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ABSTRACT

A necessary and sufficient condition is established for the strong convergence of the steepest descent approximation to a solution of equations involving quasi-accretive operators defined on a uniformly smooth Banach space.

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1. INTRODUCTION

Let E be an arbitrary Banach space. A mapping U with domain $D(U)$ and range $R(U)$ in E is called ***accretive*** [2] if the inequality

$$\|x-y\| \leq \|x-y+s(Ux-Uy)\| \tag{1}$$

holds for every x, $y \in D(U)$ and for *all* $s > 0$. If (1) holds only for *some* $s > 0$ then U is called ***monotone*** in the terminology of [20]. Let E^* denote the dual space of E and let $J : E \to 2^{E^*}$ denote the normalized duality mapping of E defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\| \cdot \|x^*\|,\ \|x^*\| = \|x\|\}\ ,$$

where $\langle \cdot\, , \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued and if E^* is uniformly convex, then J is uniformly continuous on bounded sets (see e.g. [39]). In the sequel we shall denote single-valued normalized duality mappings by j. As a consequence of a result of Kato [20], it follows from inequality (1) that U is accretive if and only if for each x, $y \in D(U)$ there exists $j(x-y) \in J(x-y)$ such that

$$Re\langle Ux-Uy, j(x-y)\rangle \geq 0\ . \tag{2}$$

Furthermore, U is called ***strongly accretive*** (see e.g., [2], [7], [20]) if there exists a constant $k > 0$ such that

$$Re\langle Ux-Uy, j(x-y)\rangle \geq k\|x-y\|^2\ . \tag{3}$$

If $E = H$, a Hilbert space, then (2) and (3) are equivalent, respectively, to the ***monotonicity*** and ***strong monotonicity*** properties of U in the sense of Minty [26].

In the sequel, we shall call the map U ***φ-strongly accretive*** if there exists a strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that the inequality

$$\langle Ux-Uy, j(x-y)\rangle \geq \varphi(\|x-y\|)\|x-y\| \tag{4}$$

holds for all $x, y \in D(U)$. Let $N(U) = \{x \in E : Ux = 0\}$. If $N(U) \neq \emptyset$, and the inequalities (2), (3) and (4) hold for any $x \in D(U)$ and $y \in N(U)$, then the corresponding operator U is said to be ***quasi-accretive***, ***strongly quasi-accretive*** and ***φ-strongly quasi-accretive***, respectively. Such operators have been studied by various authors (see e.g., [2], [3], [6], [8], [25], [27], [35]).

The accretive operators were introduced in 1967 by Browder [2] and Kato [20]. Interest in such mappings stems mainly from the fact that many physically significant problems can be modelled in terms of an initial valued problem of the form

$$\left.\begin{aligned} \frac{dx}{dt} &= -Tx \\ x(0) &= x_0 \end{aligned}\right\} \tag{5}$$

where T is either accretive, strongly accretive or φ-strongly accretive in an appropriate Banach-space. Typical examples of how such evolution equations arise are found in models involving either the heat or wave or the Schrödinger equation.

An early fundamental result in the theory of accretive operators due to Browder [2], states that the initial value problem (5) is solvable if T is *locally* Lipschitzian and accretive on E. Utilizing the existence result for (5), Browder [2] showed that if T is *locally* Lipschitzian and accretive then T is *m-accretive*, i.e. $(I+T)(E) = E$, where I denotes the identity operator on E. This result was subsequently generalized by Martin [25] to the *continuous* accretive operators, and to *demicontinuous* accretive operators by Deimling [13, Theorem 13.1], where an operator A from E to E is said to be *demicontinuous* if it is continuous from the strong topology of E to the weak topology of E.

We observe that members of $N(T)$, the kernel of T, are in fact the equilibrium points of the system (5). Consequently, considerable effort has been devoted to developing constructive techniques for the determination of the kernels of accretive operators (see e.g., [7-10], [12], [13], [14], [19], [21], [24], [26], [28], [30], [31], [32-34], [36], [37], [41], [42], [43], [44], [45], [46], [47], [48], [49]). Moreover, since a continuous accretive operator can be approximated well by a sequence of strongly accretive ones, particular attention has been devoted to the kernels of strongly accretive maps. In this connection, but in Hilbert space, Vainberg [41] and Zarantonello [49] introduced the steepest descent approximation method,

$$x_{n+1} = x_n - c_n T x_n,\ x_0 \in H,\ n = 0, 1, 2, \ldots \tag{6}$$

and proved that if,

(i) $T = I + J$ where I is the identity map of H and J is a monotone and Lipschitz map on H;

(ii) $c_n = \lambda \in (0,1)$, $n = 0, 1, 2, \ldots$,

then the sequence $\{x_n\}$ defined iteratively by (6) converges strongly to an element of $N(T)$. This result has been extended to the class of bounded monotone operators (see e.g., [6], [8], [17]). Typical of the results obtained is the following theorem:

Theorem * Let H be a Hilbert space, $T : H \to H$ a bounded strongly accretive map with a nonempty kernel, $N(T)$. Then the sequence $\{x_n\}$ defined iteratively by (6) with c_n in $\ell^2 \backslash \ell^1$ converges globally to an element of $N(T)$.

Extensions of Theorem * to more general Banach spaces have been obtained by various authors . Vainberg [42, pp. 276-284] proved the convergence of (6) in L_p spaces for $1 < p < \infty$ when T is Lipschitz continuous and strongly accretive; the author [7] obtained the same result in L_p spaces, $p \geq 2$, under less restrictive conditions. Crandall and Pazy [12] proved convergence of (6) for a continuous strongly accretive operator on an arbitrary Banach space. Reich [33], and also Liu [22] proved the convergence of (6) for an arbitrary

strongly accretive operator acting on Banach spaces whose duals are uniformly convex (or equivalently, on uniformly smooth Banach spaces).

