IC/93/23
INTERNAL REPORT
(Limited Distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON A "CONFORMAL" PERFECT FLUID IN THE CLASSICAL VACUUM

H. Culetu*
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

A possible existence of a conformal perfect fluid in the classical vacuum is investigated in this letter. It is shown, contrary to Madsen's opinion, that the scalar field stress tensor acquires a perfect fluid form even with a nonminimal coupling ($\xi = 1/6$) in the Einstein Lagrangian, provided the geometry is the Lorentzian analogue of the Euclidean Hawking wormhole. In addition, our $T_{\mu\nu}$ equals (up to a constant factor) the vacuum expectation value of the Fulling stress tensor for a massless scalar field and Visser's one concerning transversible wormholes. On the other side of the light cone, there is a coordinate system (the dimensionally reduced Witten bubble) where the stress tensor becomes diagonal.

MIRAMARE – TRIESTE
February 1993

* Permanent address: Institute of Physics and Nuclear Engineering, Institute for Atomic Physics, Bucharest, Romania.

Although there is no complete understanding of the dynamics of the scalar field, much attention has been paid to it in the last decade, especially because of its significant role in the phenomenon of inflation.

Madsen [1] was able to give a fluid structure of the scalar field stress tensor $T_{\mu\nu}$ and to derive the kinematical constraints on the evolution of the scalar field in spacetime. He has found that $T_{\mu\nu}$ reduces to a perfect fluid form only if $\xi = 0$ (minimally coupling); $\xi \neq 0$ implies the presence of both a heat flux and anisotropic stress in the space directions.

The purpose of the present letter is to prove that a conformal coupling between the gravitation and the scalar field leads to a perfect fluid form of the scalar field stress tensor when the field has an appropriate (Lorentz invariant) structure. Our scalar field is a gravitational one [2] and the spacetime is the Lorentzian version of the Euclidean Hawking wormhole [3,4]

$$ds^2 = \left(1 - \frac{b^2}{x_\alpha x^\alpha}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

with $x_\alpha x^\alpha = \eta_{\alpha\beta} x^\alpha x^\beta \equiv s^2$, $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$, $\alpha, \beta = 0,1,2,3$ and "b" is of the order of the Planck length (units are chosen such that $\hbar = c = 8\pi G = 1$).

We follow Madsen in defining the flow vector U^μ in terms of the scalar field itself. We found the conformally invariant ψ-stress tensor, and, consequently, its energy density and pressures are vanishing when $\hbar = 0$. In addition, we reach the equation of state for radiation ($\varepsilon = 3p$). An important support for our result is the fulfillment of the strong energy condition [5]

$$(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha_\alpha) U^\mu U^\nu \geq 0 \ .$$

In addition, the heat flux q_μ and the anisotropic stress tensor $\pi_{\mu\nu}$ are vanishing. We start with the action [1,2,6]

$$S = \frac{1}{2} \int (g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi + \frac{R}{6} \psi^2) \sqrt{-g} d^4x + S_m \ , \tag{2}$$

where R is the Ricci scalar, the Riemann tensor is defined as $R^\beta_{\mu\alpha\nu} = \partial_\alpha \Gamma^\beta_{\mu\nu}$-... and S_m is the action for matter. Note that the usual Einstein Lagrangian is missing since it may be generated, putting $\psi =$ const. (see [2] for details). Variation of the action S yields

$$\frac{\delta S}{\delta \psi} = 0 \ , \quad \to \Box \psi - \frac{R}{6} \psi = 0 \tag{3}$$

and

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \ , \to G_{\mu\nu} = \frac{6}{\psi^2} T^M_{\mu\nu} + T_{\mu\nu} \tag{4}$$

where $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$, $G_{\mu\nu}$ is the usual Einstein tensor, $T^M_{\mu\nu}$ is the matter stress tensor and

$$T_{\mu\nu} = \frac{-4}{\psi^2} \nabla_\mu \psi \nabla_\nu \psi + \frac{2}{\psi} \nabla_\mu \nabla_\nu \psi - \frac{2}{\psi} g_{\mu\nu} \Box \psi + \frac{1}{\psi^2} g_{\mu\nu} \nabla^\alpha \psi \nabla_\alpha \psi \tag{5}$$

is the (conserved) (when $T^M_{\mu\nu} = 0$) stress tensor associated to the conformally coupled scalar field ψ. It is clear that the particular case $\psi = const$ (or, as we shall see, when Minkowski separation S is much larger than the Planck time), Eqs. (4) become the classical Einstein equations. We deal now with the kinematics of the scalar field

