantization. These are but a few examples which demonstrate the growing interest in the semiclassical theory for the spectrum of quantized maps which is the subject of the present paper.

The spectrum of the evolution operator coincides with the zero set of the spectral determinant. When this is expressed as a series, one obtains the corresponding Selberg zeta function. In the present context the evolution operator can be represented by a finite matrix, and the Selberg zeta function reduces to a polynomial (the characteristic polynomial). One can derive the semiclassical approximation for this function in two ways, which, in principle may not yield the same answers since the semiclassical approximation is introduced in different stages of the derivation. In the first way, one starts with the characteristic polynomial, and after expressing its coefficients in terms of traces of powers of the evolution operator, the semiclassical approximation is introduced as a last step. The second method is not as straight forward, but is similar to the recent methods developed for the semiclassical quantization of hamiltonian systems [15-18]. My purpose in the present note is to compare the two methods when applied to the semiclassical quantization of maps, and to show that they remain equivalent if one requires that the semiclassical Selberg zeta function is analytic. The presentation and the discussion of this derivation has also a pedagogical merit since it illustrates some modern semiclassical methods in a relatively simple context. The semiclassical theory of quantum maps was recently discussed in [12] using the first method described above. There is therefore some degree of overlap between the present discussion and the results of [12]. This repetition is unavoidable since it provides the background for some new results and insight which is the main message of this note.

The paper is organized in the following way. The next chapter gives a summary of useful and relevant relations from the theory of matrices. The semiclassical theory will follow in Chapter 3, where two derivations of the Selberg zeta function for the quantum evolution operator will be presented.In the summary chapter I shall emphasize the connection between the present derivation, and the techniques introduced in [15-18] for the quantization of hamiltonian systems.

2 Some Useful Connections

The spectrum of a unitary operator U consists of L unimodular eigenvalues $\exp(i\theta_l)$, $l = 1,\ldots,L$, where L is the dimension of the Hilbert space. L is related to the area of the classical phase space A via $L = [A/(2\pi\hbar)]$ (here and elsewhere $[\cdot]$ stand for the integer part). One can calculate the spectrum by either searching for the zeros of the characteristic polynomial

$$P(z) = \det(z - U) = \sum_{l=0}^{L} f_l z^l, \tag{2.1}$$

on the unit circle $z = e^{i\omega}$, or alternatively, by calculating directly at the spectral density

$$d(\omega) = \sum_{l=1}^{L} \delta_p(\omega - \theta_l). \tag{2.2}$$

(δ_p stands for the 2π periodic delta function). The information content of the two functions is exactly the same.

The spectral determinant (2.1) is not real for real ω. It is useful to work with another function $Z(\omega)$ which is real on the real ω line, and vanishes at $\omega = \theta_l$. It is obtained from (2.1) by extracting a phase factor,

$$Z(\omega) = e^{-i\frac{1}{2}(L\omega+\Theta)} P(e^{i\omega}), \tag{2.3}$$

where

$$e^{i\Theta} = e^{i(\sum_{k=1}^{L}\theta_k - L\pi)} = \det(-U) \tag{2.4}$$

The coefficients f_l are homogeneous, symmetric polynomials of order $L-l$ in the eigenvalues $e^{i\theta_l}$. They satisfy an important symmetry relation which follows from the unitarity of U,

$$e^{-i\Theta/2} f_l = e^{i\Theta/2} f^*_{L-l}. \tag{2.5}$$

The f_l coefficients can be written also in terms of the traces of powers of U, via the Newton identities [19],

$$Tr(U^k) + f_{L-1}Tr(U^{k-1}) + \ldots + f_{L-k+1}Tr(U) + kf_{L-k} = 0 \tag{2.6}$$

which hold for $1 \le k \le L$. To prove (2.6), consider two functions $g(x)$ and $h(x)$ (with $g(0)=0$) such that

$$g(x) = \exp(-h(x)),$$

and

$$g(x) = \sum_{l=0}^{\infty} g_l x^l \quad ; \quad h(x) = \sum_{k=1}^{\infty} \frac{1}{k} h_k x^k.$$

