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GENERATING FUNCTIONAL FOR DONALDSON INVARIANTS AND OPERATOR ALGEBRA IN TOPOLOGICAL D = 4 YANG-MILLS THEORY

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ПРОИЗВОДЯЩИЙ ФУНКЦИОНАЛ ДЛЯ ИНВАРИАНТОВ ДОНАЛЬДСОНА И ОПЕРАТОРНАЯ АЛГЕБРА В D=4 ТОПОЛОГИЧЕСКОЙ ТЕОРИИ ЯНГА-МИЛЛСА

А.А.Иогансен

А н н о т а ц и я

Показано, что при некоторых ограничениях производящий функционал для инвариантов Дональдсона в D=4 топологической теории Янга-Миллса может быть интерпретирован как статсумма перенормируемой теории.

Abstract

It is shown, that under the certain constraints the generating functional for the Donaldson invariants in the D=4 topological Yang-Mills theory can be interpreted as a partition function for the renormalizable theory.

1 Introduction

This paper is devoted to the investigation of properties of the cohomological theories [1, 2]. These theories are characterized by a fermionic symmetry (BRST symmetry) generated by a nilpotent operator Q,

$$Q^2 = 0. \tag{1}$$

The physical observables are defined as the classes of the cohomology for the operator Q. The important property of these theories is that the energy-momentum tensor $T_{\alpha\beta}$ is Q-exact, i.e.

$$T_{\alpha\beta} = \{Q, \Lambda_{\alpha\beta}\}. \tag{2}$$

In eq. (2) $\Lambda_{\alpha\beta}$ is a composite operator globally defined on the manifold M where the fundamental fields of the theory "live". Eq.(2) leads to the fact that the correlators of the Q-invariant observables are independent on the external metric. Therefore the symmetry of these theories is very large as compared to the conventional quantum field theories.

The Lagrangian of a cohomological theory is also Q-exact (and hence, Q-invariant), i.e.

$$L = \{Q, \Lambda\}, \tag{3}$$

where Λ is a composite operator globally defined on the manifold M.

Let us consider a renormalizable topological theory with the Q-exact Lagrangian (eq.(3)). Let O_i be the operators corresponding to the observables of the theory. These operators represent the classes of cohomology of the BRST charge Q, i.e.

$$\{Q, O_i\} = 0, \;\; O_i \neq \{Q, \Lambda_i\}, \tag{4}$$

where Λ_i is a composite operator globally defined on the manifold M.

As it is shown in ref.[1] the cohomological theories are semiclassical because of the independence of the correlators of the observables O_i on the external metric. These correlators are found to be represented by integrals of the closed forms on suitable moduli space and hence, are the topological invariants of this moduli space. In the other words, the cohomological theories give the description of these invariants in terms of the quantum field model.

It is worth emphasizing that for the self-consistence of this interpretation the theory should be renormalizable. The point is that the nonrenormalizability of the theory would mean that the quantum corrections change the Lagrangian of the theory in an uncontrollable way. In general such a change modifies the structure of the corresponding moduli space. Actually in the simplest case the quantum corrections to the Lagrangian of the cohomological non-renormalizable theory are represented by the Q-exact local operators. However even in this situation it is not clear what is the interpretation of the theory. To illustrate let us consider the calculation of the Witten index in the supersymmetric theories. This index is the topological invariant [3]. It also can be interpreted as the topological invariant in the corresponding topological theory related to the supersymmetric theory through the twisting [4].

The simplest case is the supersymmetric quantum mechanics with the polynomial superpotential W(X), where X is a chiral superfield. Of course, in this theory there is no problem with renormalizability. However one can ask how does the Witten index change when modifying the superpotential. Such a modification is equivalent to adding the superinvariant operator to the action of the theory and is similar to deformation of the Lagrangian of the topological theory by Q-exact operators, in particular the deformation of the Lagrangian of the nonrenormalizable topological theory by quantum corrections. The Witten index Δ, depends on the asimptotical behaviour of the function W(X) at $X \to \infty$ [3], i.e.

$$\Delta = \begin{cases} 0, & n \text{ is odd} \\ 1, & n \text{ is even} \end{cases} \tag{5}$$

where n is the degree of the polynomial W(X). Eq.(5) means that the deformation of the superpotential modifying its asimptotical behaviour changes the Witten index. Such a situation can happen in the topological nonrenormalizable theory where Q-exact deformations of the Lagrangian are generated by quantum corrections. If so the possibility of the topological interpretation of this theory and its connection with any moduli space are not controllable.

Almost all the suggested topological theories are renormalizable. However the problem of renormalizability arises for the generating functional for the topological invariants in the renormalizable topological theory:

$$F[t] = \int [D\Phi] \exp S[\Phi, t]. \qquad (6)$$

In eq.(6) Φ represents the quantum fields of the theory, t_i are the additional coupling constants or the sources for the operators O_i, $S[\Phi, t]$ is the action of the topological theory deformed by adding the linear combination of the representatives of the classes of Q-cohomology O^i

$$S[\Phi, t] = \int_M \{Q, \Lambda\} + \sum_i t_i O_i. \qquad (7)$$

From now on we shall assume that the theory with the non-deformed action $S[\Phi, t_i = 0]$ is renormalizable. In this case the generating functional exists as the expansion in powers of the sources t_i, and each term of this expansion represents the topological invariant related to the moduli space described by the topological theory [1, 2, 5]. However when the sources are not small the question of the existence of the generating functional is not quite clear. Moreover the deformed action can correspond to the nonrenormalizable quantum field theory. In this case when taking into account the quantum corrections one gets the generating functional for the correlators of the operators O_i in a new topological theory with a new Q-exact action, and the interpretation of this theory is not clear.

