INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

DISTRIBUTIVELY GENERATED MATRIX NEAR RINGS

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ABSTRACT

It is known that if R is a near ring with identity then $(I, +)$ is abelian iff $(I^+, +)$ is abelian and $(I, +)$ is abelian iff $(I^*, +)$ is abelian [S.J. Abbasi, J.D.P. Meldrum, 1991]. This paper extends these results. We show that if R is a distributively generated near ring with identity then $(I, +) \subseteq Z(R)$, the center of R, iff $(I^+, +) \subseteq Z(M_n(R))$, the center of matrix near ring $M_n(R)$. Furthermore $(I, +) \subseteq Z(R)$ iff $(I^*, +) \subseteq Z(M_n(R))$.

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1 Introduction

(R,+,.) is a right near ring if (R,+) is a group (not necessarily abelian), (R,.) is a semigroup and $(x+y)z = xz + yz \forall x, y, z \in R$. R is zero-symmetric if $x0 = 0 \ \forall \ x \in R$. A normal subgroup (I,+) of (R,+) is an ideal of R if $IR \subseteq R$ and $x(a+y) - xy \in I \ \forall \ x, y \in R$ and $a \in I$. A group G is an R-module if R is homomorphic to M(G), the set of all functions from G to itself. A normal subgroup (H,+) of G is called an R-ideal of G if $r(h+g) - rg \in H \ \forall r \in R, h \in H$ and $g \in G$. If R is zero-symmetric and G is an R-module then any R-ideal of G is an R-submodule.

R is distributively generated (d.g.) if ((R,+) is generated as a group by (S,.), a semigroup of distributive elements of R. Every d.g. near ring is zero-symmetric [A. Fröhlich, 1958]. If X $\subseteq$ (R,S), then the ideal I of (R,S), generated by X, is the normal subgroup of (R,+) generated by $SXR = \{sxr, sx, xr, x : s \in S, x \in X, r \in R\}$.
(I_1, I_2), the commutator of ideals I_1 and I_2 of a d.g. near ring R, is an ideal of R.
All these results are available in [J. D. P. Meldrum 1985].

$M_n(R)$, the near ring of nxn matrices, is a subnear ring of $M(R^n)$, generated by the set $\{f_{ij}^r : r \in R, 1 \leq i, j, \leq n\}$ where $R^n = \oplus_1^n (R, +)$, the direct sum of n copies of (R,+), $f_{ij}^r = \iota_i f^r \pi_j$ with $f^r(x) = rx \ \forall \ x \in R$, ι_j and π_j are j-th co-ordinate injection and projection functions respectively. $\overline{I}$ is defined as a subnear ring of $M_n(R)$, generated by the set $\{f_{ij}^a : a \in I, 1 \leq i, j \leq n\}$ and I^+ is defined as an ideal generated by $\overline{I}$. We define $I^* = (I^n : R^n) = \{X \in M_n(R) : X\alpha \in I^n \ \forall \ \alpha \in R^n\}$. I^* is an ideal of $M_n(R)$ and $I^+ \subseteq I^*$. If R has an identity, then $()^+$ and $()^*$ are injections [J. D. P. Meldrum, A.P.J. Van Der Walt, 1986], [A. P. J. Van Der Walt, 1987].

2 On d.g. matrix near rings

2.1 Lemma

Let R be a d.g. near ring. Then $f_{ij}^{(x,y)} = (f_{ij}^x, f_{ij}^y)$ where $x, y \in R$ and $1 \leq i, j \leq n$.
Proof: Since $f_{ij}^{x+y} = f_{ij}^x + f_{ij}^y$ and $f_{ij}^{-x} = -f_{ij}^x$, therefore the result follows immediately by definition and simple calculation.