The coefficients g_l can be expressed in terms of the h_l by the recursion relations

$$lg_l = -\sum_{k=1}^{l} g_{l-k} h_k, \tag{2.7}$$

which can be easily derived by taking the l'th derivative of $g(x) = \exp(-h(x))$ at $x = 0$. Apply the above to $f(z) = det(I - z^{-1}U)$ and $h(z) = -Tr\log(I - z^{-1}U)$, and the identities (2.6) follow immediately. U is a zero of its own characteristic polynomial, so that $\sum_{l=0}^{L} f_l Tr U^{l+k} = 0$, for all $k \geq 0$. Hence, the expansion (2.7) in the present case is truncated after L terms, as expected. The recursion (2.6) can be solved by simple iterations. The result takes the form

$$f_{L-l} = \sum_{\boldsymbol{l}} \frac{(-1)^{\mu}}{\prod_{i=1}^{\mu} l_i} Tr(U^{l-l_1}) Tr(U^{l_1-l_2}) \dots Tr(U^{l_{\mu-1}-l_{\mu}}) Tr(U^{l_{\mu}}) \tag{2.8}$$

Here the summation goes over all the vectors $\boldsymbol{l}$ of integer entries such that $l > l_1 > l_2 > \dots > l_\mu \geq 1$ and $\mu = \mu(\boldsymbol{l})$ denotes the number of entries in $\boldsymbol{l}$. The first few f_l are given explicitly by $f_L = 1$; $f_{L-1} = -\mathrm{Tr}U$; $f_{L-2} = \frac{1}{2}((Tr(U))^2 - Tr(U^2))$, *etc.*

We now turn our attention to the spectral density (2.2). It can also be rewritten in terms of traces of U^n,

$$\begin{aligned} d(\omega) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{i\omega n} \left(Tr(U^n)\right)^* \\ &= \frac{L}{2\pi} + \frac{1}{\pi} \Re \sum_{n=1}^{\infty} e^{i\omega n} \left(Tr(U^n)\right)^* \end{aligned} \tag{2.9a}$$

The first term in (2.9a) is the mean density $\bar{d} = \frac{L}{2\pi}$. The infinite sum converges only for complex ω with $\Im\omega > 0$. A similar expansion which converges in the lower half of the complex ω plane is give by

$$d(\omega) = \frac{L}{2\pi} + \frac{1}{\pi} \Re \sum_{n=1}^{\infty} e^{-i\omega n} Tr(U^n) \tag{2.9b}$$

The spectral density can be derived from the Z function (2.3) by one of the relations

$$d(\omega) = \mp \frac{1}{\pi} \lim_{\epsilon \to 0} \Im \frac{d}{d\omega} \log Z(\omega \pm i\epsilon). \tag{2.10}$$

This equation displays an apparent "paradox", because the two sides of (2.10) depend on traces of different powers of U : $Z(\omega)$ involves only $Tr(U^l)$ with $l \leq L$, while for $d(\omega)$ (see (2.9)) one needs $Tr(U^n)$ for *all* n. The resolution of this "paradox" is in the fact that the matrix U is a zero of its characteristic polynomial. Hence U^L can be expressed in terms of the lower powers of U, which implies that all $Tr(U^n)$ with $n \geq L$ can be expressed in terms of $Tr(U^l)$ with $l < L$.

The identities quoted above will become very handy in the semiclassical discussion which follows in the next chapter.

3 Semiclassical Theory

We shall start the present section by quoting a well known semiclassical expression for $Tr(U^n)$ [20]. This is the input needed for the evaluation of the Z function or the spectral density in terms of classical information exclusively.

U is the quantum analogue of a classical mapping $\mathbf{r}' = T(\mathbf{r})$, where $\mathbf{r} = (q,p)$ is a point in our compact phase–space. T can be expressed in terms of a generating function (action) $S(q,q')$

$$p = -\frac{\partial S(q,q')}{\partial q} \qquad p' = \frac{\partial S(q,q')}{\partial q'} \tag{3.1}$$

The semiclassical approximation for $< q_f|U|q_i >$ is [8]

$$< q_f|U|q_i > \approx \frac{1}{(2\pi i\hbar)^{\frac{1}{2}}} \sum_s \left|\frac{\partial q'}{\partial p}\right|_s^{-\frac{1}{2}} e^{i(S_s/\hbar - \frac{\pi}{2}\nu_s)} \tag{3.2}$$

