

**INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS**

**QUANTUM HEISENBERG GROUPS
AND SKLYANIN ALGEBRAS**

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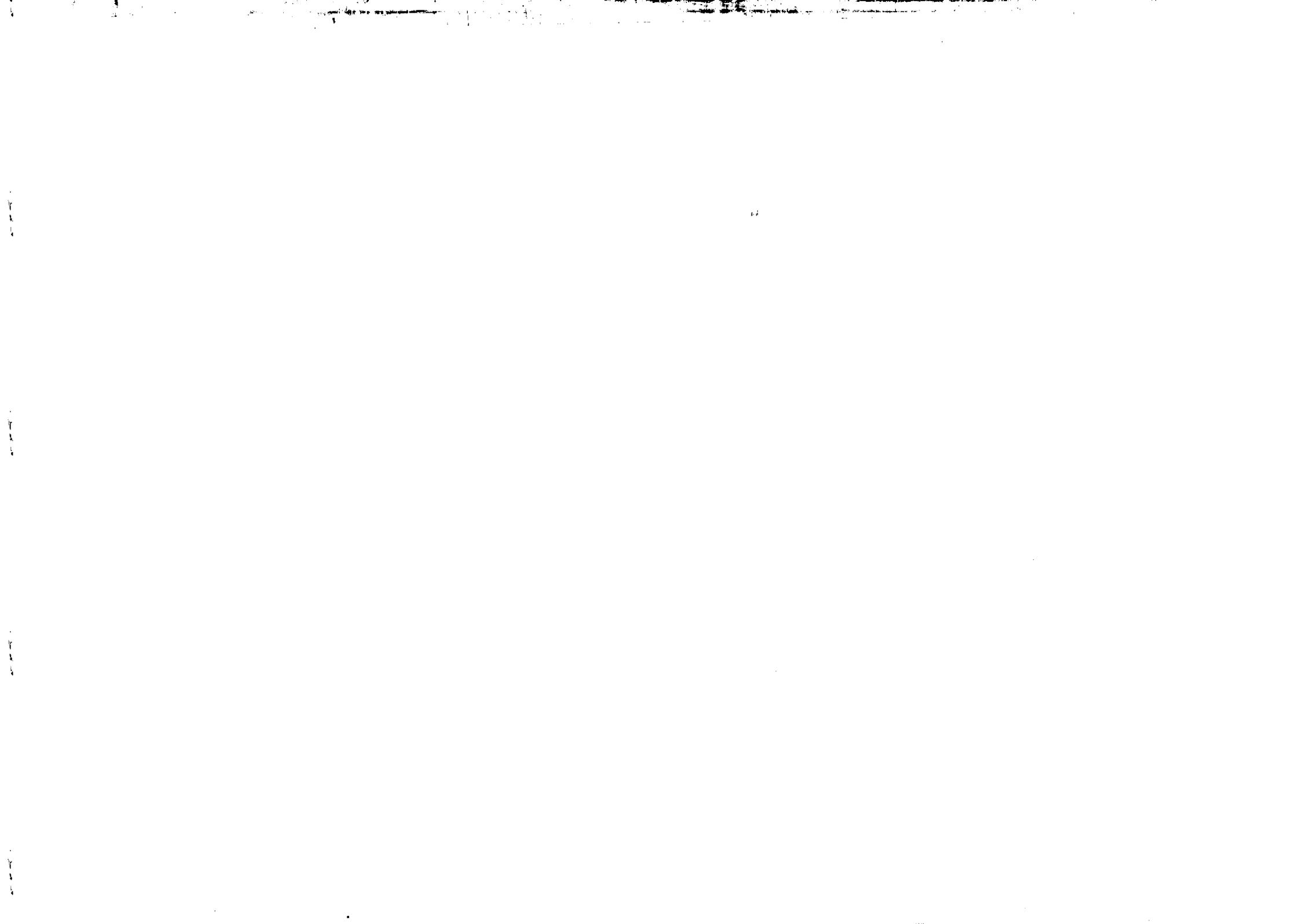


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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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AND SKLYANIN ALGEBRAS**

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MIRAMARE - TRIESTE

May 1993

We define new quantizations of the Heisenberg group by introducing new quantizations in the universal enveloping algebra of its Lie algebra. Matrix coefficients of the Stone-von Neumann representation are preserved by these new multiplications on the algebra of functions on the Heisenberg group. Some of the new quantizations provide also a new multiplication in the algebra of theta functions; we obtain in this way Sklyanin algebras.

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- * Forschungsstipendiat der Alexander von Humboldt-Stiftung.
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§1. Introduction. It is known that theta functions arise as certain matrix coefficients of the Stone-von Neumann representation (see for example [Mu]). We are interested in the algebra of thetas, i.e. the homogeneous coordinate ring [Mu, Section 10]. It is easy to see that the multiplication of thetas corresponds to the multiplication of matrix coefficients in the algebra of functions on the Heisenberg group. The purpose of this paper is to state a quantum analogue of this phenomenon. We introduce new quantizations of the Heisenberg group; some of them give rise to quantum multiplications on the ring of theta functions. We obtain in this way Sklyanin algebras. (See [Sk], [ATV], [SS], [LS], [OF]).

Besides the so-called quantum Heisenberg algebra (known to physicists for a long time, see [Ku], [JBS], [GF], [R]), a quantized enveloping algebra $U_{\hbar}(\mathfrak{g})$ of the Heisenberg-Lie algebra was introduced in [Ce et al], and (independently but later) in [ALT]. We discuss $U_{\hbar}(\mathfrak{g})$ in Section 2. The novelty is that we determine the primitive spectrum of $\mathbb{C}_{\hbar}[G]$ (the algebra of functions on the quantum Heisenberg group) and constate a bijective correspondance with the set of symplectic leaves of the corresponding Poisson structure (previous work in the semisimple case was done in [VS],[LS],[HL],[J]). It is however more convenient for our purposes to work with other quantizations of the Heisenberg group.