In this paper we shall consider the generating functional for the Donaldson invariants in the D=4 topological Yang-Mills theory (TYMT) [1]. This theory is renormalizable and describes the topological properties of the moduli space of instantons on the manifold M. In this model there is an infinite set of the observables which can be classified by their ghost numbers. It is worth emphasizing that the operators corresponding to the observables can be of any degree in the fundamental fields (roughly speaking, the degree of the operator is proportional to its ghost number). Therefore in general the action (7) is not polynomial in the fields of the theory and, hence, the problem of its renormalizability should be studied.

We shall see below that under certain constraints the theory with the action (7) is really renormalizable and therefore the generating functional for the Donaldson polynomials F[t] can be interpreted as a partition function for the renormalizable theory. To this end we shall analyse the anomalous dimensions of the representatives of the classes of the BRST cohomology and their operator algebra.

Finally it is worth noticing that the problem of the existence of the generating functional for the topological invariants in the D=4 topological Yang-Mills theory is also interesting from the physical viewpoint. As was shown in ref.[6] in the string theories there is a phase transition between the usual low-energy phase and the high-energy phase corresponding to an unusually small number of physical degrees of freedom. The topological theories are the quite natural candidates for a description of this high-energy phase. On the other hand the two-dimensional topological gravity [7, 8, 5] describes the moduli space of two dimensional surface of a fixed genus while this phase transition is related to the contribution of the two-dimensional surfaces of very high genera [6]. Though the D=4 TYMT is not directly related to the string theory it would be interesting to consider the possibility of such a phase transition between the topological and nontopological phases in this model. The point is that the topological Yang-Mills theory automatically takes into account the Yang-Mills fields of any topological charge. From the point of view of the topological interpretation of the theory the value of the topological charge is analogous to the genus of the two-dimensional surface in two-dimensional topological theory. Therefore one can hope that the D=4 TYMT is analogous to the case of the summing up of the contributions of all the genera in D=2 topological theory and try to study the possibility of the phase transition into the non-topological phase.

One can try to describe the phase transition in the D=4 TYMT in terms of the renormgroup flow in the theory that is a soft deformation of the D=4 TYMT. This description is similar to the case of deformations of the two-dimensional conformal theories [9]. The soft deformation of the D=4 TYMT should probably be renormalizable and unitary. The latter is not quite trivial since the usual connection between spin and statistics is broken in the D=4 TYMT [1]. The unitarity of the theory is due to the BRST invariance of the physical states. Therefore the deformed theory should also be invariant under a very large group of symmetry (but smaller than the topological one) to provide the unitarity of the space of physical states. The generating functional is a natural deformation of the D=4 TYMT. If the parameters of the deformation t_i in eq.(7) depend on external metric then the deformed energy-momentum tensor is not Q-exact but Q-closed. In this case the theory is not topological since the correlators depend on the external metric. In this paper we shall no more discuss the problem of the possibility to come out of the topological phase fixed point in the D=4 TYMT.

The paper is organized as follows. In sect.2 the superfield formulation of the D=4 TYMT is reviewed and the BRST cohomology is considered. In sect.3 the anomalous dimensions of observables are studied. In sect.4 the operator algebra is analysed and the constraints on the renormalizability of the deformed theory are found.

2 Superfield Formulation of the D=4 TYMT and the BRST Cohomology

The superfield formulation [10] is convenient for studying of the renormalization properties of the D=4 TYMT because the BRST symmetry related to the topological symmetry is realized linearly in this approach. In this case the D=4 TYMT contains the following superfields:

$$\begin{aligned} X_{\alpha\beta} &= \chi_{\alpha\beta} + 2\theta B_{\alpha\beta}, \\ A_\alpha &= a_\alpha + \theta\psi_\alpha, \\ A_\theta &= \xi - \theta\phi, \\ \Lambda &= \lambda - 2i\theta\eta, \end{aligned} \tag{8}$$

where θ is an anticommuting scalar coordinate, $\chi_{\alpha\beta}$, ψ_α , ξ , η are the anticommuting fields, $\chi_{\alpha\beta}$ and $B_{\alpha\beta}$ are the self-dual tensors. All the fields belong to the adjoint representation of the gauge group.

The action of the theory reads as

$$\begin{aligned} S = \frac{1}{e^2}\int_M d^4x \int d\theta\sqrt{g}\,Tr\Big\{&\frac{1}{8}X_{\alpha\beta}\nabla_\theta X^{\alpha\beta} + \frac{i}{2}X_{\alpha\beta}G_{\alpha\beta} \\ &+ \frac{1}{2}(\nabla^\alpha\Lambda)G_{\alpha\theta} - \frac{ik}{2}G_{\theta\theta}[\Lambda,\nabla_\theta\Lambda]\Big\} \end{aligned} \tag{9}$$

where the covariant derivatives and the strengths are defined as

$$\nabla_\alpha = D_\alpha - iA_\alpha\,,\ \nabla_\theta = \partial_\theta - iA_\theta, \tag{10}$$

$$G_{\alpha\beta} = i[\nabla_\alpha, \nabla_\beta]\,,\ G_{\alpha\theta} = i[\nabla_\alpha, \nabla_\theta]\,,\ G_{\theta\theta} = i\{\nabla_\theta, \nabla_\theta\}. \tag{11}$$