We remark immediately that in these results in Banach spaces, the conditions imposed on the iteration parameter c_n are not convenient in applications. For instance, Crandall and Pazy [12] required that at each iteration step, c_k be determined by

$$c_k = \delta_{k+1}/(1+\delta_{k+1}) \text{ where } \delta_{k+1} = 2^{-n_k}$$

and n_k is the least nonnegative integer such that

$$\left\| A\left(\frac{2^n}{1+2^n}x_k - \frac{1}{1+2^n}Ax_k\right) - Ax_k \right\| \leq \exp\{-(\delta_1 + \delta_2 + \ldots + \delta_k + 1)\}$$

It is clear that c_k can hardly be determined in an explicit form from the above condition. In [33], Reich imposed the additional assumption that $\sum_{n=0}^{\infty} c_n^2 \|Tx_n\|^2 < +\infty$. Again this condition causes computational difficulties.

Recently, some authors have proved convergence theorems for strongly accretive operators in which the iteration parameter c_n is easily evaluated. In this connection we have the following theorem:

Theorem 1 (Chidume [7]) Suppose K is a nonempty closed bounded and convex subset of L_p, $p \geq 2$ and $T : K \to K$ is a Lipschitz map such that $A = I - T$ is strongly accretive. Let $\{c_n\}$ be a real sequence satisfying:

(i) $0 < c_n < 1$ for all $n \geq 1$,

(ii) $\sum_{n=1}^{\infty} c_n = \infty$; and

(iii) $\sum_{n=1}^{\infty} c_n^2 < \infty$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by $x_1 \in K$,

$$x_{n+1} = x_n - c_n A x_n,\ n \geq 1$$

converges strongly to the unique solution of $Ax = 0$.

An operator T such that $(I - T)$ is strongly accretive is called ***strongly pseudo-contractive*** (see e.g. [3], [7], [21], [24], [36], [37]). It is then clear that the mapping theory for accretive operators is closely related to the fixed point theory of pseudocontractive maps. A prototype for the c_n in Theorem 1 is $c_n = 1/(n+1)$, $n \geq 0$. While Theorem 1 is restricted to L_p spaces, $p \geq 2$, the parameter c_n is easily chosen at the start of the iteration process.

Several authors have also generalized Theorem 1. In [36], Schu extended the theorem to the class of real Banach spaces with property $(U, \alpha, m+1, m)$, i.e. Banach spaces in which the inequality $\|x+y\|^{m+1} + \alpha\|x-y\|^{m+1} - 2^m(\|x\|^{m+1} + \|y\|^{m+1}) \geq 0$ holds for all $x, y \in E$, where $m+1 \geq 0$, $\alpha \in \mathbb{R}$. These Banach spaces include L_p spaces for $p \geq 2$ (see e.g. [36], [40]) and in [37] he extended the theorem to the class of *uniformly continuous* maps on *smooth* Banach spaces. Bethke [1] also announced a slight generalization of the theorem, still in L_p spaces, $p \geq 2$, while Weng [46] and also Kang [19] announced extensions of the theorem to the class of *local* strong accretive maps. Other extensions can be found in Xu, Zhang and Roach [44] and ir Xu and Roach [45].

In [44], the authors interpreted the iteration process (6) as an Euler's difference approximation to the initial value problem (5), defined a sequence of interpolation functions of the iterates $\{x_n\}$ such that the sequence of functions takes the unique solution of the initial value problem (5) as its limit point and then applied the asymptotic stability theory for the equlibrium point of the system (5) to obtain global convergence of (6) with the choice of the iteration parameter c_n being independent of the operator. Their method involves the following definitions:

For any x, $y_0 \in D(A)$, let

$$U(x) = \{y \in E : \|y\| \leq \alpha^{-1}\|Ax\|\}; M(x) = \sup\{\|(Ay)\| : \|y - y_0\| \leq \alpha^{-1}(\|Ax\| + \|Ay_0\|)\};$$

$$\beta(x) = \sup\{\tau \in \mathbb{R}^+ : \rho_E(\tau M(x))/\tau \leq 2\alpha^{-1}\|Ax\|^2/K[\alpha^{-1}\|Ax\| + \tau M(x) + \frac{1}{2}c]\}$$

where α, K and C are constants such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|x-y\|^2 ,$$

$$K = \max\{8, 128(\sqrt{3}-1)c\}$$

and

$$C = \frac{4\tau_0}{(1+\tau_0^2)^{1/2} - 1} \prod_{j=1}^{\infty} \left(1 + \frac{15\tau_0}{2^{j+2}}\right)$$

with $\tau_0 = (\sqrt{339} - 18)/30$, and ρ_E denotes the modulus of smoothness of E (see e.g. [15], [23]). With these definitions, and using the following inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + K\max\{\|x\| + \|y\|, \frac{1}{2}C\}\rho_E(\|y\|)$$

which the authors proved (see [44]) holds for each x, y in any uniformly smooth Banach space the authors proved the following therom:

Theorem XZR Let E be a real uniformly smooth Banach space and $A : D(A) = E \to E$ be a demicontinuous strongly accretive operator with $M(x_0) < +\infty$ for any element $x_0 \in E$. Suppose that $\{c_n\}_{n=0}^{\infty}$ is an arbitrary sequence of real numbers satisfying

(i) $0 < c_n < \min\{(2a)^{-1}), \beta(x_0)/M(x_0)\}$ for any $n \geq 0$;

(ii) $\sum_{n=0}^{\infty} c_n = \infty$

(iii) $c_n \to 0$ as $n \to \infty$.

Then the steepest descent approximation

$$x_{n+1} = x_n - c_n Ax_n, \; n = 0, 1, 2, \ldots$$

converges strongly to the unique solution of the equation $Ax = 0$.