We recall basic facts about the Stone-von Neumann theorem in Section 3. Let us fix a positive integer m . Then any irreducible unitary representation of the Heisenberg group, such that its center (a copy of S^1) acts by $z \mapsto z^m \text{id}$, is the tensor product of a trivial representation with $\mathcal{H}^{(m)}$, the Stone-von Neumann representation of weight m . Applying this to the tensor product $\mathcal{H}^{(m)} \otimes \mathcal{H}^{(p)}$, we see that the product of a matrix coefficient of $\mathcal{H}^{(m)}$ and a matrix coefficient of $\mathcal{H}^{(p)}$ is a matrix coefficient of $\mathcal{H}^{(m+p)}$. Here the product of matrix coefficients can be thought in the dual of $U(\mathfrak{g})$, the enveloping algebra of the Heisenberg-Lie algebra. Clearly, different theta functions arise as matrix coefficients of the various $\mathcal{H}^{(m)}$.

So our first approximation is to introduce a new comultiplication in $U(\mathfrak{g})[[\hbar]]$ with the same property. We do this in section 4. We also explain carefully there how more new comultiplications in $U(\mathfrak{g})[[\hbar]]$, preserving coefficients of the Stone-von Neumann representations, should look like. In section 6 we present these new comultiplications. They provide quantizations of the Heisenberg-Lie group in the sense of [B et al], [Dr1]. In Section 5, we discuss our new multiplication (on the algebra of functions) in a purely algebraic setting, which encompass also the presentation of Sklyanin algebras given in [ATV]. We show in Section 7 that some of the comultiplications introduced in Section 6 give rise to "quantum" multiplication in the ring of theta functions, via the identification of the later with particular matrix coefficients. Thanks to the results of Section 5, we see that we have obtained Sklyanin algebras. On another direction, we show in Section 8 that the quantum Heisenberg algebra mentioned above is a braided Hopf algebra.

§2. We want first to discuss the algebra of functions on the quantum Heisenberg group considered in [Ce et al], [ALT]. Let $\hat{\mathfrak{g}}$ (resp., \mathfrak{g}) be the extended Heisenberg Lie algebra, i.e. the Lie algebra spanned by $x_i, y_i, 1 \leq i \leq g, z, d$ with brackets

$$\begin{aligned} [x_i, y_j] &= \delta_{ij} z, & [d, x_i] &= x_i, & [d, y_i] &= -y_i \\ [x_i, x_j] &= [y_i, y_j] = [z, x_i] = [z, y_j] = [d, z] = 0. \end{aligned}$$

(resp., the subalgebra generated by x_i, y_i, z).

The quantized universal enveloping algebra $U_{\hbar}(\hat{\mathfrak{g}})$ of the extended Heisenberg Lie algebra $\hat{\mathfrak{g}}$ is the $\mathbb{C}[[\hbar]]$ -algebra generated in the \hbar -adic sense by X_i, Y_i ($1 \leq i \leq g$), Z, D subject to the relations

$$[X_i, Y_j] = \delta_{ij} \frac{\sinh \frac{\hbar}{2} Z}{\frac{\hbar}{2}}, \quad Z \text{ is central}, \quad [D, X_i] = X_i, \quad [D, Y_i] = -Y_i,$$

together with its Hopf algebra structure, defined by the comultiplication Δ_{\hbar} , the counit ε_{\hbar} and the antipode S_{\hbar} :

$$\begin{aligned} \Delta_{\hbar}(X_i) &= X_i \otimes \exp\left(\frac{\hbar}{4} Z\right) + \exp\left(-\frac{\hbar}{4} Z\right) \otimes X_i, \\ \Delta_{\hbar}(Y_i) &= Y_i \otimes \exp\left(\frac{\hbar}{4} Z\right) + \exp\left(-\frac{\hbar}{4} Z\right) \otimes Y_i, \\ \Delta_{\hbar}(Z) &= Z \otimes 1 + 1 \otimes Z, \\ \Delta_{\hbar}(D) &= D \otimes 1 + 1 \otimes D, \end{aligned}$$

$\varepsilon_{\hbar}(X_i, Y_i, Z, D) = 0$, $S_{\hbar}(X_i, Y_i, Z, D) = -\text{id}$. $U_{\hbar}(\hat{\mathfrak{g}})$ is a quasi-triangular Hopf algebra and $U_{\hbar}(\mathfrak{g})$ is, by definition, its Hopf subalgebra generated by X_i, Y_i, Z . The assignment

$$\begin{aligned} \rho(X_i).e_j &= \delta_{j,i+1} e_1, & \rho(Y_i).e_j &= \delta_{j,n+2} e_{i+1}, \\ \rho(Z).e_j &= \delta_{j,n+2} e_1, & \rho(D).e_j &= \delta_{j,1} e_1 + \delta_{j,n+2} e_{n+2}, \end{aligned}$$

defines a $g+2$ -dimensional representation of $U_{\hbar}(\hat{\mathfrak{g}})$. Let \hat{G} be the connected unipotent subgroup of $GL(g+2)$ corresponding to \mathfrak{g} . The subalgebra of the dual of $U_{\hbar}(\hat{\mathfrak{g}})$ generated by the matrix coefficients of this representation is called the "algebra of rational functions on the extended quantum Heisenberg group". It can be presented by generators $\mathcal{X}_i, \mathcal{Y}_i, \mathcal{D}, \mathcal{D}^{-1}, \mathcal{Z}$ ($i = 1, \dots, g$), with the following relations:

$$[\mathcal{X}_i, \mathcal{X}_j] = [\mathcal{X}_i, \mathcal{Y}_j] = [\mathcal{Y}_i, \mathcal{Y}_j] = 0, \quad \{\mathcal{X}_i, \mathcal{Z}\} = \frac{\hbar}{2} \mathcal{X}_i, \quad \{\mathcal{Y}_i, \mathcal{Z}\} = \frac{\hbar}{2} \mathcal{Y}_i, \quad \mathcal{D} \mathcal{D}^{-1} = 1,$$