In eqs.(9), (10) and (11) D_α is the covariant derivative on the manifold M with metric $g_{\alpha\beta}$; $\partial_\theta = \partial/\partial\theta$; e^2 and k are the coupling constants. The action (9) is invariant under the transformation $\theta \rightarrow \theta + \epsilon$, where ϵ is a constant anticommuting parameter. This transformation is generated by the BRST charge $Q = \partial_\theta$ which corresponds to the gauge fixing of the topological symmetry [10, 11, 13, 12, 14]. The action (9) is also invariant under the supergauge transformations of the superfields (8) represented in ref.[10] with the parameter Y that is the commuting scalar superfield. The superfields A_α and A_θ transform as the superconnections. The superfields $X_{\alpha\beta}$ and Λ, the supercovariant derivatives in eq.(10) and the superstrengths in eq.(11) covariantly transform under the supergauge transformations:

$$X_{\alpha\beta} \rightarrow e^Y X_{\alpha\beta}\, e^{-Y},\ \Lambda \rightarrow e^Y \Lambda\, e^{-Y},$$

$$G_{\alpha\beta} \rightarrow e^Y G_{\alpha\beta}\, e^{-Y},\ G_{\alpha\theta} \rightarrow e^Y G_{\alpha\theta}\, e^{-Y},\ G_{\theta\theta} \rightarrow e^Y G_{\theta\theta}\, e^{-Y},$$

$$\nabla_\alpha \rightarrow e^Y \nabla_\alpha e^{-Y},\ \nabla_\theta \rightarrow e^Y \nabla_\theta\, e^{-Y}, \tag{12}$$

As it is shown in ref.[1] the operators corresponding to the observables are the classes of cohomology of the BRST operator Q. These observables can be constructed as it was suggested in refs. [15, 16]. Let us consider the operator

$$\hat{d} = d + \partial_\theta, \tag{13}$$

where d is the exterior derivative operator, and by definition

$$\{d, \partial_\theta\} = 0. \tag{14}$$

Obviously,

$$\hat{d}^2 = 0. \tag{15}$$

Let us introduce the covariant version of the operator $\hat{d}$:

$$\hat{\nabla} = \nabla + \nabla_\theta, \tag{16}$$

where $\nabla = dx_\mu \nabla_\mu$. The superstrengths (11) enter the 2- superform:

$$\hat{G} = i\{\hat{\nabla}, \hat{\nabla}\}. \tag{17}$$

It is easy to check that the Bianci identity is valid in this situation

$$[\hat{\nabla}, \hat{G}] = 0. \tag{18}$$

Then one gets the following identity

$$\hat{d}Tr(\hat{G})^{2n} = 0, \; n = 1, 2, 3... \tag{19}$$

Expanding the expression in eq.(19) in powers of the usual exterior forms we obtain the set of identities

$$dH^n_{p-1} + \partial_\theta H^n_p = 0 \qquad p = 0, 1, 2, 3, 4, \tag{20}$$

where p is a degree of the exterior form, $H^n_{-1} = 0$ by definition and

$$\hat{H}^n = \frac{1}{2n} Tr(\hat{G})^{2n} = H^n_0 + H^n_1 + H^n_2 + H^n_3 + H^n_4 \tag{21}$$

The exterior forms H^n_p are gauge invariant. The integrals over the non-trivial cycles γ_p ($\partial\gamma_p = 0$) determine the Q-invariant operators which correspond to the physical observables:

$$O^n_p = \int_{\gamma_p} H^n_p. \tag{22}$$

These operators do not depend on θ due to eq.(20)

$$\partial_\theta O^n_p = \int_{\gamma_p} \partial_\theta H^n_p = -\int_{\gamma_p} d\, H^n_{p-1} = 0, \tag{23}$$

and, in particular, $\partial_\theta O^n_0(x) = 0$.

In what follows it will be important that the operator $\hat{H}^n$ in eq.(21) can be locally represented as the $\hat{d}$ derivative of the superfield generalization of the Chern-Simons form $\hat{K}^n$:

$$\hat{d}\hat{K}^n = \hat{H}^n. \tag{24}$$

The superform $\hat{K}^n$ is defined as follows

$$\hat{K}^n = \int_0^1 dt Tr \hat{A} (\hat{G}_t)^{2n-1}. \quad (25)$$

where $\hat{A} = A_\alpha dx_\alpha + A_\theta$, $\hat{G}_t$ is defined by eq.(17) with the modified superconnections $A_{\alpha,\theta} \to t A_{\alpha,\theta}$.

Let us expand the superform $\hat{K}^n$ in powers of the usual exterior forms:

$$\hat{K}^n = K_0^n + K_1^n + K_2^n + K_3^n + K_4^n. \quad (26)$$

Then the operators H_p^n can be locally represented as

$$H_p^n = \partial_\theta K_p^n + dK_{p-1}^n. \quad (27)$$

It is worth noticing that for n=1 the form K_3^2 is the conventional Chern-Simons form, while $K_4^2 = 0$. However in general $K_4^n \neq 0$ for $n > 2$. The operators $\hat{H}^n$ represent the non-trivial classes of the $\hat{d}$ cohomology since they are the $\hat{d}$ derivatives of the gauge non-invariant operators. Hence, the operators O_p^n are the non-trivial elements [1] of the ∂_θ-cohomology in the sense that they can not be represented as the ∂_θ-derivative of any gauge invariant operator (see eqs.(21),(22)).

It is worth emphasizing that the operators O_p^n are reduced to the Witten's ones [1] when going to the Wess-Zumino-like supergauge [10]. For example, $O_0^n \sim Tr\phi^n$. In the Wess-Zumino gauge the descent equations (20) correspond to the descent eqs. given in ref.[1].