For our next theorem we shall need the following lemma:

Lemma 1 (Reich [32]) Let E be a uniformly smooth Banach space. Then there exists a nondecreasing continuous function

$$b : [0, \infty) \to [0, \infty)$$

satisfying:

(i) $b(ct) \leq cb(t)$ for all $c \geq 1$

(ii) $\lim_{t \to 0^+} b(t) = 0$, and

(iii) $\|x + y\|^2 \leq \|x\|^2 + 2Re\langle y, j(x)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|)$

for all $x, y \in E$.

With b as defined in this lemma, a more general result than Theorem 1 which was discovered independently by the author [11] and Osilike [30] is the following theorem:

Theorem 2 Let E be a real uniformly smooth Banach space and K be a nonempty closed convex and bounded subset of E. Suppose $T : K \to K$ is a continuous strongly accretive map. For a given $f \in K$, define the sequence $\{x_n\}_{n=0}^{\infty}$ in K iteratively by $x_0 \in K$,

$$x_{n+1} = x_n - c_n(Tx_n - f), \; n \geq 0 ,$$

where $\{c_n\}_{n=0}^{\infty}$ is a real sequence satisfying:

(i) $0 \leq c_n < 1$ for all n,

(ii) $\sum_{n=0}^{\infty} c_n = +\infty$; and

(iii) $\sum_{n=0}^{\infty} c_n b(c_n) < +\infty$.

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Tx = f$.

Remark 1 Nevanlinna and Reich [29] have shown that for any given continuous nondecreasing function $\beta(t)$ with $\beta(0) = 0$, sequences $\{c_n\}_{n=0}^{\infty}$ always exist satisfying condition (i)-(iii) of Theorem 2. If $E = L_p$, $1 < p < \infty$, we can choose any sequence $\{c_n\}_{n=0}^{\infty}$ in $\ell^s \backslash \ell^1$ with $s = p$ if $1 < p \leq 2$ and $s = 2$ if $p \geq 2$. It is then clear how the iteration parameter c_n can be chosen in Theorem 2.

In the special case when T is a strongly pseudocontractive operator or equivalently $A = (I - T)$ is strongly accretive (in particular, when T is ***nonexpansive***, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in the domain of T), several authors have used the iteration process: $x_0 \in D(A)$ given by,

$$x_{n+1} = x_n - c_n A x_n \ ,$$

with $c_n \in (0, \infty)$, $\sum_{n=0}^{\infty} c_n = +\infty$, $\lim_{n\to\infty} c_n = 0$ to determine solutions of the equation $Ax = 0$ (see e.g. [2], [4], [5], [6], [8], [12], [16], [18], [32-34], [38], [41-42], [45], [49]).

Basically, the authors have proved the following typical results:

(a) Let E be a uniformly smooth Banach space (or equivalently, E is a Banach space whose dual is uniformly convex) and A be a bounded, strongly accretive map. Then there exists a real number $T(x_0) > 0$ such that the iteration scheme (6) with $c_n \leq T(x_0)$ for each n, converges strongly to the unique solution of $Ax = 0$ (when it exists) [45, 47, 48].

(b) If $A = I - T$, where for a closed convex subset K of E, $T : K \to K$ is a nonexpansive map with a nonempty fixed point set $F(T)$, then the iterative scheme (6), with $c_n \in [0, 1)$ for each n, converges strongly to a fixed point of T provided T satisfies the condition

$$\|x - Tx\| \geq f(d(x, F(T))) \ , \tag{7}$$

where $f : [0, \infty) \to [0, \infty)$, $f(0) = 0$ is a strictly increasing function, and $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$ (see e.g. [18], [38]).

As a consequence of (a) and (b), Xu and Roach [45] studied the following two natural questions:

(i) Can the iterative process (6) be used for quasi-accretive maps rather than just for strongly accretive ones?

(ii) For nonexpansive maps, is the condition (7) necessary for the convergenc of the iteration process (6)?

They gave an affirmative answer to both questions. However, while the conditions imposed on the iteration parameters c_n in their results do not involve the norm of the operator, they involve the computation of an infinite product and also they depend explicitly on an estimate of the modulus of smoothness of the space.

It is our purpose in this paper first to give a proof of the results of Xu and Roach [45] in which the iteration parameter does not depend on an estimate of the modulus of smoothness of the space and in which the computation of an infinite product is not required. Then, we shall also prove a theorem which extends Theorem XZR to the class of φ-strongly accretive maps. Moreover, no interpolation theory or stability theory of ordinary differential equations will be needed for our theorem.

2. MAIN RESULTS

Following [45], we shall say that a quasi-accretive operator A on a Banach space E satisfies ***Condition I***, if, for any $x \in D(A)$, $x^* \in N(A)$ and any $j(x - x^*) \in J(x - x^*)$, the equality $\langle Ax, j(x - x^*)\rangle = 0$ holds if and only if $Ax = Ax^* = 0$. It is clear from this definition that any strongly quasi-accretive operator satisfies Condition I. Furthermore, as has been shown in [45], if T is a nonexpansive map defined on a uniformly convex Banach space then (I-T) satisfies Condition I.

Using the technique of [45] and the inequality of Reich, we prove the following theorems:

Theorem 3 Let E be a real uniformly smooth Banach space and let $A : D(A) = E \to E$ be a quasi-accretive, bounded operator which satisfies Condition I. Then, for any initial value $x_0 \in D(A)$ there exist positive real numbers $T(x_0)$ such that the steepest descent approximation method

$$x_{n+1} = x_n - c_n\, A x_n, \quad n \geq 0 \tag{8}$$

with

(i) $0 < c_n \leq T(x_0)$ for each n;

(ii) $\sum_{n=0}^{\infty} c_n = \infty$; and

(iii) $\lim_{n\to\infty} c_n = 0$,

(iv) $\sum_{n=0}^{\infty} c_n\, b(c_n) < \infty$

converges strongly to a solution x^* of the equation $Ax = 0$ if and only if there exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(0) = 0$ such that

$$\langle Ax_n - Ax^*, j(x_n - x^*)\rangle \geq \varphi(\|x_n - x^*\|)\|x_n - x^*\| \tag{9}$$

Proof. Necessity This follows exactly as in [45].