($i, j = 1, \dots, n$) and $\mathcal{D}, \mathcal{D}^{-1}$ are central elements ([Ce et al], [ALT]). The "algebra of rational functions on the quantum Heisenberg group" \mathcal{A} is the quotient of the last by a suitable ideal; it is isomorphic to the subalgebra generated by $\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}$. The determination of the Lie algebra structure on \mathfrak{g}^* follows easily from [ALT, Prop. 1] (which generalizes [Dr1, Ex. 2.2]). It turns out that $\mathcal{A} \simeq U(\mathfrak{g}^*_{\hbar})$, where the bracket of \mathfrak{g}^*_{\hbar} is that of \mathfrak{g}^* multiplied by \hbar . As \mathfrak{g}^*_{\hbar} is solvable, it follows from the Kostant-Kirillov "orbit method" (cf. [Di]) that the primitive spectrum of $\mathcal{A} \simeq U(\mathfrak{g}^*_{\hbar})$ is in natural correspondence with the set of coadjoint orbits of \mathfrak{g}^* . But we know that the linearization at the identity of the left dressing action is the coadjoint representation [LW, Th. 2.4]. As \mathfrak{g} is nilpotent, the exponential map is a bijection and therefore the symplectic leaves of \hat{G} parametrize naturally the primitive spectrum of \mathcal{A} (compare with [LS], [VS], [HL], [J]).

§3. We are, however, more interested in quantum versions of another subalgebra of $U(\mathfrak{g})^*$, which will be discussed in this section.

If V is a real vector space, we shall denote by $V[[\hbar]]$ the \hbar -adic completion of $V \otimes \mathbb{C}[[\hbar]]$. It is easy to see that there exists an isomorphism of $\mathbb{C}[[\hbar]]$ -algebras $U_{\hbar}(\hat{\mathfrak{g}}) \rightarrow U(\hat{\mathfrak{g}})[[\hbar]]$. Therefore we can work with $U(\mathfrak{g})[[\hbar]]^*$.

Let \mathbf{G} be the real Heisenberg group; as a variety, $\mathbf{G} = S^1 \times \mathbb{R}^{2g}$; the universal covering of \mathbf{G} is of course $\tilde{\mathbf{G}}(\mathbb{R})$. Let $(U^{(m)}, \mathcal{H}^{(m)})$ be the Stone-von Neumann representation of the Heisenberg group \mathbf{G} of weight m ; it is the unique (up to isomorphism) unitary irreducible representation of \mathbf{G} such that $U_{\lambda}^{(m)} = \lambda^m \text{id}$ for $\lambda \in S^1$. Moreover, if (V, \mathcal{H}_1) is any unitary representation of \mathbf{G} such that $V_{\lambda} = \lambda^m \text{id}$ for all $\lambda \in S^1$, then \mathcal{H}_1 is isomorphic, as an unitary \mathbf{G} -module, to $\mathcal{H}^{(m)} \hat{\otimes} \mathcal{H}_0$, where \mathcal{H}_0 is some Hilbert space acted upon trivially by \mathbf{G} . We denote in particular \mathcal{H} instead of $\mathcal{H}^{(1)}$. We shall apply the preceding to the (diagonal) representation of \mathbf{G} on $\mathcal{H} \hat{\otimes} \dots \hat{\otimes} \mathcal{H}$, or more generally, to the tensor product $\mathcal{H}^{(m)} \hat{\otimes} \mathcal{H}^{(p)}$, m, p positive integers.

Let \mathcal{H}_{∞} (resp., $\mathcal{H}_{-\infty}$) be the Harish-Chandra submodule of \mathcal{H} (resp., its continuous dual, cf. [Mu, pp. 21 and 24]): \mathcal{H}_{∞} is a dense subspace of \mathcal{H} which carries a representation of \mathfrak{g} . Let $v, w \in \mathcal{H}_{\infty}$, $\ell, h \in \mathcal{H}_{-\infty}$. Let $\tilde{\phi}_{\ell, v} : \mathbf{G} \rightarrow \mathbb{C}$ denote the matrix coefficient $\tilde{\phi}_{\ell, v}(x) = \langle \ell, xv \rangle$. We will be mainly concerned with $\phi_{\ell, v} = \tilde{\phi}_{\ell, v} \circ s : \mathbb{R}^{2g} \rightarrow \mathbb{C}$, where s is the continuous section $s : \mathbb{R}^{2g} \rightarrow \mathbf{G}$, $s(x) = (1, x)$. In the same vein, consider also the restriction $\phi_{\ell \otimes h, v \otimes w}$ of a matrix coefficient of \mathbf{G} ; it is clear that in the algebra of functions from \mathbb{R}^{2g} to \mathbb{C} the following equality holds:

$$(1) \quad \phi_{\ell, v} \phi_{h, w} = \phi_{\ell \otimes h, v \otimes w}.$$

Alternatively, we can consider $\tilde{\phi}_{\ell, v}$ as an element of $U(\mathfrak{g})^*$, and correspondingly, the equality (1) still holds in this space, which is an algebra via the comultiplication of $U(\mathfrak{g})$.

It follows from these remarks that the span of the matrix coefficients $\tilde{\phi}_{\ell, v}$, for all $v \in \mathcal{H}_{\infty}^{(m)}$, $\ell \in \mathcal{H}_{-\infty}^{(m)}$, $m \in \mathbb{N}$, is an associative algebra (without unity). What happens when replacing the usual comultiplication of $U(\mathfrak{g})$ by that pushed forward from $U_{\hbar}(\hat{\mathfrak{g}})$ via the naive isomorphism alluded above? The main obstacle is that the elements of the center of \mathfrak{g} are no more primitive. It follows from [Dr2] that $U(\hat{\mathfrak{g}})[[\hbar]]$ is isomorphic (as a Hopf algebra with the new comultiplication), up to a "gauge" transformation by an element $F \in U(\hat{\mathfrak{g}})[[\hbar]] \hat{\otimes} U(\hat{\mathfrak{g}})[[\hbar]]$, to the "standard" quantization of the pair (\mathfrak{g}, t) ; here t is the invariant symmetric 2-tensor which arises as the classical limit of $U(\hat{\mathfrak{g}})[[\hbar]]$. The explicit isomorphism and F seem to be difficult to compute; for this reason we introduce in the next sections new quantized enveloping algebras of $\hat{\mathfrak{g}}$.

§4. We define in this section a new quantization of $U(\mathfrak{g})$ and motivate the introduction of further quantizations in subsequent sections.