The summation goes over all the values of p for which the classical evolution equations (3.1) have a solution with $q = q_i$ and $q' = q_f$. To calculate U^n semiclassically one performs the intermediate sums by replacing the discrete sums by integrals (using the Poisson identity) and by integrating the results using the saddle point approximation. The trace operation is also approximated by a saddle point integration, and the final result is

$$Tr(U^n) \approx \sum_{r_p n_p = n} \frac{g_p n_p}{\left|\det(M_p^{r_p} - I)\right|^{\frac{1}{2}}} e^{i(S_p r_p/\hbar - \frac{\pi}{2}\nu_p r_p)} \tag{3.3}$$

where the sum is over all (primitive) periodic orbits p of T with periods n_p which are divisors of n. The action S_p, the Maslov index ν_p and the monodromy matrix M_p all relate to the primitive periodic orbit. g_p count the degeneracy of the periodic orbit, that is, the number of ***distinct*** periodic orbits which are related to each other by the symmetries of the system. If a periodic orbit is conjugate to itself by the symmetry, the corresponding value of g_p is 1. Detailed derivation of (3.3) can be found in Tabor's paper [20].

At this point we have to choose between two alternative courses of action. One way is to substitute the semiclassical result (3.3) in (2.6) and obtain the Selberg zeta function directly. The other possibility is to substitute (3.3) in (2.8) which gives a semiclassical approximation to the spectral density $d(\omega)$. The semiclassical Selberg zeta function can be derived from $d(\omega)$ by using (2.10). If the semiclassical approximation were exact, the two routes should be equivalent. However, the property that $Tr(U^n)$ with $n \geq L$ can be expressed in terms of $Tr(U^n)$ with $n < L$ is not guaranteed by the semiclassical approximation. Therefore the equivalence of the two routes proposed above in the semiclassical approximation needs further scrutiny. The problem is even more acute because the inaccuracies in the semiclassical approximation for $Tr(U^n)$ may jeopardized the convergence of the series, which is problematic especially when $\Im\omega \to 0$. The main purpose of this paper is to show that the equivalence of the two methods can be restored only if analyticity is ***imposed*** on the semiclassical zeta function calculated in the second way.

When we choose the direct approach, and use (3.3) to get the Z function, we may take advantage of the symmetry (2.5) and reduce the effort by a half. However, one has to calculate the phase factor $e^{i\Theta}$ (2.4). This can be achieved by using (2.5) with $l = [L/2]$,

$$e^{i\Theta} = \frac{f_{[L/2]}}{f^*_{L-[L/2]}} \qquad (3.4)$$

Some caution should be exercised when the semiclassical approximation for f_l is substituted in (3.4). Consider e.g., the case when L is odd. Then, ***there is no guarantee that*** the expression (3.4) gives a unimodular complex number. We can argue, however, that in the semiclassical limit $\Theta \to 0$. The eigenphases θ_l for a classically chaotic mapping are distributed according to the predictions of random matrix theory. Denoting by $< \cdot >$ the ensemble average, we find $< \theta_l > = \frac{2\pi}{L}(l - \frac{1}{2})$ and therefore $< \sum_{i=1}^{L} \theta_l > = L\pi$, which implies that $< \Theta > = 0$. Due to the rigidity of the spectrum, the variance of Θ decrease as $\frac{1}{L}\log L$ [21]. Hence, for a typical system, one can replace Θ by its mean value, which vanishes in the semiclassical limit.

The semiclassical expression for f_{L-l} involves a complicated sum over the integer vectors $\boldsymbol{l}$ defined above, (see eq (2.8)) and for each vector of integers $\boldsymbol{l}$ one has to collect all the contributions to each $Tr(U^{l_i - l_{i+1}})$ which are due to periodic orbits with primitive periods n_{p_i} and repetitions r_{p_i} such that $n_{p_i} r_{p_i} = l_i - l_{i+1}$. The resulting expression for f_{L-l} is a sum of terms, each of them is due to a collection of periodic orbits and their repetitions such that $\sum_{i=1}^{\mu} n_{p_i} r_{p_i} = l$. ($\mu = \mu(\boldsymbol{l})$ is the number of positive entries in the vector $\boldsymbol{l}$). This set of periodic orbits defines a *composite* or *pseudo orbit* [17]. Each composite orbit contributes a term which is a complex number. Its phase is the *composite action* $\Phi = \sum r_{p_i} S_{p_i}$ (a *composite Maslov index* is included in the action). The amplitude can be worked out from the corresponding coefficients in the semiclassical expression (3.3). Once this is done, and the symmetry (2.5) is used, one gets the semiclassical Z function