Sufficiency Suppose now that inequality (9) holds. We prove that there exists $T(x_0) > 0$ such that the iteration process (8) with $c_n \le T(x_0)$ for every n converges strongly to x^*, a solution of the equation $Ax = 0$. In the sequel, we shall make repeated use of Lemma 1.

Let $M(x_0) = \sup\{\|Au\| : \|u - x_0\| \le 3\varphi^{-1}(\|Ax_0\|)\}$.

Let τ be the largest positive real number such that

$$b(\tau) \le \frac{2\varphi^{-1}(\|Ax_0\|)\|Ax_0\|}{\max\{2\varphi^{-1}(\|Ax_0\|), 1\}\max\{M(x_0), 1\}M(x_0)} , \tag{10}$$

where b is the function defined in Lemma 1.

(Note that (10) is possible because b is continuous and $b(0) = 0$).

Set, $T(x_0) = \min\{\tau, \frac{\varphi^{-1}(\|Ax_0\|)}{M(x_0)}\}$.

We now prove the following claims:

Claim 1 The sequence $\{x_n\}$ defined by (8) with $c_n \le T(x_0)$ is bounded. The proof of this claim is by contradiction. So assume that $\{x_n\}$ is not bounded. Then there are two possible cases.

Case 1 There exist integers $n_0 > 0$ such that

$$\|x_n - x^*\| > \varphi^{-1}(\|Ax_0\|) \quad \forall n \ge n_0 \tag{11}$$

Without loss of generality we may assume n_0 is the smallest integer for which (11) holds. Consequently,

$$\|x_{n_0-1} - x^*\| \le \varphi^{-1}(\|Ax_0\|) \tag{12}$$

Furthermore,

$$\|x_{n_0} - x^*\| \le \|x_{n_0-1} - x^*\| + c_{n_0-1}\|Ax_{n_0-1}\| \le \varphi^{-1}(\|Ax_0\|) + T(x_0)\|Ax_{n_0-1}\| \tag{13}$$

But since,

$$\|x_{n_0-1} - x_0\| \le \|x_{n_0-1} - x^*\| + \|x^* - x_0\| \le \varphi^{-1}(\|Ax_0\|) + \varphi^{-1}(\|Ax_0\|) = 2\varphi^{-1}(\|Ax_0\|)$$

it follows that $\|Ax_{n_0-1}\| \le M(x_0)$. Hence from (13)

$$\|x_{n_0} - x^*\| \le \varphi^{-1}(\|Ax_0\|) + T(x_0)M(x_0) \le 2\varphi^{-1}(\|Ax_0\|) \tag{14}$$

Consequently,

$$\|x_{n_0} - x_0\| = \|x_{n_0} - x^*\| + \|x^* - x_0\| \le 2\varphi^{-1}(\|Ax_0\|) + \varphi^{-1}(\|Ax_0\|) = 3\varphi^{-1}(\|Ax_0\|) \tag{15}$$

so that $\|Ax_{n_0}\| \le M(x_0)$. We now compute:

$$\begin{aligned}\|x_{n_0+1} - x^*\|^2 &= \|x_{n_0} - x^* - c_{n_0}Ax_{n_0}\|^2 \\ &\le \|x_{n_0} - x^*\|^2 - 2c_{n_0}\langle Ax_{n_0}, j(x_{n_0} - x^*)\rangle + \\ &+ \max\{\|x_{n_0} - x^*\|, 1\}c_{n_0}\|Ax_{n_0}\|b(c_{n_0}\|Ax_{n_0}\|) \\ &\le \|x_{n_0} - x^*\|^2 - 2c_{n_0}\varphi(\|x_{n_0} - x^*\|)\|x_{n_0} - x^*\| \\ &+ \max\{\|x_{n_0} - x^*\|, 1\}c_{n_0}\|Ax_{n_0}\|\max\{\|Ax_{n_0}\|, 1\}b(c_{n_0})\end{aligned} \tag{16}$$

Recall that $\|x_{n_0} - x^*\| > \varphi^{-1}(\|Ax_{n_0}\|)$ so that $\varphi(\|x_{n_0} - x^*\|) > \|Ax_{n_0}\|$.

Hence (16) reduces to:

$$\begin{aligned}\|x_{n_0+1} - x^*\|^2 &\le \|x_{n_0} - x^*\|^2 - 2c_{n_0}\|Ax_0\|\varphi^{-1}(\|Ax_0\|) \\ &+ \max\{2\varphi^{-1}(\|Ax_0\|), 1\}c_{n_0}M(x_0)\max\{M(x_0), 1\}b(c_{n_0}) \\ &\le \|x_{n_0} - x^*\|^2 - 2c_{n_0}\|Ax_0\|\varphi^{-1}(\|Ax_0\|) \\ &+ \max\{2\varphi^{-1}(\|Ax_0\|), 1\}c_{n_0}M(x_0)\max\{M(x_0), 1\}b(T(x_0))\end{aligned}$$

since b is nondecreasing, we also have $b(T(x_0)) \le b(\tau)$ so that

$$\|x_{n_0+1} - x^*\|^2 \le \|x_{n_0} - x^*\|^2 - c_{n_0}\{2\varphi^{-1}(\|Ax_0\|)\|Ax_0\| - Kb(\tau)\} \tag{17}$$

where $K = \max\{2\varphi^{-1}(\|Ax_0\|, 1\}M(x_0)\max\{M(x_0), 1\}$.

Inequalities (17) and (10) yield

$$\|x_{n_0+1} - x^*\| \le \|x_{n_0} - x^*\| .$$

In the same way one can show that for all $n \ge n_0$,

$$\|x_{n+1} - x^*\| \le \|x_n - x^*\| \le \ldots \le \|x_{n_0} - x^*\| ,$$

which contradicts the assumption that $\{x_n\}$ is not bounded.