Let $\Delta_I : U(\hat{\mathfrak{g}})[[\hbar]] \rightarrow U(\hat{\mathfrak{g}})[[\hbar]] \otimes U(\hat{\mathfrak{g}})[[\hbar]]$ be the application defined by

$$\begin{aligned} \Delta_I(x_i) &= x_i \otimes \exp\left(\frac{\hbar}{2}z\right) + \exp\left(-\frac{\hbar}{2}z\right) \otimes x_i, \\ \Delta_I(y_i) &= y_i \otimes \exp\left(-\frac{\hbar}{2}z\right) + \exp\left(\frac{\hbar}{2}z\right) \otimes y_i, \\ \Delta_I(z) &= z \otimes 1 + 1 \otimes z, \quad \Delta_I(d) = d \otimes 1 + 1 \otimes d. \end{aligned}$$

Lemma 1. Δ_I is well defined and provides $U(\hat{\mathfrak{g}})[[\hbar]]$ a Hopf algebra structure, together with the antipode S_I and the counit ε_I . Its classical limit is the Lie bialgebra structure on $\hat{\mathfrak{g}}$ given by

$$(2) \quad \delta(x_i) = x_i \wedge z, \quad \delta(y_i) = -y_i \wedge z, \quad \delta(z) = \delta(d) = 0.$$

Proof. It is straightforward.

Recall now a well-known realization of the Stone-von Neumann representation, and of its Harish-Chandra module \mathcal{H}_{∞} . Let v be the representation of $\hat{\mathfrak{g}}$ on the Schwarz algebra on \mathbb{R}^g (denoted $\mathcal{S}(\mathbb{R}^g)$) given by

$$v(x_i)f = \frac{\partial f}{\partial t_i}, \quad v(y_i)f = t_i f, \quad v(z)f = f, \quad v(d)f = - \sum_{1 \leq i \leq g} t_i \frac{\partial f}{\partial t_i}.$$

Then $\mathcal{S}(\mathbb{R}^g)$ can be identified with $(\mathcal{H})_{\infty}$. This v is the "derivative" of the representation U of \mathbf{G} on $L^2(\mathbb{R}^g)$ given by

$$U_{\lambda, y} F(x) = \lambda \exp(i\pi(2x \cdot y_2 + y_1 \cdot y_2)) F(x + y_1).$$

$(L^2(\mathbb{R}^g), U)$ is isomorphic to the Stone-von Neumann representation of \mathbf{G} . More generally, the Stone-von Neumann representation of weight m is $(L^2(\mathbb{R}^g), U^{(m)})$, where $U^{(m)}$ is

$$U_{\lambda, y}^{(m)} F(x) = \lambda^m \exp(i\pi m(2x \cdot y_2 + y_1 \cdot y_2)) F(x + y_1).$$

The derivative of $U^{(m)}$ is the representation $v^{(m)}$ of \mathfrak{g} on $\mathcal{S}(\mathbb{R}^g)$ given by

$$\begin{aligned} v^{(m)}(x_i)f &= \frac{\partial f}{\partial t_i}, & v^{(m)}(y_i)f &= m t_i f, \\ v^{(m)}(z)f &= m f, & v^{(m)}(d)f &= - \sum_{1 \leq i \leq g} t_i \frac{\partial f}{\partial t_i}. \end{aligned}$$

We shall name for brevity $\mathcal{H}_{\infty}^{(m)}$ instead of $(\mathcal{S}(\mathbb{R}^g), U^{(m)})$.

Let us now identify $(\mathcal{H})_{\infty} \otimes (\mathcal{H})_{\infty}$ with an algebra of C^{∞} -functions on \mathbb{R}^{2g} . According with our identification $\mathbb{R}^{2g} \simeq \mathbb{R}^g \times \mathbb{R}^g$, we use the variables in \mathbb{R}^{2g} $u_1, \dots, u_g, v_1, \dots, v_g$. Then the diagonal action of \mathfrak{g} is given by

$$\begin{aligned} v_0(x_i)f &= \left(\frac{\partial}{\partial u_i} + \frac{\partial}{\partial v_i}\right)f, & v_0(y_i)f &= (u_i + v_i)f, \\ v_0(z)f &= 2f, & v_0(d)f &= - \sum_{1 \leq i \leq g} (u_i \frac{\partial}{\partial u_i} + v_i \frac{\partial}{\partial v_i})f. \end{aligned}$$

The new coproduct allows us to define "twisted" tensor product of representations. In particular, we use Δ_I to define, for a fixed value of \hbar , a new representation ν_I of $\hat{\mathfrak{g}}$ on $\mathcal{H}_\infty \otimes \mathcal{H}_\infty$. If $q = \exp(\frac{\hbar}{2})$, then

$$\begin{aligned} \nu_I(x_i)f &= (q \frac{\partial}{\partial u_i} + q^{-1} \frac{\partial}{\partial v_i})f, & \nu_I(y_i)f &= (q^{-1}u_i + qv_i)f, \\ \nu_I(z)f &= 2f, & \nu_I(d)f &= df. \end{aligned}$$

Notice that there exists a linear isomorphism $\mathcal{G} : \mathcal{H}_\infty \otimes \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty \otimes \mathcal{H}_\infty$ such that

$$(3) \quad \mathcal{G}(\nu_0(x)f) = \nu_I(x)\mathcal{G}(f),$$

for all $x \in \hat{\mathfrak{g}}$, $f \in \mathcal{S}(\mathbb{R}^g) \otimes \mathcal{S}(\mathbb{R}^g)$. Indeed,

$$\mathcal{G}(f)(u, v) = f(q^{-1}u, qv).$$

Note now that \mathcal{G} is in fact well defined as an application $\mathcal{G} : L^2(\mathbb{R}^{2g}) \rightarrow L^2(\mathbb{R}^{2g})$ and hence allows us to define an action V of the Heisenberg group \mathbb{G} on $L^2(\mathbb{R}^{2g})$ such that (3) still holds. Explicitly,

$$(V_{\lambda, y}f)(u, v) = \lambda^2 \exp(2i\pi(q^{-1}u + qv + y_1)y_2)f(u + qy_1, v + q^{-1}y_1).$$