$$\psi = (1 - b^2 s^{-2})^{-1} \tag{6}$$

in the vacuum, i.e. $T^M_{\mu\nu} = 0$. The spacetime is (1). We are looking for the structure of the energy-momentum tensor $T_{\mu\nu}$. By defining, according to Madsen

$$-U^\mu = (\partial^\alpha\psi\partial_\alpha\psi)^{-\frac{1}{2}}\partial^\mu\psi \ . \tag{7}$$

we get, using Eqs. (1) and (6)

$$U^\mu = S^{-1}\psi x^\mu \tag{8}$$

(We restrict to the region, $S^2 > 0$). U^μ is time-like and has unit magnitude (notice that the flow vector U^μ cannot preserve the above properties on the other side of the light cone, i.e. $S^2 < 0$).

The acceleration, vector, defined as $A^\mu = U^\alpha\nabla_\alpha U^\mu$, can be written in the form:

$$A^\mu = (\partial^\alpha\psi\partial_\alpha\psi)^{-1}\partial^\nu\psi\nabla_\nu\nabla^\mu\psi - (\partial^\alpha\psi\partial_\alpha\psi)^{-2}\partial^\mu\psi\partial^\nu\psi\partial^\tau\psi\nabla_\nu\nabla_\tau\psi \ .$$

It is an easy task to find that A^μ does vanish in the spacetime (1), with ψ from Eq.(6). In other words, the flow lines defined as the integral curves of U^μ, are geodesics. Consider now the tensor projected onto the spacelike hypersurface orthogonal to $\partial^\mu\psi$

$$h_{\mu\nu} = g_{\mu\nu} - U_\mu U_\nu \ .$$

with $h_{\mu\nu}U^\mu = 0$. The expansion tensor

$$\theta_{\mu\nu} = V_{(\mu\nu)}$$

where

$$V_{\mu\nu} = h^\alpha_\mu h^\beta_\nu \nabla_\alpha U_\beta \tag{9}$$

equals $V_{\mu\nu}$ when U^μ is given by (7). Making use of the fact that

$$\nabla_\mu U_\nu = -(\partial^\alpha\psi\partial_\alpha\psi)^{-\frac{1}{2}}\nabla_\mu\nabla_\nu\psi + (\partial^\alpha\psi\partial_\alpha\psi)^{-\frac{3}{2}}\partial_\nu\psi\partial^\beta\psi\nabla_\mu\nabla_\beta\psi$$

we obtain in the spacetime (1) and after some algebra

$$\theta_{\mu\nu} = S^{-1}(1 + \frac{b^2}{s^2})(\eta_{\mu\nu} - S^{-2}X_\mu X_\nu) \tag{10}$$

It is worth to note that when $b \ll S$ (i.e. far from the light cone or for large intervals compared to Planck's), $\theta_{\mu\nu}$ does not vanish. It means the gravitational potential ψ, which

plays a significant role at the Planck scale only, induces a macroscopic expansion. Taking the trace, Eq. (10) leads to

$$\theta^\alpha_\alpha = 3s^{-1}\psi^2(1 + \frac{b^2}{s^2}) \ ,$$

which means $\theta^\alpha_\alpha > 0$, in accordance with inflationary scenarios (see [1]). The well-known formulae for vorticity and shear

$$\omega_{\mu\nu} = V_{[\mu,\nu]}; \quad \sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3}h_{\mu\nu}\theta^\alpha_\alpha \ ,$$

become, in our particular case

$$\omega_{\mu\nu} = 0; \quad \sigma_{\mu\nu} = 0 \ .$$

Let us now analyze the structure of the ψ-stress tensor. It may generally be written as

$$T_{\mu\nu} = \varepsilon U_\mu U_\nu + q_\mu U_\nu + q_\nu U_\mu - p h_{\mu\nu} - \pi_{\mu\nu} \tag{11}$$

where ε and p are, respectively, the energy density and pressure, q_μ is the heat flux (with $U_\mu q^\mu = 0$) and $\pi_{\mu\nu}$ is the anisotropic stress tensor. The above quantities can be extracted from $T_{\mu\nu}$ as the following projections.