Case 2 The interval $[0, \varphi^{-1}(\|Ax_0\|)]$ contains infinitely many $\|x_n - x^*\|$ of the sequence $\{\|x_n - x^*\|\}$. Compactness of this interval implies that in this case, there exists some real number $\rho \ge 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - x^*\| \to \rho$ as $k \to \infty$. But by assumption $\{\|x_n - x^*\|\}$ is not bounded. Hence $\mathbb{R}\backslash[0, 2\varphi^{-1}(\|Ax_0\|)]$ must also contain infinitely many $\|x_n - x^*\|$. Consequently, the sequence $\{\|x_n - x^*\|\}$ must pass through the interval

$$\left[\rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|), \rho + \frac{4}{5}\varphi^{-1}(\|Ax_0\|)\right]$$

infinitely many times. So we can find two subsequences of $\{x_n\}$, say, $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that

(a) $n_k < m_k$

(b) $\|x_{n_k-1} - x^*\| \le \rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|)$ and $\|x_{n_k} - x^*\| > \rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|)$

(c) $\|x_{m_k-1} - x^*\| \le \rho + \frac{4}{5}\varphi^{-1}(\|Ax_0\|)$ and $\|x_{m_k} - x^*\| > \rho + \frac{4}{5}\varphi^{-1}(\|Ax_0\|)$

Let $M_0 = \sup\{\|Au\| : \|u - x^*\| \le \rho + \varphi^{-1}(\|Ax_0\|)\}$ and observe that since $\|x_{n_k-1} - x^*\| \le \rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|) \le \rho + \varphi^{-1}(\|Ax_0\|)$ we have

$$\|x_{n_k} - x_{n_k-1}\| = c_{n_k-1}\|Ax_{n_k-1}\| \le c_{n_k-1}M_0 \to 0 \text{ as } k \to \infty .$$

Hence,

$$\|x_{n_k} - x^*\| \le \|x_{n_k} - x_{n_k-1}\| + \|x_{n_k-1} - x^*\| \to \rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|) \tag{18}$$

On the other hand, for every $n \in [n_k, m_k - 1]$ we have:

$$\|x_{n+1} - x^*\|^2 = \|x_n - c_nAx_n - x^*\|^2$$

$$\le \|x_n - x^*\|^2 - 2c_n(Ax_n, j(x_n - x^*)) + \max\{\|x_n - x^*\|, 1\}c_n\|Ax_n\|b(c_n\|Ax_n\|)$$

$$\le \|x_n - x^*\|^2 - 2c_n\varphi(\|x_n - x^*\|)\|x_n - x^*\| + \max\{\|x_n - x^*\|, 1\}c_n\|Ax_n\| \max\{\|Ax_n\|, 1\}b(c_n)$$

$$\le \|x_n - x^*\|^2 - 2c_n\varphi[\rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|)][\rho + \frac{1}{5}\varphi^{-1}(\|Ax_0)\|]$$

$$+ \max\{\rho + \frac{4}{5}\varphi^{-1}(\|Ax_0\|), 1\}c_n\|Ax_n\| max\{\|Ax_n\|, 1\}b(c_n)$$

$$\le \|x_n - x^*\|^2 - 2c_n\varphi[\rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|)][\rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|]$$

$$+ \max\{\rho + \frac{4}{5}\varphi^{-1}(\|Ax_0\|)\}M_0 \max\{M_0, 1\}c_nb(c_n)$$

$$\le \|x_n - x^*\|^2 - c_n[2\varphi\{\rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|)\}\{\rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|)\} - K_2b(c_n)] , \tag{19}$$

where $K_2 = \max\{\rho + \frac{4}{5}\varphi^{-1}(\|Ax_0\|)\}M_0 \max\{M_0, 1\}$ Since $c_n \to 0$, b is continuous and $b(0) = 0$, it follows that there exists $N^* > 0$ such that for all $n \ge N^*$,

$$2\varphi\{\rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|)\}\{\rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|)\} - K_2b(c_n) \ge 0$$

Hence, for all $n \ge N^*$, it follows from (19) that

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 \quad \text{for all} \quad n \in [n_j, m_j - 1], n_j \ge N^* .$$

So, in particular,

$$\|x_{m_j} - x^*\| \le \|x_{m_j-1} - x^*\| \le \ldots \le \|x_{n_j} - x^*\| \quad \text{for all} \quad n_j \ge N^*.$$

But then, using the second part of condition (c) and inequality (18), we must have:

$$\rho + \frac{4}{5}\varphi^{-1}(\|Ax_0\|) \le \limsup_j \|x_{m_j} - x^*\| \le \ldots \le \lim \|x_{n_j} - x^*\| \le \rho + \frac{1}{5}\varphi^{-1}(\|Ax_0\|)$$

which is a contradiction. This, together with Case 1 completes the proof that $\{x_n\}$ is bounded.

Claim 2 The sequence $\{\|x_n - x^*\|\}$ is convergent. Assume this is not the case. The boundedness of $\{x_n\}$ then implies that there exist at least two real numbers ρ_1 and ρ_2 and subsequences $\{x_{n_i}\}$, $\{x_{n_j}\}$ such that

$$\|x_{n_i} - x^*\| \to \rho_1 \ (i \to \infty); \quad \|x_{n_j} - x^*\| \to \rho_2 \ (j \to \infty) .$$

Without loss of generality we may assume $\rho_1 > \rho_2$. Thus the sequence $\{\|x_n - x^*\|\}$ must pass through the interval $[\rho_2 + \frac{1}{5}\rho_1, \rho_2 + \frac{4}{5}\rho_1]$ infinitely many times. Consequently, an argument similar to that used in Case 2 of Claim 1 shows that this is impossible. Hence the sequence $\{\|x_n - n^*\|\}$ must be convergent. Let $\|x_n - x^*\| \to \delta \ge 0$ as $n \to \infty$.