More generally, Δ_I allows us to define new representations $\nu_I^{(m,p)}$ of $\hat{\mathfrak{g}}$ on $\mathcal{H}_\infty^{(m)} \otimes \mathcal{H}_\infty^{(p)}$:

$$\begin{aligned} \nu_I^{(m,p)}(x_i)f &= (q^p \frac{\partial}{\partial u_i} + q^{-m} \frac{\partial}{\partial v_i})f, \\ \nu_I^{(m,p)}(y_i)f &= (q^{-p}u_i + q^m v_i)f, \\ \nu_I^{(m,p)}(z)f &= (m+p)f \end{aligned}$$

Again, we have an intertwining operator $\mathcal{G}^{(m,p)}$ between $\nu_I^{(m,p)}$ and $\nu_0^{(m)} \otimes \nu_0^{(p)}$:

$$(4) \quad \mathcal{G}^{(m,p)}(f)(u, v) = f(q^{-p}u, q^m v).$$

Obviously, $\mathcal{G}^{(m,p)}$ extends to an application $L^2(\mathbb{R}^{2g}) \rightarrow L^2(\mathbb{R}^{2g})$ (named identically) and hence we have a representation $V^{(m,p)}$ of the Heisenberg group \mathbb{G} on $L^2(\mathbb{R}^{2g})$ by

$$\begin{aligned} (V_{\lambda, y}^{(m,p)}f)(u, v) \\ = \lambda^{m+p} \exp(i\pi(2mq^{-p}u + 2pq^m v + (m+p)y_1)y_2)f(u + q^p y_1, v + q^{-m} y_1). \end{aligned}$$

Let now $\ell \in \mathcal{H}_{-\infty}^{(m)}$, $h \in \mathcal{H}_{-\infty}^{(p)}$, $f \in \mathcal{H}_\infty^{(m)}$, $g \in \mathcal{H}_\infty^{(p)}$. Then Δ_I provides us a product of (restriction of) matrix coefficients:

$$\begin{aligned} (5) \quad \phi_{\ell, f, g} \phi_{h, g}(y) &= (\ell \otimes h, V_{1, y}^{(m,p)} f \otimes g) = \\ &= (\ell, U_{1, (q^p y_1, q^{-p} y_2)}^{(m)} f) (h, U_{1, (q^{-m} y_1, q^m y_2)}^{(p)} g) = \phi_{\ell, f}(q^p y_1, q^{-p} y_2) \cdot \phi_{h, g}(q^{-m} y_1, q^m y_2). \end{aligned}$$

Notice that the analogue of (5) for $\tilde{\phi}$ is a new multiplication on a ring of matrix coefficients of unitary representations of \mathbb{G} of positive weight, that is, an algebra of functions on \mathbb{G} .

§5. Algebraically, the multiplication (5) is a special case of the following fact. Let Γ be an abelian group, $A = \bigoplus_{m \in \Gamma} A_m$ a Γ -graded algebra, and $\tau, \sigma : \Gamma \rightarrow \text{Aut } A$ be two representations of Γ by automorphisms of Γ -graded algebras; suppose in addition that $\tau_r \sigma_p = \sigma_p \tau_r$, for any $r, p \in \Gamma$. We introduce an associative multiplication \circ on A which still satisfies $A_m \circ A_p \subseteq A_{m+p}$, by the rule

$$(6) \quad F \circ G = \tau_p(F)\sigma_m(G),$$

$F \in A_m, G \in A_p$. It is easy to see that \circ is associative and whit unit $1 \in A_0$. This bi-twisted product is a generalization of the twisted product defined in [ATV]. There, they consider $\Gamma = \mathbb{Z}$, $\tau_p = id$ and $\sigma_r = \sigma^r$, where σ is a \mathbb{Z} -graded automorphism of a \mathbb{Z} -graded algebra A . We denote this new algebra structure by A_σ .

Here is another example: let again $\Gamma = \mathbb{Z}$, A a \mathbb{Z} -graded algebra and τ a graded automorphism of A . Let $\tau_p = \tau^p$ and $\sigma_m = \tau^{-m}$. The resulting algebra is denoted by A^τ . This is a generalization of (5).

Proposition 1. *If $\sigma = \tau^{-2}$, then $A_\sigma \cong A^\tau$ via the isomorphism: $\phi : A_\sigma \rightarrow A^\tau$, defined by $\phi(a) = \tau^m(a)$ for $a \in A_m$.*

The proof is straightforward.

Remark 1. Let X be a non-empty set, $M : X \rightarrow X$ a bijection, \mathbf{k} a commutative ring. Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$, where A_n is a copy of the algebra of functions from X to \mathbf{k} . Let $\tau_1(F)(x) = F(Mx)$, $\tau_p = \tau_1^p$ and $\sigma_p = \tau_p^{-1}$. If $F \in A_m, G \in A_p$ we have

$$(7) \quad (F \circ G)(x) = F(M^p x)G(M^{-m} x), \quad x \in X.$$

This algebra is commutative if and only if $M^2 = id$. This construction extends obviously to any subalgebra of the algebra of functions on X with values in \mathbf{k} , stable by the transpose of M .

§6. The preceding suggests the following construction. We begin by reversing the reasoning used to obtain (5). We obtain, for each pair of positive integers m and p , a representation of \mathbb{G} on $L^2(\mathbb{R}^g)$ by the formula $U_{(\lambda, M^p y)}^{(m)} \otimes U_{(\lambda, M^{-m} y)}^{(p)}$ and we seek for an intertwining operator between it and the usual tensor product representation on $\mathcal{H}_\infty^{(m)} \otimes \mathcal{H}_\infty^{(p)}$. That is, we are looking for an operator \mathcal{G} making commutative the following diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}^{2g}) & \xrightarrow{\mathcal{G}} & L^2(\mathbb{R}^{2g}) \\ U_{(\lambda, y)}^{(m)} \otimes U_{(\lambda, y)}^{(p)} \downarrow & & \downarrow U_{(\lambda, M^p y)}^{(m)} \otimes U_{(\lambda, M^{-m} y)}^{(p)} \\ L^2(\mathbb{R}^{2g}) & \xrightarrow{\mathcal{G}} & L^2(\mathbb{R}^{2g}). \end{array}$$