$$\varepsilon = T_{\mu\nu}U^\mu U^\nu \ , \quad q_\mu = T_{\alpha\beta}U^\alpha h^\beta_\mu \ , \quad p = \frac{1}{3}\Pi^\alpha_\alpha \tag{12}$$

with $\Pi_{\alpha\beta} \equiv p h_{\alpha\beta} + \pi_{\alpha\beta} = -T_{\mu\nu}h^\mu_\alpha h^\nu_\beta$. We obtain after some algebra:

$$T_{\mu\nu} = -4\psi^2 S^{-4}(\eta_{\mu\nu} - 4S^{-2}X_\mu X_\nu) \tag{13}$$

$$\varepsilon = 12\psi^4 s^{-4}, \ p = 4\psi^4 s^{-4} = \varepsilon/3 \ . \tag{14a, b}$$

$$q_\mu = 0, \quad \pi_{\mu\nu} = 0 \tag{14c}$$

$$T^\alpha_\alpha = -6\psi^{-1}\Box\psi = 0 \tag{14d}$$

In other words, we reached, in our particular spacetime, a perfect fluid stress tensor

$$T_{\mu\nu} = (p + \varepsilon)U_\mu U_\nu - p g_{\mu\nu} \ ,$$

in spite of the fact that the coupling constant $\xi = 1/6 \neq 0$. In addition, the strong energy condition

$$(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\alpha_\alpha)U^\mu U^\nu = 12\psi^4 S^{-4}(1 + b^2 s^{-2}) =$$

$$= (1/2)(\varepsilon + 3p) \geq 0$$

is obeyed.

We note our $T_{\mu\nu}$ equals, up to a constant factor, the vacuum expectation value of the stress tensor of a massless scalar field calculated by Fulling [7]

$$\langle 0|T_{\mu\nu}(x,x')|0\rangle = \frac{-1}{8\pi_a^2\sigma^2}(\eta_{\mu\nu} - 4t_\mu t_\nu) \qquad (15)$$

Using the point-splitting method, (note that his result is valid in the flat space, when $\psi = 1$). In his notation, $\sigma \equiv (t-t')^2 - (\vec{x} - \vec{x}')^2$ and $t^\mu = -\eta^{\mu\nu}\partial_\nu\sigma/2\sqrt{|\sigma|}$ is the unit tangent vector to the geodesic joining the two points, i.e. our $U^\mu = s^{-1}X^\mu$.

In addition, our stress tensor preserves its perfect fluid form in any frame, in contrast with Fulling's. Moreover, it has physical significance, as long as we obtained it on a curved spacetime, where any energy does matter, not only the difference of energies.

Surprisingly, a similar result reached Visser [8], concerning his researches on traversible wormholes. He studied the effect that the wormhole geometry has on the propagation of quantum fields. Considering a quantum massless scalar field, he obtained:

$$\langle 0|T_{\mu\nu}|0\rangle = -\frac{\hbar\kappa}{\delta^4}(\eta_{\mu\nu} - 4n_\mu n_\nu)$$

where $n_\mu = \delta_\mu/\delta$ is the tangent to the closed spacelike geodesic threading the wormhole throat, $\delta = \sqrt{\ell^2 - T^2}$ is the invariant length of the geodesic and κ-a constant. "ℓ" is the distance between the wormhole mouths and $T-$ the time shift.

All the physical parameters from Eqs.(14) are vanishing when $\hbar = 0$, and have low values far from Planck's world. For example, at the origin of coordinates ($x^i = 0$, $i = 1,2,3$) and, say, at $x^0 = 10^{-10}s$ (taken from an arbitrary origin), we have:

$$\varepsilon(0) \simeq 12\hbar c(cx^0)^{-4} = (4/9)10^{-17}erg/cm^3$$

a value far from being measured nowadays. We think the geometry (1) is more suitable for the classical vacuum than the Minkowski geometry.

We look now for the form of $T_{\mu\nu}$ in the region $S^2 < 0$, i.e. on the other side of the light cone. The flow vector U^μ cannot be defined there using Madsen's prescription (7). Therefore, we look for something else.

Let us now consider the Witten bubble spacetime [9]

$$ds^2 = g^2r^2dt^2 - (1 - \frac{R^2}{r^2})^{-1}dr^2 - r^2\cos h^2 gtd\Omega^2 - (1 - \frac{R^2}{r^2})d\chi^2 \qquad (16)$$

with $R \le r < \infty$, $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and χ - the coordinate of the compactified fifth dimension. Using the above geometry, Witten has studied the decaying process (an expanding bubble) of the ground state of the original Kaluza-Klein geometry, which, although

stable classically, is unstable against a semiclassical barrier penetration (see [10,11]). We are now interested in the 4-dimensional subspace χ=const.