Claim 3 $\delta = 0$. Suppose this is not the case (i.e. suppose that $\delta > 0$). Let $N > 0$ be such that $\|x_j - x^*\| \ge \delta/2$ for all $j \ge N$. Thus, in particular,

$$\liminf \varphi(\|x_j - x^*\|) \ge \varphi(\delta/2) > 0 \tag{20}$$

Furthermore, since A is a bounded operator (by hypothesis) and $\{x_n\}$ is bounded, we can set $M^* = \sup_{n\ge1} \|Ax_n\|$ and compute as follows:

$$\|x_{n+1} - x^*\|^2 = \|x_n - x^* - c_nAx_n\|^2$$

$$\le \|x_n - x^*\|^2 - 2c_n(Ax_n, j(x_n - x^*)) + \max\{\|x_n - x^*\|, 1\}c_n\|Ax_n\|b(c_n\|Ax_n\|)$$

$$\le \|x_n - x^*\|^2 - 2c_n\varphi(\|x_n - x^*\|)\|x_n - x^*\| \max\{\|x_n - x^*\|, 1\}c_n\|Ax_n\| \max\{\|Ax_n\|, 1\}b(c_n)$$

$$\le \|x_n - x^*\|^2 - 2c_n\varphi(\|x_n - x^*\|)\|x_n - x^*\| + \max\{\|x_n - x^*\|, 1\}M^* \max\{M^*, 1\}c_nb(c_n)$$

Hence, for some constant $M > 0$, we obtain:

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 - 2c_n\varphi(\|x_n - x^*\|)\|x_n - x^*\| + Mc_nb(c_n)$$

Iteration of this inequality from 1 to N yields:

$$\|x_{N+1} - x^*\|^2 \le \|x_1 - x^*\|^2 - 2\sum_{j=1}^{N} c_j\varphi(\|x_j - x^*\|)\|x_j - x^*\| + M\sum_{j=1}^{N} c_jb(c_j)$$

As $N \to \infty$, using Condition (iv) and the fact that $\{\|x_n - x^*\|\}$ is bounded we obtain:

$$\sum_{j=1}^{\infty} c_j\varphi(\|x_j - x^*\|)\|x_j - x^*\| < +\infty$$

which then implies (by Condition (ii) and the assumption that $\|x_n - x^*\| \to \delta > 0$) that

$$\liminf \varphi(\|x_j - x^*\|) = 0 \ ,$$

contradicting (20). So, $\delta = 0$. This completes the proof of the Theorem.

Remark 1 In [45], the authors proved Theorem 3 by using the inequality (**) and showed that the iteration process

$$x_{n+1} = x_n - c_n A x_n, n = 0, 1, 2 \ldots$$

with $\{c_n\} \subseteq (0, T(x_0))$, $\sum_n c_n = +\infty$, $c_n \to 0$ as $n \to \infty$ converges strongly to the unique solution x^* of $Ax = 0$, where

$$T(x_0) = \min\{\beta, \varphi^{-1}(\|Ax_0\|)/M(x_0)\} \ ,$$

$$M(x_0) = \sup\{\|Au\| : \|u - x_0\| \leq 3\varphi^{-1}(\|Ax_0\|)\} \ ,$$

φ is the function given in inequality (9), β is the largest positive real number such that

$$\beta^{-1}\rho_E(\beta M(x_0)) \leq \frac{2\varphi^{-1}(\|Ax_0\|)\|Ax_0\|}{K_1[3\varphi^{-1}(\|Ax_0\|) + \frac{1}{2}C]}, \tag{21}$$

$$K_1 = \max\{8, \sqrt{128}(\sqrt{3} - 1)C\}$$

and $C = \frac{4\tau_0}{(1+\tau_0^2)^{1/2}-1} \sum_{i=1}^{\infty}(1 + \frac{15}{2^{i+2}}\ \tau_0)$ with $\tau_0 = \frac{\sqrt{339}-18}{30}$.

As has been remarked earlier, it is clear that to use the method of [45] one has to first compute the infinite product $\prod_{i=1}^{\infty} (1 + \frac{15}{2^{i+2}}\ \tau_0)$. (Note that in [45] the series $\sum_{i=1}^{\infty} \ (1 + \frac{15}{2^{i+2}}\ \tau_0)$ is written instead of the infinite product. This is probably a typographical error. In fact, the *series* does not even converge!) Then using the values of K_1 and C (as above) one has to get an estimate of β from inequality (21). This inequality depends explicitly on the modulus of smoothness of the space. This modulus is, in general, not easy to compute. Our method does not require the computation of any infinite product and does not depend explicitly on the modulus of smoothness of the space.

We now present the following immediate Corollary of Theorem 3. We shall need the following definitions:

For any $x, y_0 \in D(A)$, let

$$U(x) = \{u \in E : \|u\| \leq \varphi^{-1}(\|Ax\|)\}$$

$$M(x) = \sup\{\|Au\| : \|u - y_0\| \leq \varphi^{-1}(\|Ax\|) + \varphi^{-1}(\|Ay_0\|)\}$$

and let $T(x_0)$ be as defined in the proof of Theorem 3. Then we have the following

Corollary

Let E be a real uniformly smooth Banach space, $A : D(A) = E \to E$ be a φ-strongly quasi-accretive operator such that $M(x_0) < +\infty$ for an element $x_0 \in E$. Suppose that $\{c_n\}$ is a real sequence satisfying the following conditions:

(i) $0 < c_n \leq T(x_0)$,

(ii) $\sum_{n=0}^{\infty} c_n = +\infty$

(iii) $\lim_{n\to\infty} c_n = 0$.

(iv) $\sum_{n=0}^{\infty} c_n\, b(c_n) < \infty$.

Then the steepest descent approximation

$$x_{n+1} = x_n - c_n A x_n, \ n \geq 0, \tag{22}$$

converges strongly to the unique solution of the equation $Ax = 0$.