$$Z_{scl}(\omega) = \sum_{l=0}^{[L/2]-\epsilon_L} \left(F_{L-l}^{*} e^{i(l-L/2)\omega} + F_{L-l} e^{-i(l-L/2)\omega} \right) + \frac{\epsilon_L}{2}(F_{\frac{L}{2}} + F_{\frac{L}{2}}^{*}), \qquad (3.5)$$

where ϵ_L takes the values 1 or 0 if L is even or odd. The coefficints F_{L-l} are the semiclassical approximants to the f_{L-l}, expressed in terms of composite orbits with a composite period $l \leq [L/2]$.

The above derivation of the semiclassical zeta function (3.5) was discussed previously in [12] and similar derivations were also given in [15,16]. We shall proceed to construct the zeta function in the alternative route, starting from a semiclassical expression for the spectral density. This is the way one usually takes in quantizing general hamiltonian systems with dynamics which is continuous in time. We substitute (3.3) in (2.9a) and obtain an expression for the spectral density which is the analogue of Gutzwiller's expression for the spectral density of a chaotic hamiltonian system [22].

$$d(\omega) \approx \frac{L}{2\pi} + \frac{1}{\pi}\Re \sum_{n=1}^{\infty} \sum_{r_p n_p = n} \frac{g_p n_p}{\left|det(M_p^{r_p} - I)\right|^{\frac{1}{2}}} e^{\left(i\omega n_p r_p - i(S_p r_p/\hbar - \frac{\pi}{2}\nu_p r_p)\right)} \tag{3.6}$$

We emphasize again that the summation over n is over all the integers. Hence, to calculate $d(\omega)$ one must identify all the periodic orbits of the map and calculate their stabilities, Maslov indices and actions. The exponential proliferation of periodic orbits poses severe difficulties in the applications of (3.6) even for simple systems. Moreover, one can show (see below) that the series converge in the absolute sense only for ω with $\Im\omega > \eta$, where η is the (positive) "entropy barrier". Starting from (2.9b) one can find a semiclassical expression for $d(\omega)$ which converges only in the lower half plane $\Im\omega < -\eta$. The exclusion of the strip about the real ω axis is rather embarrassing, since this is exactly where one needs the spectral density most! Nevertheless, we shall show that this difficulty can be circumvented, and a semiclassical expression which is based on a *finite* number of periodic orbits can be used to get all the relevant spectral information.

Let us proceed for the time being with the expression (3.6) and consider it in its domain of absolute convergence. We can now use a standard procedure to derive the Selberg zeta function from the spectral density [23]. This is done by expanding the denominators in (3.6) as

$$det(M_p^{r_p} - I)^{-\frac{1}{2}} = (TrM_p^{r_p} - 2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \exp\left(-\lambda_p n_p r_p (k + \frac{1}{2}) - i\sigma_p r_p k\right), \tag{3.7}$$

where λ_p is the Lyapunov exponent for the primitive periodic orbit and σ_p takes the values 0 if the periodic orbit is hyperbolic or π if it is hyperbolic with reflection. Inserting this in (3.6) and summing first over all the repetitions r_p one finally gets an expression for $d(\omega)$ of the form

$$d(\omega) \approx -\frac{1}{\pi}\Im\frac{d}{d\omega}\log Z_{+}(\omega) \quad \Im\omega > \eta \tag{3.8a}$$

with

$$Z_+(\omega) = e^{-i\omega L/2}\prod_{k=0}^{\infty}\prod_p\left(1-\exp\left(-\lambda_p n_p(k+\frac{1}{2}) - i\sigma_p k + i\omega n_p - i(S_p/\hbar - \nu_p\frac{\pi}{2})\right)\right)^{g_p} \tag{3.9a}$$