Notice that there is the Poincare'-dual description of the classes of the Q-cohomology by the operators

$$\tilde{O}_p^n = \int_M \sigma_{4-p} H_p^n, \quad (28)$$

where σ_{4-p} is a closed (4-p)-form independent on θ. Clearly the operators $\tilde{O}_p^n$ do not depend on θ

$$\partial_\theta \tilde{O}_p^n = \int_M \sigma_{4-p} \partial_\theta H_p^n = - \int_M \sigma_{4-p} dH_p^n = 0, \quad (29)$$

[1] As it is pointed out in ref.[17, 16] the observables are the equivariant cohomology of the Q-algebra while the non-triviality of the correlators of the observables is due to the non-triviality of the space of ground states [18]. However these subtleties are not relevant in the present analysis.

i.e. $\tilde{O}_p^n$ are the BRST-invariant operators. One can consider the deformation of the topological theory by adding the operators $\tilde{O}_p^n$ (eq.(28)) to the action. These operators are the 4-dimensional integrals and, hence, the deformed Lagrangian is the local 4-dimensional Q-invariant operator. Now one can study the deformed local theory. It is interesting that the exterior forms σ_{4-p} can depend on the external metric. Therefore the deformation with the operators $\tilde{O}_p^n$ is more convenient to study the non-topological deformations of the theory since the deformed energy-momentum tensor is already not Q-exact. However in the following we shall use the representation (22) which is more convenient for the analysis of the operator algebra.

It is usefull to put down the ghost numbers and the scale dimensions of the fields and the covariant derivatives [1, 10]:

$$\begin{array}{ll} g(X_{\alpha\beta}) = -1, & d(X_{\alpha\beta}) = 2 \\ g(\Lambda) = -2, & d(\Lambda) = 2 \\ g(\nabla_\theta) = 1, & d(\nabla_\theta) = 0 \\ g(G_{\theta\theta}) = 2, & d(G_{\theta\theta}) = 0 \\ g(G_{\alpha\theta}) = 1, & d(G_{\alpha\theta}) = 1 \\ g(G_{\alpha\beta}) = 0, & d(G_{\alpha\beta}) = 2 \\ g(\nabla_\alpha) = 0, & d(\nabla_\alpha) = 1. \end{array} \quad (30)$$

From eqs.(30) it is easy to get the quantum numbers of the operators H_p^n and O_p^n:

$$g(H_p^n) = 2n - p, \; d(H_p^n) = p, \; g(O_p^n) = 2n - p, \; d(O_p^n) = 0 \quad (31)$$

Let us notice that all the fields have non-negative scale dimensions $d \geq 0$, and the operators O_p^n are dimensionless.

3 Anomalous Dimensions

In this section we shall analyze the anomalous dimensions of the operators O_p^n using the supergauge invariance of the theory, the conservation of the ghost number at the classical level, the scale dimensions of the fields and the background field formalism [19].

To find the renormalization of the operators O_p^n it is convenient to consider the calculation of the matrix element of the operators H_p^n in the external fields with momenta smaller than the scale of an infrared cutoff (it is assumed to be supergauge invariant). Then in the background field formalism one gets the following expression for the matrix element of the operators H_p^n

$$\langle H_p^n \rangle = \partial_\theta R_p^n + dR_{p-1}^n, \tag{32}$$

where $R_p^n = \langle K_p^n \rangle$ represents the local supergauge invariant operator constructed of the external fields. The ghost number and the scale dimension of the operator R_p^n are the following

$$g(R_p^n) = 2n - p - 1,\ d(R_p^n) = p. \tag{33}$$

From eq.(32) one can see that the integral of $\langle H_p^n \rangle$ over the cycle γ_p is BRST-exact:

$$\int_{\gamma_p} \langle H_p^n \rangle = \partial_\theta \int_{\gamma_p} R_p^n. \tag{34}$$

Therefore in the superfield formulation of the topological theory the operators O_p^n have the vanishing anomalous dimensions [2]. The renormalization of the operator O_p^n is due to the mixing with the BRST-exact operators and does not change the class of the cohomology of the operator O_p^n. This means that the quantum corrections do not affect the calculation of the correlators of the operators O_p^n.

Actually one can study this mixing of the operators O_p^n in more details when taking into account the quantum numbers of the mixing operators. Let us analyse the scale dimensions and the ghost numbers of the operators R_p^n.

1. Let $p = 0$. Then the operator R_0^n has the following form

$$R_0^n = M^k T_{0,k}^n, \tag{35}$$

where M is the parameter of the ultraviolet cutoff, $T_{0,k}^n$ is a local gauge invariant operator with the scale dimension $d = -k$ and the ghost number $g = 2n-1$. Obviously the case of $k < 0$ is not interesting since it corresponds to the suppression of the contribution of the operator R_0^n in eq.(32) by the factor $(\frac{1}{M})^{|k|}$. However since the scale dimensions of the fields in (30) are non-negative one has $-k \geq 0$. Therefore $k = 0$, and it is easy to see

[2]Notice that this conclusion generalizes the non-renormalizability of the topological charge at the operator level.

that $R_0^n \sim \partial_\theta Tr(G_{\theta\theta})^{n-1} = 0$ and , hence, the operator R_0^n vanishes due to eq.(23). Therefore the operator O_0^n is not mixed with the BRST-exact operators.

2. Let $p = 1$. Then one has

$$(H)_1^n = \partial_\theta R_1^n + dR_0^n = \partial_\theta R_1^n, \; R_1^n = M^p T_{1,k}^n, \tag{36}$$

where $T_{1,k}^n$ is a local gauge invariant operator with the quantum numbers $g = 2n-2$, $d = 1-k$. Like the previous case one can see that $k \geq 0$ and $1-k \geq 0$ and hence, $k = 0, 1$. At $k = 0$ the typical form of the operator R_1^n is as follows

$$R_1^n \sim Tr(G_{\theta\theta})^m \partial_\theta Tr(G_{\theta\theta})^l G_{\alpha\theta} dx_\alpha, \; m + l = n - 2. \tag{37}$$

Then one gets

$$(H)_1^n = \partial_\theta R_1^n = 0, \tag{38}$$

i.e. the operator O_1^n is not mixed with the BRST-exact operators.