Here is a solution: take $U \in GL(\mathbb{R}^g)$ and set $M(y) = (Uy_1, U^{-1}y_2)$, $\mathcal{G}(f)(u, v) = f(U^{-p}u, U^m v)$. Now we conjugate by \mathcal{G} the representation of the Heisenberg Lie algebra on $\mathcal{H}_\infty^{(m)} \otimes \mathcal{H}_\infty^{(p)}$ and obtain the following formulas as the derivative of the above representation:

$$\begin{aligned}
(8) \quad & x_i \text{ acts as } \sum_j \left\{ (U^p)_{ji} \frac{\partial}{\partial u_j} + (U^{-m})_{ji} \frac{\partial}{\partial v_j} \right\} \\
& y_i \text{ acts as multiplication by } \sum_j \{ m(U^{-p})_{ij} u_j + p(U^m)_{ij} v_j \} \\
& z \text{ acts as multiplication by } m + p.
\end{aligned}$$

We introduce now a new comultiplication in $U(\mathfrak{g})[[\hbar]]$ which explains the preceding representations, but we want first to make explicit some straightforward notation. Let B be a matrix in $M_g(\mathbb{C})$ and consider the matrix $\hbar z B \in M_g(\mathbb{C}[[\hbar]][z])$ and consequently the element $\exp(\hbar z B)_{ij}$ of $U(\mathfrak{g})[[\hbar]]$.

Let $B \in M_g(\mathbb{C})$. We define first a Lie bialgebra (\mathfrak{g}_B, δ) , which is an extension of \mathfrak{g} with the additional property that $\delta(\mathfrak{g}) \subset \mathfrak{g} \otimes \mathfrak{g}$, thus (\mathfrak{g}, δ) is a sub-bialgebra of (\mathfrak{g}_B, δ) .

Definition. (\mathfrak{g}_B, δ) is the Lie bialgebra generated by x_i, y_i, z, d ($i = 1, \dots, n$), with the following relations:

$$[x_i, y_j] = \delta_{ij} z, \quad [d, x_i] = x_i, \quad [d, y_i] = -y_i,$$

$[x_i, x_j] = [y_i, y_j] = 0$ and z is central. The structure of Lie coalgebra is given by

$$(9) \quad \delta(x_i) = \sum_j B_{ji} x_j \wedge z, \quad \delta(y_i) = -\sum_j B_{ij} y_j \wedge z, \quad \delta(z) = \delta(d) = 0.$$

Remark 2. $\mathfrak{g}_B = \hat{\mathfrak{g}}$ as Lie algebra.

Now we show a quantization of (\mathfrak{g}_B, δ) .

Lemma 2. Let $B \in M_g(\mathbb{C})$ and let $\Delta_B : U(\mathfrak{g}_B)[[\hbar]] \rightarrow U(\mathfrak{g}_B)[[\hbar]] \otimes U(\mathfrak{g}_B)[[\hbar]]$ be defined by

$$\begin{aligned}
(10) \quad \Delta_B(x_i) &= \sum_j \{ x_j \otimes \exp(\frac{\hbar}{2} z B)_{ji} + \exp(-\frac{\hbar}{2} z B)_{ji} \otimes x_j \}, \\
\Delta_B(y_i) &= \sum_j \{ y_j \otimes \exp(-\frac{\hbar}{2} z B)_{ij} + \exp(\frac{\hbar}{2} z B)_{ij} \otimes y_j \}, \\
\Delta_B(z) &= z \otimes 1 + 1 \otimes z, \quad \Delta_B(d) = d \otimes 1 + 1 \otimes d.
\end{aligned}$$

It is well defined and provides $U(\mathfrak{g}_B)[[\hbar]]$ a Hopf algebra structure, together with the antipode S and the counit ε defined by $S(u) = -u$ and $\varepsilon(u) = 0$ for all $u \in \mathfrak{g}_B$. Its classical limit is the Lie bialgebra structure on \mathfrak{g}_B given by (9).

Proof. We denote $U := \exp(\frac{\hbar}{2} z B)$ and $\Delta := \Delta_B$. We prove the well-definiteness of Δ :

$$\Delta[d, x_i] = \Delta(x_i) = \sum_j \{ [d, x_j] \otimes U_{ji} + U_{ji}^{-1} \otimes [d, x_j] \} = [\Delta(d), \Delta(x_i)].$$

Furthermore

$$\begin{aligned}
[\Delta(x_i), \Delta(y_j)] &= \sum_{k,r} \{ [x_k, y_r] \otimes U_{ki} U_{jr}^{-1} + U_{ki}^{-1} U_{jr} \otimes [x_k, y_r] \} \\
&= z \otimes \left(\sum_k U_{ki} U_{jk}^{-1} \right) + \left(\sum_k U_{ki}^{-1} U_{jk} \right) \otimes z = z \otimes \delta_{ij} + \delta_{ij} \otimes z = \Delta[x_i, y_j].
\end{aligned}$$

It is possible to verify the other relations in the same way. We now prove the co-associativity. It relies in the following elementary remark: if B_1, B_2 commute, $a_1, a_2 \in \mathbb{C}[[\hbar]]$, then $\exp(a_1(z \otimes 1)B_1 + a_2(1 \otimes z)B_2)_{ij} = \sum_k \exp(a_1 z B_1)_{ik} \otimes \exp(a_2 z B_2)_{kj}$.

$$(\Delta \otimes 1)\Delta(x_i) = \sum_k \left\{ \sum_j x_k \otimes U_{ks} \otimes U_{si} + \sum_j U_{ks}^{-1} \otimes x_k \otimes U_{si} + \sum_j U_{ks}^{-1} \otimes U_{si}^{-1} \otimes x_k \right\};$$

and

$$(1 \otimes \Delta)\Delta(x_i) = \sum_k \left\{ \sum_j x_k \otimes U_{ks} \otimes U_{si} + \sum_j U_{si}^{-1} \otimes x_k \otimes U_{ks} + \sum_j U_{si}^{-1} \otimes U_{ks}^{-1} \otimes x_k \right\}.$$