In a similar way we could start with (2.9b) and obtain

$$d(\omega) \approx \frac{1}{\pi}\Im\frac{d}{d\omega}\log Z_-(\omega) \quad \Im\omega < -\eta \tag{3.8b}$$

with

$$Z_-(\omega) = e^{+i\omega L/2}\prod_{k=0}^{\infty}\prod_p\left(1-\exp\left(-\lambda_p n_p(k+\frac{1}{2}) + i\sigma_p k - i\omega n_p + i(S_p/\hbar - \nu_p\frac{\pi}{2})\right)\right)^{g_p} \tag{3.9b}$$

The two functions $Z_\pm(\omega)$ defined above are related by

$$(Z_+(\omega))^* = Z_-(\omega^*), \quad \Im\omega > \eta \ . \tag{3.10}$$

At this point we have to make the crucial asumption that the semiclassical approximation preserves the analyticity of the Selberg zeta function. That is, we assume that there exists an analytic function $\tilde{Z}_{scl}(\omega)$ which coincides with $Z_\pm(\omega)$ in their domains of absolute convergence. Because of (3.10), $\tilde{Z}_{scl}(\omega)$ is real on the real ω axis, and, therefore, it is the zeta function of our operator U within the semiclassical approximation. The "paradox" which was discussed at the end of the previous chapter emerges again in a different guise: In $\tilde{Z}_{scl}(\omega)$ one uses the entire set of periodic orbits, without any restriction on their length, while for the direct calculation of $Z_{scl}(\omega)$ we needed only periods of length $l \leq [L]/2$. To resolve this contradiction we must return to (3.9), to study their convergence properties and to show that by a proper analytical continuation one obtains a zeta function which depends on periodic orbits of *finite* length only.

Consider the inner products in (3.9a). It is clear that the factor which could cause the most serious convergence problems is the one with $k = 0$. It does not converge absolutely on the real ω axis because the number of primitive orbits of period n_p increases as $\approx \exp(hn_p)/n_p$, where h is the topological entropy. This exceeds the size of a typical term with a given n_p which is of order $\exp(-\lambda_p n_p/2)$. Thus, the absolute convergence of (3.8a) is restricted to the half plane $\Im\omega > \eta = h - \lambda/2 > 0$, where λ is the mean Lyapunov exponent. This is the "entropy barrier" which was mentioned above, and it arises because of the exponential proliferation of periodic orbits in chaotic systems. Since $h \approx \lambda$ the products with $k > 0$ converge even for real values of ω. A similar argument shows that the absolute convergence of (3.9b) is restricted to the lower half plane $\Im\omega < \eta$.

But for the factors $\exp(\pm i\omega L/2)$, the functions $Z_{\pm}(\omega)$ bear some similarity to the representation of the Riemann zeta function in terms of the Euler product. In the domain of absolute convergence, the infinite products can be converted into infinite sums which are the analogue of the Dirichlet sum for the Riemann ζ . This is done in two steps. The products over k are taken by using the Euler identity

$$\prod_{k=0}^{\infty}(1 - yx^k) = 1 + \sum_{r=1}^{\infty} \frac{(-1)^r y^r x^{\frac{r(r-1)}{2}}}{\prod_{j=1}^{r}(1 - x^j)}. \tag{3.11}$$

For each primitive orbit p one obtains an infinite series, and when all of them are multiplied together one gets,

$$Z_{+}(\omega) = \exp(-i\omega L/2) \sum_{n=0}^{\infty} A_n \exp(i\omega n) \quad \Im\omega > \eta \tag{3.12a}$$

and

$$Z_{-}(\omega) = \exp(i\omega L/2) \sum_{n=0}^{\infty} A_n^* \exp(-i\omega n) \quad \Im\omega < -\eta \tag{3.12b}$$