3. Similar to the previous cases one can analyse the possible form of the operators R_2^n, R_3^n and R_4^n. It is easily to see that the operator R_2^n can not contain any power divergence, however it can include the logarithmically divergent contributions. These contributions can be proportional, for example, to the operator like $TrX_{\alpha\beta}(G_{\theta\theta})^{n-1}$. Thus the operator can be logarithmically renormalizable due to the mixing with the BRST-exact operator. The operators R_3^n and R_4^n can contain the quadratically and logarithmically divergent contributions. In the case of the operator R_3^n the quadratically divergent contributions are found to be BRST-exact and therefore they give no contribution to the r.h.s. of eq.(32). For the operator R_4^n the quadratically divergent contributions are not in general BRST-exact (for example, the contributions of the form $M^2 O_0^p H_1^q$, $p+q = n-1$). Therefore the operator R_4^n can be renormalizable due to the counterterms related to these quadratically divergent contributions. The appropriate operators corresponding to the logarithmical divergent counterterms for the operator TrR_3^n and R_4^n have the form like $X_{\alpha\beta}(G_{\theta\theta})^k G_{\gamma\theta}$ and $TrX_{\alpha\beta}\nabla_\theta X_{\alpha\beta}(G_{\theta\theta})^k$, $X_{\alpha\beta}(G_{\theta\theta})^k G_{\gamma\delta}$, respectively. They lead to the logarithmical renormalization of the operator O_3^n and O_4^n.

Thus the operators O_0^n and O_1^n are nonrenormalizable (and hence, are "marginal").The operators O_2^n and O_3^n can get only the logarithmically divergent counterterms. The quadratically (and logarithmically) divergent counterterms are possible for the operator O_4^*. In all the cases the counterterms are BRST-exact.

4 Operator Algebra and the Renormalizability of the Deformed Theory

Now one can hope that the adding to the Lagrangian (9), at least, of the operators O_0^* and O_1^* does not spoil the renormalizability of the theory since their quantum dimensions vanish. In this section we shall see that such a deformation of the model is really renormalizable due to the symmetries of the theory. In the case of more general deformations the analysis of the symmetries is not enough for the proof of the renormalizability.

It is worth explaining what is meant by the renormalizability in this situation. All the operators O_p^* are dimensionless and, hence, the Lagrangian of the deformed theory contains no dimensional coupling constants. However in the theory there is the dimensionless field ϕ. Therefore the quantum corrections can generate infinity of counterterms proportional to the operators of the scale dimension 0 and an arbitrary ghost number. As it is shown below in general these counterterms are not forbidden by the symmetries of the theory (the scale invariance, the conservation of the ghost number at the classical level and the superinvariance of the theory). A possible manifestation of the nonrenormalizability will be discussed in more detail at the end of this section.

To analyse the renormalization properties of the deformed theory one should consider the diagrams including an arbitrary number of the insertions of the operators O_p^*. On the other hand to do this it is enough to consider the Wilson-like expansions of the products of a finite number of the operators $O_p^* = \int_{\gamma_p} H_p^*$. Obviously the ultraviolet divergences induced by the insertions of the operator $O_p^* = \int_{\gamma_p} H_p^*$ correspond to the integration of the contributions in the operator expansion which are singular for the coinciding coordinates of the local operators H_p^*. The renormalizability of the theory would mean that the divergent part of the expansion of the product of the operators O_p^* contains only those operators which enter the bare Lagrangian of the deformed theory. Therefore the problem is reduced to the analysis of the algebra generated by the operators O_p^n.

The simplest case is the product of a finite number of the operators $O_0^n(x)$. Let us consider the following product

$$O_0^n(x)\,O_0^m(y) \sim \sum_s \frac{1}{(x-y)^s} R_s^{n,m}(x) + O(|x-y|), \qquad (39)$$

where $R_s^{n,m}$ (x) is the local gauge invariant operator with the quantum numbers $g = 2(n+m)$, $d = -s$, in eq.(39) the Lorentz indices are omitted. The crucial point here is that the dimensions of the fields are non-negative. From eq.(30) we get $d = -s \geq 0$ and therefore the product (39) is not singular at $x \to y$. It is worth emphasizing that in the case $s = 0$ there is also no logarithmical divergence because the r.h.s. of eq.(39) at $x \to y$ can contain only a local operator with the quantum numbers $g = 2(n+m)$ and $d = 0$. The unique operator of this type is O_0^{n+m} and, hence,

$$O_0^n(x)\,O_0^m(y) \sim O_0^{n+m}(x), \;\; x \to y. \qquad (40)$$

The coefficient on the r.h.s. does not contain even a logarithmical dependence on the coordinates $ln|x-y|$ because of independence of the correlator $\langle O_0^n(x)\,O_0^m(y)\rangle$ on the coordinates in the sector with a nontrivial topological charge of the Yang-Mills field. The nonsingularity of the expression (40) means that the coefficient in the r.h.s. of the eq.(40) is related to the global properties of the theory [1].

It is worth noticing that this situation is similar to the case of the N=2 D=2 superconformal theories where one considers the ring of the chiral primary operators [20] instead of the operator O_0^n. In the both cases the closure of the operator algebra and the nonsingularity of the coefficient functions are due to the constraints coming from the conservation of the ghost number and the analysis of the scale dimension of the operator.