Then we must prove that

$$(11) \quad \sum_j U_{ks}^{-1} \otimes U_{si} = \sum_j U_{si}^{-1} \otimes U_{ks}$$

and

$$(12) \quad \sum_j U_{ks}^{-1} \otimes U_{si}^{-1} = \sum_j U_{si}^{-1} \otimes U_{ks}^{-1}.$$

Now,

$$\begin{aligned}
\sum_j U_{si}^{-1} \otimes U_{ks} &= \sum_j (U^t)_{is}^{-1} \otimes U_{sk}^t = \exp(\frac{\hbar}{2} (-z \otimes 1) B^t + \frac{\hbar}{2} (1 \otimes z) B^t)_{ik} \\
&= \exp(\frac{\hbar}{2} (-z \otimes 1) B + \frac{\hbar}{2} (1 \otimes z) B)_{ki} = \sum_j U_{ks}^{-1} \otimes U_{si},
\end{aligned}$$

i.e. formula (11) holds. In analogous way we get (12).

So, we have:

Proposition 2. Let $B \in M_g(\mathbb{C})$. Consider the Hopf algebra $(U(\mathfrak{g}_B)[[\hbar]], \Delta_B)$ with a fixed value of \hbar and denote $U := \exp(\frac{\hbar}{2} B)$. Then the tensor product (via Δ_B) of $v^{(m)}$ and $v^{(p)}$ is given by the formulas (8) and is denoted by $v_B^{(m,p)}$.

Remark 3. By the same reasoning as above, there exists an intertwining operator between $v_B^{(m,p)}$ and $v_0^{(m,p)}$.

§7. We defined in §6 a family of new coproducts in \mathfrak{g} (parametrized by $B \in M_g(\mathbb{C})$) providing new products in the algebra of matrix coefficients of unitary representations of positive weight. In this section, we will see that some of these products provide a new multiplication on the ring of theta functions.

Let $T \in \mathbb{C}^{\theta \times \theta}$ be symmetric and such that $\text{Im } T$ is positive definite; i.e. T belongs to the Siegel space \mathfrak{H}_θ . Let $c : \mathbb{C}^\theta \simeq \mathbb{R}^{2\theta}$ be the isomorphism provided by T . The complex structure on $\mathbb{R}^{2\theta}$ provided by T is

$$J(x_1, x_2) = ((\text{Im } T)^{-1}(\text{Re } T x_1 + x_2), -\text{Im } T x_1 - \text{Re } T(\text{Im } T)^{-1}(\text{Re } T x_1 + x_2))$$

and the isomorphism is given by $\underline{x} = Tx_1 + x_2 \leftrightarrow (x_1, x_2)$. Let Γ_m be the space of holomorphic functions on \mathbb{C}^g such that

$$(\theta_m) \quad f(z) = \exp(i\pi m(-n_1, n_2 + n_1(2z + \underline{n}))) f(z + \underline{n}).$$

(Observe that the product of functions satisfying (θ_m) and (θ_p) respectively satisfies (θ_{m+p}) .)

Given a function f satisfying (θ_m) , we want to find some $M : \mathbb{C}^g \rightarrow \mathbb{C}^g$ such that fM^p satisfies (θ_m) again ($p \in \mathbb{Z}$). Thus we can apply (7) to define the bi-twisted product. An answer is the following. Let $U \in SO_T(\mathbb{Z}) := O(T) \cap SL(g, \mathbb{Z})$ (here $O(T)$ means the group of all $U \in GL(\mathbb{C}^g)$ such that ${}^tUTU = T$). If $T = \text{Im } T$, the group $SO_T(\mathbb{Z})$, being compact and discrete, is finite. Let M be the translation by \mathbf{c} of the application $\mathbb{R}^{2g} \rightarrow \mathbb{R}^{2g}$, $x \mapsto (Ux_1, {}^tU^{-1}x_2)$, i.e. $M : Tx_1 + x_2 \mapsto TUx_1 + {}^tU^{-1}x_2$.

We claim that this M does the job. Indeed, as $U \in SL(g, \mathbb{Z})$,

$$fM^p(z + \underline{n}) = f(M^p z + M^p \underline{n}) = \exp(i\pi \{mU^p n_1, {}^tU^{-p} n_2 - mU^p n_1, (2M^p z + M^p \underline{n})\}) fM^p(z).$$

We are hence restricted to show $mU^p n_1, (2M^p z + M^p \underline{n}) \stackrel{?}{=} mn_1, (2z + \underline{n})$; but this follows from the requirement $U \in O(T)$.

Remark 4. By Proposition 1, we conclude that we obtain in this way twisted algebras in the sense of [ATV] (also called Sklyanin algebras [Sk]), via a quantum multiplication on matrix coefficients of Stone-von Neumann representations with positive weight.

Example 1. Let $T = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, then $SO_T(\mathbb{Z}) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$. So we have two non-isomorphic rings of theta functions, the classical one and the following: if f satisfies (θ_m) and g satisfies (θ_p) , then $f \circ g = (-1)^m fg$ (this product is obtained via the isomorphism of Proposition 1).

§8. Besides the quantum deformations of the Heisenberg group considered above, there is another one that has been recently object of attention: the quantum Heisenberg algebra \mathbb{H}_q . This is the $\mathbb{C}[q, q^{-1}]$ -algebra generated by elements X, Y, Z with relations

$$XY - qYX = Z, \quad XZ = ZX, \quad YZ = ZY.$$

There is no evident way to define a Hopf algebra structure on \mathbb{H}_q . We shall show that however the situation is different if one considers a natural twisted algebra structure in the tensor product.

Let $(A, m, 1)$ be an algebra. Let $S : A \otimes A \rightarrow A \otimes A$ be an invertible linear transformation such that

- (i) $S^{12} S^{23} S^{12} = S^{23} S^{12} S^{23}$, i.e. S is a solution of the braided relation (sometimes called the quantum Yang-Baxter equation).
- (ii) $S(b \otimes 1) = 1 \otimes b$ and $S(1 \otimes b) = b \otimes 1$.
- (iii) $S(m \otimes id) = (id \otimes m) S^{12} S^{23}$ and $S(id \otimes m) = (m \otimes id) S^{23} S^{12}$.
- (iv) $S^2 = 1$

The pair (A, S) is called a *braided algebra*. Suppose that in addition

$$(v) \quad mS = m.$$

Then the pair (A, S) is called a *S-commutative* or *generalized commutative* or *braided commutative algebra*. We prefer this last term. See [GRR], [Ma], [Mn].