To understand the meaning of the last expressions it is useful to have in mind the analogy with the Riemann ζ function: the Dirichlet product extends over the primes, while the Euler sum goes over the integers, which are *composite numbers* expressed in terms of the primes and their repetitions. In our case, the primitive periodic orbits are the analogue of the primes. The corresponding *composite orbits* are the analogues of the integers [17]. We see that the composite orbits emerge also in the present approach. Here, however, the prescription for the construction of the contribution of a term which corresponds to a composite orbit is different from the one described in connection with (3.5). Given a value of n, it can be decomposed in the form

$$n = \sum r_{p_i} n_{p_i} \qquad 0 < n_{p_1}, n_{p_2}, \ldots \leq n \qquad r_{p_i} \geq 0, \tag{3.13}$$

where $\{p_i\}$ denotes a group of periodic orbits, $p_i \neq p_j$. (Note that for the present purpose, orbits which are related by symmetry are considered as distinct orbits). Each (non trivial) decomposition defines a ***composite periodic orbit***, built from the primitive orbits $\{p_i\}$ with repetitions r_{p_i}. The ***composite period*** is N. Each composite orbit contributes a term to the coefficient A_N. Its phase is given by

$$e^{-i \sum_{\{p_i\}} r_{p_i} (S_{p_i}/\hbar - \frac{\pi}{2} \nu_{p_i})} \tag{3.14}$$

where one can note the ***composite action*** and the ***composite Maslov index***. The main difference between the discussion of the composite orbits which appear in (3.5) and the present one is in the definition of the amplitudes. Here, they are obtained from the Euler product expression. They depend on the stability exponents associated with the periodic orbits which contribute to the composite orbit. Their explicit expression will not be given here. It can be easily checked that $A_0 = 1$.

A final comment concerning the functions $Z_{\pm}(\omega)$ defined above is that their domain of convergence might be larger than their domain of absolute convergence [18]. We shall not make use of this property here, and therefore shall not discuss this point any further.

The above tedious manipulations were carried out since the forms (3.12) are essential for the next and final step in this discussion. We shall assume that the analytic function $\tilde{Z}_{scl}(\omega)$ coincides with $Z_{\pm}(\omega)$ in the respective domains of absolute convergence, and use the Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(z')}{z' - z} dz',$$

to analytically continue $\tilde{Z}_{scl}(\omega)$ to the real ω axis. The contour of integration is the rectangle with horizontal sides on the lines $\Im\omega = y$ and $\Im\omega = -y$, with $y > \eta$, and vertical sides with $\Re\omega = -\omega_0$ and $\Re\omega = \omega_0$, where ω_0 is an arbitrarily large real number. For ω on the real line we get

$$\tilde{Z}_{scl}(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dt \sum_{n=0}^{\infty} e^{-y(n-L/2)} \left(-\frac{A_n e^{it(n-L/2)}}{t-\omega+iy} + \frac{A_n^* e^{-it(n-L/2)}}{t-\omega-iy} \right) \qquad (3.15)$$

where the contributions from the two vertical sides of the rectangular contour were not included since they vanish in the limit $\omega_0 \to \infty$. Using

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} dt \frac{e^{itR}}{t-x+iy} = \begin{cases} 0 & \text{if } R>0 \\ -\frac{1}{2} & \text{if } R=0 \\ -e^{i(x-iy)R} & \text{if } R<0 \end{cases} \qquad (3.16)$$

we finally get,

$$\tilde{Z}_{scl}(\omega) = \sum_{l=0}^{[L/2]-\epsilon_L} \left(A_l e^{i(l-L/2)\omega} + A_l^* e^{-i(l-L/2)\omega} \right) + \frac{\epsilon_L}{2}(A_{\frac{L}{2}} + A_{\frac{L}{2}}^*) \qquad (3.17)$$

Thus, we were able to show that $\tilde{Z}_{scl}(\omega)$ has the same form as $Z_{scl}(\omega)$ (see (3.5)) which was derived by the application of the semiclassical approximation directly to the exact zeta function (2.3). As a matter of fact, $\tilde{Z}_{scl}(\omega)$ and $Z_{scl}(\omega)$ are identical. This can be proved by making use of the relation (2.7) , for $h(z) = -Tr\log(I - z^{-1}U)$, but this time, $Tr(U^k)$ should be replaced by its semiclassical approximation (3.3). The algorithm to calculate the coefficients of the resulting Z function is exactly the same as the one used for the calculation of the coefficients F_{L-l}. We know from the previous discussion that the series obtained in this way does not converge for real ω, but the algorithm by which the terms are constructed is formally correct. The requirement that Z_{scl} is analytic was crucial, since it imposed the truncation of the series, so that the formal manipulations become meaningful. We found therefore that in spite of the fact that the composite orbits were introduced in (3.5) and (3.17) in quite different ways, the resulting coefficients of the Z function are the same. It is quite a good exercise to compare explicitely the first few terms and see how the the two methods yield the same coefficients. The calculation of the semiclassical approximation of $F_{L-3} = -\frac{1}{3}Tr(U^3) + \frac{1}{2}Tr(U^2)TrU - \frac{1}{6}(TrU)^3$ and the corresponding A_3, is recommended.