Proof Let x^* be an element of $N(A)$. Clearly A is quasi-accretive, satisfies inequality (9) and consequently satisfies Condition I. It then follows as in the proof of Theorem 3 that the sequence $\{x_n\}$ defined by (22) is bounded and that $\{\|x_n - x^*\|\}_{n=0}^{\infty}$ converges. (Note that the boundedness assumption on the operator A in Theroem 3 is needed only in the proof of Claim 3). Now let $\|x_n - x^*\| \to \delta \geq 0$ as $n \to \infty$.

Claim. $\delta = 0$ Suppose this is not the case. Let $N > 0$ be such that $\|x_j - x^*\| \geq \delta/2$ for all $j \geq N$. Then

$$\liminf \varphi(\|x_j - x^*\|) \geq \varphi(\delta/2) > 0 \ .$$

Observe that since A is φ-strongly quasi-accretive,

$$\|Ay_0\| \cdot \|y_0 - x^*\| \geq \langle Ay_0 - Ax^*, j(y_0 - x^*)\rangle \geq \varphi(\|y_0 - x^*\|)\|y_0 - x^*\|$$

so that

$$\|y_0 - x^*\| \leq \varphi^{-1}(\|Ay_0\|)$$

Similarly,

$$\|x_0 - x^*\| \leq \varphi^{-1}(\|Ax_0\|)$$

hence,

$$\|x_n - y_0\| \leq \|x_n - x^*\| + \|y_0 - x^*\| \leq \varphi^{-1}(\|Ax_0\|) + \varphi^{-1}(\|Ay_0\|)$$

and hence, since $M(x_0) < +\infty$, $\sup_{n\geq 1} \|Ax_n\|$ exists. Let $M^* = \sup_{n\geq 1} \|Ax_n\|$, and the claim now follows as in the proof of Theorem 3, completing proof of the Corollary.

Remark 3 The above Corollary is a significant generalization of the main result of [44], Theorem XZR, in the sense that the Corollary extends the main result of [44] from the class of strongly accretive operators to the more general class of φ-strongly quasi-accretive maps (It suffices to define $\varphi : [0,\infty) \to [0,\infty)$ in the Corollary by $\varphi(t) = kt$ for each $t \in [0,\infty)$). Moreover, our proof is direct and does not involve any interpolation theory or stability theory of ordinary differential equations.

We now conclude this paper with the following theorem which shows that in the special case in which $A = I - T$ with T a nonexpansive mapping with a nonempty fixed point set, $F(T)$, in a certain sense, the condition $\|x - Tx\| \geq g(d(x, F(T)))$, where $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, is a strictly increasing function and $d(x, F(T)) = \inf\ \{\|x - x^*\| : x^* \in F(T)\}$ is a necessary and sufficient condition for the convergence of the iteration process (8) to a fixed point of T.

Theorem 4 Let E be a uniformly smooth banach space, let $K \subseteq E$ be a nonempty closed convex subset of E, and let $T : K \to K$ be a quasi-nonexpansive mapping (i.e. $F(T) \neq \emptyset$ and $\|Tx - Tx^*\| \leq \|x - x^*\|$ for all $x \in K$, $x^* \in F(T)$). Then for any initial value $x_0 \in K$, the iteration process (8) with $A = I - T$ and the real sequence $\{c_n\}$ satisfying:

(i) $0 < c_n < 1$ for all n,

(ii) $\sum_{n=0}^{\infty} c_n = \infty$;

(iii) $\lim_{n\to\infty} c_n = 0$

(iv) $\sum_{n=0}^{\infty} c_n\, b(c_n) < \infty$

converges strongly to a fixed point x^* of T if and only if there exists a strictly increasing function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\|x_n - Tx\| \geq g(d(x, F(t))), \quad n \geq 0\ .$$

The proof of Theroem 4 which makes use of Theorem 3 follows exactly as in [45].

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REFERENCES

1. M. Bethke, "*Approximation von Fixpunkten streng pseudokontraktiver operatoren*" (Approximation of fixed points of strongly pseudocontractive operators), Math. Naturwiss. Fak. **27**, No. 2 (1989), 263-270.

2. F.E. Browder, "*Nonlinear mappings of nonexpansive and accretive type in Banach spaces*", Bull. Amer. Math. Soc. **73** (1967), 875-882.

3. F.E. Browder, "*Nonlinear operators and nonlinear equations of evolution in Banach spaces*" in Proc. Symp. Pure Math., Vol. 18, Part 2, Amer. Math. Soc., Providence RI, 1976.

4. F.E. Browder, "*Nonlinear monotone and accretive operators in Banach spaces*", Proc. Nat. Acad. Sci. USA **61** (1968), 388-393.

5. F.E. Browder and W.V. Petryshyn, "*Construction of fixed points of nonlinear mappings in Hilbert space*", J. Math. Anal. Appl. **20** (1967), 197-228.

6. R.E. Bruck, "*The iterative solution of the equation $y \in x + Tx$ for a monotone operator T in Hilbert space*", Bull. Amer. Math. Soc. **79** (1973), 1258-1261.

7. C.E. Chidume, "*Iterative approximation of fixed points of Lipschitz strictly pseudocontractive mappings*", Proc. Amer. Math. Soc. **99** (1987), 283-288.

8. C.E. Chidume, "*The iterative solution of the equation of $x + Tx$ for a monotone operator T in L^p spaces*", J. Math. Anal. Appl. **116** (1986), 531-537.

9. C.E. Chidume, "*Iterative solution of nonlinear equations of the monotone type in Banach spaces*", Bull. Austral. Math. Soc., Vol. 42 (1990), 21-31.

10. C.E. Chidume, "*Approximation of fixed points of stringly pseudo-contractive mappings*", Proc. Amer. Math. Soc. (to appear.

11. C.E. Chidume, "*Iterative solution of nonlinear equations with strongly accretive operators*", ICTP Trieste, preprint IC/91/361 (1991).