$$ds^2 = g^2r^2dt^2 - (1 - \frac{R^2}{r^2})^{-1}dr^2 - r^2\cos h^2 gtd\Omega^2 \qquad (17)$$

of the spacetime (16) ("g" is the rest-system acceleration). (17) is the ordinary Minkowski space provided $r \gg R$, but written in the spherical Rindler coordinates (a spherical distribution of uniformly accelerated observers uses such type of hyperbolic coordinates). It is well-known [12] that the singularity at $r = R$ is a coordinate singularity, as can be seen from the isotropical form of (17), by using a new radial coordinate ρ:

$$r = \rho + \frac{R^2}{4\rho} \ ,$$

with $R/2 \le \rho < \infty$. The metric (17) becomes:

$$ds^2 = (1 + \frac{R^2}{4\rho^2})^2(g^2\rho^2dt^2 - d\rho^2 - \rho^2\cos h^2 gtd\Omega^2) \qquad (18)$$

Under the coordinate transformation:

$$x^1 = \rho\cos hgt\sin\theta\cos\varphi \ , x^2 = \rho\cos ht\sin\theta\sin\varphi$$

$$x^3 = \rho\cos hgt\cos\theta \ , x^0 = \rho\sin hgt$$

the above metric becomes:

$$ds^2 = \left(1 + \frac{b^2}{x_\alpha x^\alpha}\right)^2 \eta_{\mu\nu}dx^\mu dx^\nu \ .$$

which is (1), but here we have $\rho^2 = x_\alpha x^\alpha \ge b^2$. We put $R = 2b$, keeping in mind that both of them are of the order of the Planck length. We are now in a position to find the stress tensor $T_{\mu\nu}$ of the conformal scalar field ψ in the region $S^2 < 0$. Since the geometry (18) is conformally flat, it is a solution of the conformally invariant vacuum Einstein eqs. (4) ($T^M_{\mu\nu} = 0$). The problem reduces to the calculation of $R_{\mu\nu}$. We may use the formula (see [13]):

$$R_{\mu\nu} = R^{(R)}_{\mu\nu} - 2(\nabla_\mu\nabla_\nu\sigma - \partial_\mu\sigma\partial_\nu\sigma) - g^{(R)}_{\mu\nu}(\Box\sigma + 2\partial_\alpha\sigma\partial^\alpha\sigma) \qquad (19)$$

to compute the Ricci tensor in the metric $g_{\mu\nu} = e^{2\sigma}g^{(R)}_{\mu\nu}$; $g^{(R)}_{\mu\nu}$ is the spherical Rindler spacetime

$$ds^2_R = +g^2\rho^2dt^2 - d\rho^2 - \rho^2\cos h^2 gtd\,\Omega^2 \ .$$

with $\sigma \equiv \ell n(1 + b^2\rho^{-2})$. We have, of course $R^{(R)}_{\mu\nu} = 0$ and:

$$\Box\sigma \equiv g^{\alpha\beta}_{(R)}\nabla_\alpha\nabla_\beta\sigma = -4b^4\rho^{-2}(\rho^2 + b^2)^{-2}$$

It is now an easy task to find the components of $R_{\mu\nu}$. Eqs. (19) yield

$$R^t_t = R^\theta_\theta = R^\varphi_\varphi = -\frac{1}{3}R^\rho_\rho = 4b^2(\rho^2 + b^2)^{-2} .$$

All the other components are vanishing. In addition, $R^\alpha_\alpha = 0$, as it would be. Then, from (4), we have $T_{\mu\nu} = R_{\mu\nu}$. As $T_{\mu\nu}$ is diagonal, the particles of the fluid are comoving with the Witten bubble surface, which is expanding hyperbolically. Using (13) and the fact that $U^\alpha = (g^{-1}\rho(\rho^2 + b^2)^{-1}, 0, 0, 0), U^\alpha U_\alpha = 1$, we have

$$\varepsilon = \frac{-4b^2}{(\rho^2 + b^2)^2}; p = \frac{-\varepsilon}{3}; q_\mu = 0$$

$$\pi^\nu_\mu = \mathrm{diag}\left(0, \frac{32b^2}{3(\rho^2 + b^2)^2}, \frac{-16b^2}{3(\rho^2 + b^2)^2}, \frac{-16b^2}{3(\rho^2 + b^2)^2}\right) .$$

In conclusion, we found the conformal coupling between the scalar field ψ and gravitation leads to a perfect fluid ψ-stress tensor in the region $S^2 > 0$. By contrast, the fluid is anisotropic, when $S^2 < 0$. Moreover, the energy density is negative. Up to a constant factor, our $T_{\mu\nu}$ equals Fulling's and Visser's stress tensor characterizing a traversible wormhole.

Acknowledgments

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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