2.2 Lemma

Let I_1 and I_2 be ideals of a d.g. near ring R. Then $\overline{(I_1, I_2)} \subseteq (I_1^+, I_2^+)$.
Proof: Let $A \in \overline{(I_1.I_2)}$. We use induction on the weight, w(A), of A. If $w(A) = 1$, and $A = f_{ij}^x$ where $x \in (I_1, I_2)$, $1 \leq i.j \leq n$ then $A \in (I_1^+, I_2^+)$, by simple calculation and lemma 1.
Now suppose that the result is true for all elements of (I_1, I_2) of weight less than m, $m \in N, m \geq 2$. If w(A) = m then $A = A_1 + A_2$ or $A = A_1 A_2$, where $w(A_1), w(A_2) < w(A)$. Case 1 follows simply by induction. For case 2, since $(I_1^+, I_2^+)M_n(R) \subseteq (I_1^+, I_2^+)$, therefore $A \in (I_1^+, I_2^+)$. The result now follows by induction.
Remark: For a d.g. near ring, $I^+ = Gp < \mathcal{S}\overline{I}M_n(R) >^{M_n(R)}$, where $\mathcal{S} = \{f_{ij}^s : s \in S, 1 \leq i, j \leq n\}$

2.3 Lemma

$(I_1^+, I_2^+) \supseteq (I_1, I_2)^+$.
Proof; As (I_1^+, I_2^+) is an ideal of $M_n(R)$ so $(I_1^+, I_2^+) \supseteq \mathcal{S}(I_1^+, I_2^+)M_n(R) \supseteq \mathcal{S}\overline{(I_1, I_2)}M_n(R)$, by lemma 2. Now since $((I_1^+, I_2^+), +)$ is a normal subgroup of $(M_n(R), +)$, therefore $(I_1^+, I_2^+) \supseteq \mathcal{S}\overline{(I_1, I_2)}M_n(R)$ iff $(I_1^+, I_2^+) \supseteq \text{Gp} < \mathcal{S}\overline{(I_1, I_2)}M_n(R) >^{M_n(R)} = (I_1, I_2)^+$.

2.4 Lemma

$(I_1^*, I_2^*) \subseteq (I_1, I_2)^*$.
Proof: Let (A,B) $\in (I_1^*, I_2^*)$ where $A \in I_1^*, B \in I_2^*$. Take $\alpha \in R^n$. Then $(A, B)\alpha \in (I_1, I_2)^n$, by definition of commutator and the fact that $(I_1^n, I_2^n) = (I_1, I_2)^n$. So $(A, B) \in (I_1, I_2)^*$. Let $U \in (I_1^*, I_2^*)$. Then $U = \pm C_1 \pm \ldots \pm C_k \in (I_1, I_2)^*$ as $C_t \in (I_1.I_2)^* \ \forall\, t, 1 \leq t \leq k$. This completes the proof.

2.5 Theorem

Let R be a d.g. near ring with identity. If $I \subseteq Z(R)$, the centre of R, then $I^* \subseteq Z(M_n(R)$.
Proof: Since $(R^*, I^*) \subseteq (R, I)^* = \{0\}^* = 0$ and $R^* = M_n(R)$, therefore $(M_n(R), I^*) = 0$.

2.6 Theorem

Let R be a d.g. near ring with identity. If $I \subseteq Z(R)$, then $I^+ \subseteq Z(M_n(R))$.
Proof: $I^+ \subseteq I^* \subseteq Z(M_n(R))$.

2.7 Theorem

Let R be a d.g. near ring with identity. $I \subseteq Z(R)$ iff $I^+ \subseteq Z(M_n(R))$.
Proof: Since $(I, R)^+ \subseteq (I^+, R^+) = (I^+, M_n(R)) = 0 = \{0\}^+$, therefore $(I, R) = \{0\}$. Hence $I \subseteq Z(R)$.

2.8 Theorem

Let R be a d.g. near ring with identity. $I \subseteq Z(R)$ iff $I^* \subseteq Z(M_n(R))$.
Proof: Since $(I, R)^+ \subseteq (I^+, R^+) \subseteq (I^*, M_n(R)) = 0 = \{0\}^+$, therefore $(I, R) = \{0\}$. This completes the proof.

3 Open Problem

The author does not know whether these results are true or false for non d.g. near rings, and for d.g. near rings without identity.

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