The analysis of the operator algebra including the operators O_p^n at $p > 0$ is less trivial. In the following eq.(27) will be important. Let us consider the product of the operators O_p^n and O_q^m. To calculate the coefficients in the operator expansion it is necessary to compute a certain set of the Feynman diagrams. We shall not make these calculations in an explicit way but the use of the background field formalism [19] will be implied below.

The ultraviolet renormalisation is related to the local properties of the theory. Therefore in eq.(27) one can omit the second term [3] which is the total derivative and vanishes after integration over the closed cycle. Then the operator expansion of the product $O^n_p O^m_q$ is as follows

$$O^n_p O^m_q = \int_{\gamma_p} \int_{\gamma'_q} \partial_\theta K^n_p(x)\, \partial_{\theta'} K^m_q(y)$$

$$= \sum_s \int_{\gamma_p} \int_{\gamma'_q} \frac{1}{(x-y)^s} \partial_\theta \partial_{\theta'} [S^s_{p,q}(x,\theta) + (\theta - \theta') A^s_{p,q}(x,\theta)]$$

$$= \sum_s \int_{\gamma_p} \int_{\gamma'_q} \frac{1}{(x-y)^s} \partial_\theta A^s_{p,q}(x,y), \tag{41}$$

where $A^s_{p,q}(x,\theta)$ is a local gauge invariant operator with the ghost number $g = 2(n+m) - 1 - p - q$ and $d = p + q - s$; in eq.(41) the Lorentz indices are omitted. Thus all the divergent contributions to the r h.s. of eq.(41) are the BRST exact.

Let us analyse the possible form of the operator $A^s_{p,q}$. First of all the *case $p = q = 0$ was already studied above. Consider the more complicated* cases.

1. Let $p = 0$, $q = 1$. When the argument x of the operator $O^n_0(x)$ approaches the cycle γ'_1, one has

$$O^n_0(x)\, O^m_1 = \partial_\theta \int_{\gamma'_1} \sum_s \frac{1}{(x-y)^s} A^s_{01}(x,\theta). \tag{42}$$

where y is the variable of integration. The operator A^s_{01} has the ghost number $g = 2(n+m-1)$ and the scale dimension $d = 1 - s$. From eqs.(30) one gets $d \geq 0$ and, hence, $s \leq 1$. If $s = 1$ then one can expect the logarithmical divergence when integrating over the cycle γ'_1. However it is easy to see that the unique dimensionless operator with the even ghost number is the operator like O^k_0 which does not contribute to the r.h.s. of eq.(42) due to the differentiation in θ. If $s = 0$ there is no divergence because the logarithmical singularity is integrable. Moreover one can show

[3] Actually this term induces the contributions into the operator expansions which are the integrals of the total derivatives of the singular functions and, hence, could be non-vanishing. However these contributions are the integrals of the exact forms and vanish since the ultraviolet regularisation is introduced.

that the operator with the scale dimension $d = 1$ and the even ghost number $g = 2(n+m-1)$ has the form like $O_0^k(x)\, dO_0^{n+m-k-1}(x)$ and does not contribute to eq.(42) due to the differention in θ. Thus the product $O_0^n O_1^m$ does not contain any ultraviolet divergences.

2. Let $p = q = 1$. In this case one gets

$$O_1^n O_1^m = \partial_\theta \sum_s \int_{\gamma_1} \int_{\gamma_1'} \frac{1}{(x-y)^s} A_{11}^s(x,\theta). \tag{43}$$

In eq.(43) we integrate over both x (the cycle γ_1) and y (the cycle γ_1'). The operator $A_{1,1}^s$ has the scale dimension $d = 2 - s$ and the ghost number $g = 2(n+m) - 3$, while $s \leq 2$ due to eq.(30). Obviously an operator with the scale dimension $d = 0$ and the odd ghost number can not be constructed and, hence, $s \leq 1$. Let $s = 1$. Then one can expect the logarithmical divergence when integrating over the cycle γ_1' for $\gamma_1 = \gamma_1'$ (or for $dim\gamma_1 \cap \gamma_1' = 1$). In this case the r.h.s. of eq.(43) can be represented as follows

$$O_1^n O_1^m = \partial_\theta \int_{\gamma_1} dx_\mu \int_{\gamma_1'} dy_\nu \frac{(x-y)_\alpha}{(x-y)^2} \sum_k f_{\alpha\beta,\mu\nu}^k \, O_0^k(x)\, H_\beta^{n+m-k}(x), \tag{44}$$

where $f_{\alpha\beta,\mu\nu}^k$ is a constant (or logarithmically dependent on $|x-y|$) tensor. Obviously the integral over y is finite because the integrand is odd under the change $(x-y) \to -(x-y)$.

Thus the product $O_1^n O_1^m$ does not contain any ultraviolet divergences.

Therefore the algebra generated by the operators O_0^n and O_1^m has the finite structure constants which are related to the global properties of the theory.

3. In an analogous way one can show that the products $O_0^n O_2^m$, $O_0^n O_3^m$, $O_0^n O_4^m$, $O_1^n O_2^m$, $O_1^n O_3^m$ and $O_1^n O_4^m$ are finite.