To get some insight into (3.17), we may consider the contribution of the $l = 0$ term. It is due to composite orbits of period 0, and hence is expected to give the smooth part of the spectrum. This is indeed the case since its contribution to the zeta function is $2\cos\frac{\omega L}{2}$ which vanishes whenever $\omega = \frac{2\pi}{L}(l - \frac{1}{2})$. The rest of the terms introduce the spectral fluctuations about this mean spectrum.

Further properties of (3.17) will be discussed in the next section.

4 Summary and Discussion

The most important feature of the semiclassical zeta function, is that it is expressed in terms of composite orbits with composite periods which do not exceed $[L/2]$. We gave two derivations for the semiclassical Z function. The first one, leading to (3.5) was derived in the more straight forward and natural way. That is, the semiclassical approximation was applied at the last stage of the derivation. The inherent quantum mechanical ingredients – the unitarity of U and the finite dimension of the Hilbert space – where imposed at the outset. The road to (3.17) was much more devious. In this approach, the semiclassical spectral density is the starting point, and by *imposing* analyticity on the semiclassical zeta function, one obtains the zeta function in the domain of interest, where the original semiclassical expression does not converge. This approach shows that the analyticity imposed on the semiclassical zeta function revive the features which are due to the unitarity of U and its finite dimension, which where otherwise lost.

It is interesting to compare (3.17) with the expression (3.12a) for $Z_+(\omega)$, which was the starting point for the derivation of (3.17). The analytic continuation of (3.12) to the real ω line is achieved by taking from (3.12a) the leading $[L/2] - \epsilon_L$ terms and replacing the rest of the series (which does not converge for real ω) by the complex conjugate of the leading expression. Thus, the analytic continuation amounts to a resummations of the diverging "tail" of an infinite series which is expressed in terms of its "head". This is the phenomenon of resurgence discussed by Berry [24], and a prominent example where it occurs is the Riemann- Siegel approximation for the Riemann zeta function [24].

The derivation of (3.17) is similar in spirit to the method used by Keating and Berry [17] to derive a Riemann Siegel "lookalike" expression for the dynamical zeta function. Their result is also expressed in terms of a finite number of composite orbits, with composite periods which are shorter than half the Heisenberg time – the time which is needed to resolve an energy interval which equals the mean level spacing. The present problem is somewhat simpler than the one discussed in [17]. The truncation of the zeta function at $N = [L/2]$ *is exact, whereas in [17] it is only approximate.* (One of the important achievements in [17] is the calculation of the difference between the exact zeta function and its truncated form.) Technically speaking, the main difference between the Berry Keating problem and the present one is in the way that the parameter ω (the energy E in [17]) appears in the semiclassical zeta function. In [17], E enters via the actions S_p of the periodic orbits and through the smooth spectral counting function $\bar{N}(E)$. The dependence on E of these functions is usually non trivial. In our case, the smooth spectral counting function is simply $\bar{N}(\omega) = L\frac{\omega}{2\pi}$. Otherwise ω appears in the semiclassical expression as a part of the effective action $S_p - \hbar\omega n_p$ (see e.g., (3.6)). The linear dependence of these two functions on ω is the reason why we could use the simple Cauchy Integral Method to analytically continue the zeta function, and did not have to introduce a weight function similar to the one used in [17].

Another implication of the analytcity assumption can be formulated by recalling that due to analyticity, $\tilde{Z}_{scl}(\omega)$ satisfies a "functional equation" of the type (3.10). A similar functional equation is satisfied by the Riemann zeta function, and it is crucial for the derivation of the Riemann-Siegel approximation.

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