12. M.G. Crandall and A. Pazy, "*On the range of accretive operators*", Israel J. Math. **27** (1977), 235-246.

13. K. Deimling, *Nonlienar Functional Analysis* (Springer-Verlag, Berlin, 1985).

14. K. Deimling, "*Zeros of accretive operators*" Manuscripta Math. **13** (1974), 365-374.

15. J. Diestel, *Geometry of Banach spaces - Selected Topics*, Lecture Notes in Mathematics **485**, Springer Verlag, New York 1975.

16. W.G. Dotson, "*On the Mann iteration process*", Trans. Amer. Math. Soc. **149** (1970), 65-73.

17. A. A. Fonarev, "*The solution of nonlinear equations with monotone transformations*" (Russian), Differential'nye Uravneija **14:A** (1978).

18. S. Ishikawa, "*Fixed points and iteration of a nonexpansive mapping an a Banach space*", Proc. Amer. Math. Soc. **59** (1976), 65-71.

19. Zon Bao Kang, "*Iterative approximation of the solution of a locally Lipschitzian equation*", Appl. Math. Mech. (English Ed.) **12** (1991), No. 4, 409-414; translated from Appl. Math. Mech. **12** (1991), No. **4**, 385-389.

20. T. kato, "*Nonlienar semigroups and evolution equations*", J. Math. Soc. Japan **19** (1967), 508-520.

21. W.A. Kirk, *Remarks on pseudocontractive mappings*", Proc. Amer. Math. Soc. **25** (1970), 820-823.

22. Kunkun Kiu, "*Iterative method for constructing zeros of accretive sets*", M.Sc. Thesis, Xi'an Jaiotong University 1984.

23. J. Lindenstrauss and L. Tzafriri, "*Classical Banach spaces II- Function space*" (Springer-Verlag, Berlin, 1979).

24. C. Morales, "*Surjectivity theroems for multi-valued mappings of accretive type*", Comment. Math. Univ. Carolin. **26**, 2 (1985).

25. R.H. Martin, "*A global existence theorem for autonomous differential equations in Banach spaces*, Proc. Amer. Math. Soc. **26** (1970), 307-314.

26. G.J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. **29** (1962), 541-546.

27. E. Mitidieri and G. Morosaru, "*Asymptotic behaviour of the solution of second order difference equations associated to monotone operators*", (384) (1985-86), 419-434.

28. O. Nevanlinna, "*Global iteration schemes for monotone operators*" Nonlinear Anal **3** No. 4 (1979).

29. O. Nevanlinna and S. Reich, "*Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces*", Israel J. Math. **32** (1979) 44-58.

30. M.O. Osilike, "*Ishikawa and Mann iteration methods for nonlinear strongly accretive mappings*", Bull. Austral. Math. Soc., Vol. **46** (1992), 411-422.

31. D. Pascali and S. Sbarlan, "*Nonlinear mappings of monotone type*" Editure Academiai Bucaresti, Romania, 1978.

32. S. Reich, "*An iterative procedure for constructing zeros of accretive sets in Banach space*", Nonlinear Anal. **2** (1978), 85-92.

33. S. Reich, "*Constructive techniques for accretive and monotone operators*", in Applied Nonlinear Analysis, pp. 335-345, Academic Press, New York 1979.

34. S. Reich, "*Constructing zeros of accretive operators, I, II*", Appl. Anal. **8** (1979), 349-352; Appl. Anal. **9** (1979), 159-163.

35. R.T. Rockajellar, "*Monotone operator and the proximal point algorithm*", SIAM J. Control. Optim. **14**, No. 5 (1976).

36. J. Schu, "*On a theorem of C.E. Chidume concerning the iterative approxiamtion of fixed points*", Math. Nachr. **153** (1991), 313-319.

37. J. Schu, "*Iterative construction of fixed points of strictly pseudocontractive mappings*", Applicable Analysis, Vol. 40 (1991), 67-72.

38. H.F. Senter and W.G. Dotson, "*Approximating fixed points of nonexpansive mappings*", Proc. Amer. Math. Soc. **44** (1974), 375-380.

39. V.L. Smul'yan, "*On the derivative of the norm in a Banach space*, Dokl. Acad, Nauk SSSR **27** (1940), 255-258.

40. Yu. V. Trubnikov, "*Hanner inequality and convergence of iteration procedures*", Soviet Math. (Izvestiya) **31** (7) (1987), 74-83.

41. M.M. Vainberg, "*On the convergence of the method of steepest desceng for nonlinear equations*", Sibirsk. Math. Z. **2** (1961), 201-220.

42. M.M. Vainberg, *Variational Method and Method of Monotone Operators in the Theory of Nonlinear Equations*, John Wiley and Sons, 1973.

43. Zongben Xu and G.F., Roach, "*Characteristic inequalities fo uniformly convex and uniformly smooth Banach spaces*", J. Math. Anal. Appl. **157** (1991), 189-210.

44. Zongben Xu, Bo Zhang and G.F. Roach, "*On the steepest descent approximation to solutions of nonlinear strongly accretive operator equations*" J. Comput. Math., Vol. 7. No. 2 (1992).

45. Zongben Xu and G.F. Roach, "*A necessary and sufficient condition for convergence of steepest descent approximation to accretive operator equations*", J. Math. Anal. Appl. **167** (1992), 340-354.

46. Xinlong Weng, Proc. Amer. Math. Soc., Vol. II, No. 3 (1991), 727-731.

47. Zhao You and Zongben Xu, "*A class of iteration methods for a strongly monotone operator equation and application to finite element approximate solution of nonlinear elliptic boundary value problem*", J. Comput. Math. **2** (1984), 112-121.

48. Zhaoyong You, Zongben Xu and Kunkun Liu, *The region contraction algorithm for constructing zeros of a quasi-strongly monone operator*", J. Engr. Math. **1** (1984), 11-28.

49. E.H. Zarantonello, "*The closure of the numerical range contains the spectrum*", Bull. Amer. Math. Soc. (N.S.) 70 (1964), 781-783.