4. Let $p = q = 2$. The operator $A_{2,2}^s$ has the scale dimension $d = 4 - s \geq 0$ and the ghost number $g = 2(n+m) - 5$. The case $s = 4$ is excluded due to the observation that the dimensionless operator with the odd ghost number can not be constructed. If $s = 3$ then there is the linearly divergent integral over y when $\gamma_2 = \gamma_2'$ or $dim\gamma_2 \cap \gamma_2' = 2$. However the

integrand is odd under the change $(x-y) \to -(x-y)$ and therefore the real degree of the divergence is smaller but depends on the definition of the divergent integral. For $s=2$ (and $dim\gamma_2 \cap \gamma_2' = 2$) in general there is the logarithmical divergence related to the operators $A^3_{2,2}$ of the form $\partial_\alpha Tr(G_{\theta\theta})^k Tr(G_{\theta\theta})^{n+m-k-3} G_{\alpha\theta}$, $Tr(G_{\theta\theta})^{n+m-2} X_{\alpha\beta}$ and so on.

5. In the case $p=2$, $q=3$ there is no divergence. However for $p=2$, $q=4$ $\gamma_4' = M$ and there could be the linear divergence ($s=5$) related to the operators $A^s_{2,4}$ like $Tr(G_{\theta\theta})^k Tr(G_{\theta\theta}^{n+m-k-4} G_{\alpha\theta}$, and the logarithmical divergence corresponding to the operators $\partial_\alpha Tr(G^k_{\theta\theta} Tr(G_{\theta\theta})^l G_{\alpha\theta}$, $k+l = n+m-4$, $tr(G_{\theta\theta})^{n+m-3} X_{\alpha\beta}$ and so on.

6. Let $p=q=3$. The operator $A^s_{3,3}$ has the scale dimension $d=6-s$ and the ghost number $g=2(n+m)-7$. Obviously at $s=5$ there is the quadratically divergent integral when $\gamma_3 = \gamma_3'$ (or $dim\gamma_3 \cap \gamma_3' = 3$). The real degree of the divergence is smaller because the integrand is odd under the change $(x-y) \to -(x-y)$. However the integral depends on the definition. This contribution corresponds to the operators $A^5_{3,3}$ of the form like $Tr(G_{\theta\theta})^k G_{\alpha\theta}$. For $s=4$ the integral in eq.(41) is linearly divergent and the operator $A^4_{3,3}$ can be of the form $Tr(G_{\theta\theta})^{n+m-3} X_{\alpha\beta}$, $\partial_\alpha Tr(G_{\theta\theta})^k Tr(G_{\theta\theta})^{n+m-4-k} G_{\alpha\theta}$ and so on. For $s=3$ the formal logarithmical divergence vanishes because the integrand is odd under the change $(x-y) \to -(x-y)$.

7. In the case $p=3$, $q=4$ the linear divergence is possible ($s=5$), corresponding to the operators $A^5_{3,4}$ like $Tr(G_{\theta\theta})^{n+m-4} G_{\alpha\beta}$, $Tr(G_{\theta\theta})^{n+m-3}\Lambda$, $Tr(G_{\theta\theta})^k G_{\alpha\theta}(G_{\theta\theta})^{n+m-5-k} G_{\beta\theta}$. At $s=4$ there is the logarithmically divergent integral in eq.(41), corresponding to the operators similar to $Tr(G_{\theta\theta})^{n+m-4} G_{\alpha\theta} X_{\beta\gamma}$, $Tr(G_{\theta\theta})^{n+m-3}(\nabla_\alpha \Lambda)$ and so on.

8. Finally, at $p=q=4$ the operator $A^s_{4,4}$ has the scale dimension $d=8-s$ and the ghost number $g=2(n+m)-9$. For $s=7$ the integral in eq.(41) is formally cubically divergent and the operator $A^7_{4,4}$ is of the form like $Tr(G_{\theta\theta})^{n+m-5} G_{\alpha\theta}$. At $s=6$ there is the quadratically divergent contributions related to the operators similar to $Tr(G_{\theta\theta})^{n+m-4} X_{\alpha\beta}$, $\partial_\alpha Tr(G_{\theta\theta})^k Tr(G_{\theta\theta})^{n+m-4-k} G_{\theta\alpha}$. The linear divergences at $s=5$ correspond to the operators $A^5_{4,4}$, for example, of the form $Tr(G_{\theta\theta})^{n+m-5} G_{\alpha\beta} G_{\gamma\theta}$, $Tr(G_{\theta\theta})^k X_{\alpha\beta} \partial_\gamma Tr(G_{\theta\theta})^{n+m-4-k}$, $Tr(G_{\theta\theta})^{n+m-4} G_{\alpha\theta}\Lambda$. If $s=4$ there is the logarithmically divergent the integral and the operator $A^4_{4,4}$ can be of the form like $Tr(G_{\theta\theta})^{n+m-3} X_{\alpha\beta} \nabla_\theta X^{\alpha\beta}$, $Tr(G_{\theta\theta})^{n+m-3} [\Lambda, \nabla_\theta \Lambda]$ and so on.

Let us turn to the problem of the renormalizability of the deformed theory. As it is shown above the products $O_0^n O_0^m$, $O_0^n O_1^m$ and $O_1^n O_1^m$ are finite. In this case the associativity of the operator algebra leads to the conclusion that the product of any finite number of this operators is finite. Really, one should consider the product of the finite number of the local operators $O_0^{n_i}(x_i)$ and $H_1^{m_j}(y_j)$. Because of the associativity of the operator algebra this product has singularities for the arguments $x_i \to y_j$ or $y_i \to y_j$. However as it is pointed out above these singularities are at most like $1/(x_i - y_j)$ or $1/(y_i - y_j)$ and integrable. Hence, they do not induce any divergences when integrating over the 1-cycles. Therefore the deformation of the theory (9) by adding the operators O_0^n andO_1^m to the action does not spoil the renormalizability of the theory. Moreover no new counterterms are induced by the deformation as compared to the non-deformed theory. Thus the generating functional for the Donaldson invariants can be interpreted as the partitial function of the renormalizable theory.