The following example was found by Demidov (see [Mn]) and is known as the "quantum plane" (notice that related skew-fields are known to algebraists since long time ago). Let \mathbb{P}_q be the quotient of the tensor algebra $T(V)$ (where V is a 2-dimensional vector space with basis X, Y) by the 2-sided ideal generated by $X \otimes Y - qY \otimes X$. Now let I be the ideal of $\mathbb{P}_q \otimes \mathbb{P}_q$ generated by $X \otimes Y - qY \otimes X$. Let $S : \mathbb{P}_q \otimes \mathbb{P}_q \rightarrow \mathbb{P}_q \otimes \mathbb{P}_q$ be the unique linear application such that

$$X \otimes X \mapsto X \otimes X, \quad X \otimes Y \mapsto qY \otimes X, \quad Y \otimes X \mapsto q^{-1}X \otimes Y, \quad Y \otimes Y \mapsto Y \otimes Y,$$

and S preserves I . S is a linear isomorphism; moreover (\mathbb{P}_q, S) is easily seen to be braided commutative. Observe that $S(X^n Y^p \otimes X^m Y^r) = q^{nr-pm} (X^m Y^r \otimes X^n Y^p)$. (It is known that \mathbb{P}_q generalizes to higher dimensional q -affine spaces; the definition of S and the following results are also valid for them).

The following Lemma was communicated to the first author by P. Cartier (see also [GRR], [Ma], [Mn]).

Lemma 3. *If (A, S) is a braided algebra then $A \otimes A$, with the product:*

$$(a \otimes b) * (c \otimes d) = (m \otimes m)(a \otimes S(b \otimes c) \otimes d)$$

is an associative algebra with unit $1 \otimes 1$. It will be denoted by $A \underline{\otimes} A$.

Proof. Let $a, b, c, d, e, f \in A$ and denote $d_i \otimes e^i = S(d \otimes e)$; then

$$\begin{aligned} (a \otimes b) * ((c \otimes d) * (e \otimes f)) &= (m \otimes m)(a \otimes S(b \otimes cd_i) \otimes e^i f) = \\ &= (m \otimes m)(id \otimes m \otimes id \otimes id)(a \otimes S^{23} S^{12}(b \otimes c \otimes d_i) \otimes e^i f) = \\ &= (m \otimes m)(id \otimes m \otimes id \otimes m)(a \otimes S^{23} S^{12} S^{34}(b \otimes c \otimes d \otimes e) \otimes f). \end{aligned}$$

In analogous way we can show that:

$$((a \otimes b) * (c \otimes d)) * (e \otimes f) = (m \otimes m)(m \otimes id \otimes m \otimes id)(a \otimes S^{23} S^{34} S^{12}(b \otimes c \otimes d \otimes e) \otimes f).$$

Using the associativity of m is enough to prove that:

$$S^{23} S^{12} S^{34} = S^{23} S^{34} S^{12},$$

and this is true because $S^{12} S^{34} = S^{34} S^{12}$.

Notice that $\mathbb{P}_q \underline{\otimes} \mathbb{P}_q$ is isomorphic to the 4-dimensional q -affine space.

Lemma 4. *There exists a unique algebra homomorphism $\Delta : \mathbb{P}_q \rightarrow \mathbb{P}_q \underline{\otimes} \mathbb{P}_q$ such that*

$$(13) \quad \Delta(X) = X \underline{\otimes} 1 + 1 \underline{\otimes} X, \quad \Delta(Y) = Y \underline{\otimes} 1 + 1 \underline{\otimes} Y.$$

Moreover $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$. That is, \mathbb{P}_q is a braided bialgebra (the counit ϵ is defined by $\epsilon(X) = \epsilon(Y) = 0$).

Now we pass to \mathbb{H}_q . Let $\tilde{S} : \mathbb{H}_q \otimes \mathbb{H}_q \rightarrow \mathbb{H}_q \otimes \mathbb{H}_q$ be defined by

$$(14) \quad \tilde{S} : X^n Y^p Z^t \otimes X^m Y^r Z^v \mapsto q^{nr-mp} X^m Y^r Z^v \otimes X^n Y^p Z^t.$$

Proposition 3. (i) $(\mathbb{H}_q, \tilde{S})$ is a braided algebra and $\mathbb{H}_q \rightarrow \mathbb{F}_q, Z \rightarrow 0$ is a morphism of braided algebras.

(ii) Let $\Delta : \mathbb{H}_q \rightarrow \mathbb{H}_q \otimes \mathbb{H}_q$ be defined by (13) and $\Delta(Z) = 1 \otimes Z + Z \otimes 1$. Then (\mathbb{H}_q, Δ) is a braided Hopf algebra (the counit takes the 0 value in X, Y, Z). Let \mathbb{H}_q^{op} be \mathbb{H}_q with the multiplication $m\tilde{S}$, when m is the multiplication. Then there exists a unique morphism of algebras $\mathcal{S} : \mathbb{H}_q \rightarrow \mathbb{H}_q^{\text{op}}$ such that $\mathcal{S}(X) = -X, \mathcal{S}(Y) = -Y$ and $\mathcal{S}(Z) = -Z$. It is easy to verify that \mathcal{S} is the antipode of \mathbb{H}_q .

One has thus an exact sequence of braided Hopf algebras

$$0 \rightarrow k[Z] \rightarrow \mathbb{H}_q \rightarrow \mathbb{F}_q \rightarrow 0.$$

ACKNOWLEDGMENTS

The work of the first author was supported by the Alexander von Humboldt-Stiftung and was done at the Max Planck Institut (Bonn). We also want to thank P. Slodowy for his interest on this work; the third author acknowledges the invitation of P. Slodowy to Bonn in November 92, when the seminal ideas of this paper were found at. (J.D.) and (A.T.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency, UNESCO and the International Centre for Theoretical Physics, Trieste, for support.

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