It is worth noticing that above we assumed that the possible counterterms were represented by the integrals of the local operators. This is certainly right in the renormalizable theory with a polynomial Lagrangian and means that the infrared singular contributions to the effective action can be summarized into the logarithmical ones. In this situation the arguments of the running constants in the effective action depend on the fields entering the theory. However in our case the Lagrangian is non-polynomial and the number of the constants of the theory is infinite. Therefore the assumption of locality of the counterterms corresponds to a certain definition of the theory.

Let us discuss now the generating functional for the correlators including the operators O_2^n, O_3^n and O_4^n. Let us assume that their algebra is not closed [4] in the sense that it generates an infinite set of the Q-exact counterterms according to the analysis given above. In the theory (even in the deformed one) there are no dimensional coupling constants. Therefore the counterterms are classified according to their ghost numbers. Of course the ghost number is not limited from above if the deformation violates the conservation of the ghost number. It is worth emphasizing that the infinite set of the counterterms is generated even when the theory is deformed by adding a *finite number* of the operators with a non-trivial ghost numbers to

[4] This algebra can be trivial if the theory involves hidden symmetries.

the action because of the additivity of the ghost number. In particular the deformations corresponding to the adding of the operators O_4^* to the action (9) generates the following contributions into the effective action:

$$Tr[f_1(G_{\theta\theta})X_{\alpha\beta}\nabla_\theta X_{\alpha\beta} + f_2(G_{\theta\theta})X_{\alpha\beta}G^{\alpha\beta} + ...], \qquad (45)$$

where $f_1, f_2...$ are arbitrary functions. In this case the effective kinetic term of the Yang-Mills field (in components) is as follows

$$Tr\, f(\phi)\, F_{\alpha\beta}F_{\alpha\beta}, \qquad (46)$$

where $f(\phi)$ is an arbitrary function, $F_{\alpha\beta}$ is the Yang-Mills strength. Thus we get the topological theory with a new Q-exact action and the generating functional for the correlators of the operators O_p^* in this theory. The classical equations of motion for the Yang-Mills field contain the coefficients depending on the scalar field φ. Therefore, the connection of this theory with the moduli space of instanton is unclear and should be studied.

In conclusion it is worth noticing that the constraints found above on the renormalizability of the deformed theory are due to the fact that the $\hat{d}$-cohomology is splitted by the quantum corrections into the d- and ∂_θ-cohomologies. In principle one can look for a more symmetric version of the $D = 4$ TYMT explicitly respecting the $\hat{d}$-cohomology at the quantum level. For such a version the different exterior forms entering the expansion of a class of the $\hat{d}$-cohomology are the components of a supermultiplet. Therefore we can expect that the total operator algebra generated by these exterior forms does not induce any ultraviolet divergences since the algebra of the zero-forms has no divergences. In this case the deformed theory would be renormalizable.

A sufficient condition for the non-renormalizability of the operators O_p^* can be found as follows. First notice that one can consider the d-cohomology as generated by the one-form operator

$$P = dx_\mu P_\mu, \qquad (47)$$

where P_μ is the operator of momentum, i.e.

$$P_\mu = \int d^3x\, T_{\mu 0} = \{Q\, \Lambda_\mu\}. \qquad (48)$$

In eq.(48) $T_{\mu\nu}$ is the energy-momentum tensor, and $\Lambda_\mu = \int d^3x \Lambda_{\mu 0}$ while the operator $\Lambda_{\mu\nu}$ is defined in eq.(2).

Clearly the operator Λ_μ is commuting with P_μ since the operator of momentum does not depend on time. Therefore it can be easily checked that the local superform operator $\hat{H}^{*}$ in eq.(21) can be represented as follows

$$\hat{H}^{*} = e^{\Lambda} H_0^{*} e^{-\Lambda}, \tag{49}$$

where $\Lambda = dx_\mu \Lambda_\mu$. The operator $\hat{H}^{*}$ is closed in the $(P+Q)$-cohomology

$$\{P+Q, \hat{H}^{*}\} = 0. \tag{50}$$

The renormalization of the operator O_p^{*} can be found by differentiating in the normalization point m of the operators. Obviously the operator P does not depend on m while in general the operator Λ can depend on m. However the derivative $\partial\Lambda/\partial m$ is reduced to the Q-exact operator since the operator P is non-renormalizable

$$\partial\Lambda/\partial m = \{Q, A\}. \tag{51}$$

In this eq. $A = dx_\mu A_\mu$, $A_\mu = \int d^3x \, a_\mu(x)$, a_μ is a local operator. On the other hand in this paper we have found that the operator H_0^{*} is non-renormalizable. Therefore it can be easily checked that

$$\partial\hat{H}^{*}/\partial m = \{P+Q, [\hat{f}, \hat{H}^{*}]\}, \tag{52}$$

where

$$\hat{f} = \int_0^1 dt \, e^{-t\Lambda} A e^{t\Lambda} = A - 1/2\,[\Lambda, A] + \ldots \tag{53}$$

From this eq. we conclude that the operators O_p^{*} can be renormalizable due to the mixing with the Q-exact operators. This renormalization vanishes if $A = 0$, i.e. the operator Λ is non-renormalizable. In this case the operator Λ would generate a symmetry of the theory and the operators H_p^{*} would be really the components of the Λ-"supermultiplet" H_p^{*}. Of course, the operator corresponds to the twisted N=2 multiplet of supergenerators (in the case $k = 1/8$ in eq.(9)) [1]. However in general the symmetry generated by these operators is broken since the external metric is not superinvariant. Therefore we should expect that the operator Λ is renormalizable.

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