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## Abstract

*A microscopically derived Langevin equation is applied to thermally induced nuclear fission. An important memory effect is pointed out and discussed. A strong friction coefficient, estimated from microscopic quantities, tends to decrease the stationary limit of the fission rate and to increase the transient time. The calculations are performed with a collective mass depending on the collective variable and with a constant mass. Fission rates calculated at different temperatures are shown and compared with previous available results.*

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# 1 Introduction

At moderate excitation energies ($E^*/A$=1-2MeV), there are mainly two ways of deexcitation of hot nuclei, which are competiting, neutron evaporation and fission. A dynamical approach is necessary to determine the most rapid channel the nucleus will follow. We will focus our work on the fission process. More than fifty years ago, H. A. Kramers, in a cornerstone paper [1], proposed to describe induced nuclear fission and other physical phenomenas, such as chemical reactions, as the diffusion over a barrier of a collective variable and to use a Langevin Equation (LE) for the calculations. The nucleus is originally caught in a potential well and escape in the course of time by passing over the fission barrier. The aim is to calculate the probability of escape as a function of time. Kramers studied analytically the stationary flow over the barrier with a Fokker-Planck Equation (FPE), equivalent to the LE. He derived an expression for the stationary fission rate (or reaction rate in chemistry) as a function of the temperature, the friction coefficient and the shape of the potential. In the case of nuclear fission, more complete studies of this diffusion process have been performed with a FPE [2] and with a LE [3]. Both groups pointed out a transient time which is very important to understand the competition between fission and neutron evaporation. These approaches are phenomenological in the sense that the parameters entering the equations have been adjusted in order to reproduce experimental data. A friction coefficient has been extracted from this comparison and hence a value of the nuclear viscosity.

A Langevin equation for a collective variable has been derived in reference [4] from a microscopic model, the Boltzmann-Langevin-Equation (BLE) and all the parameters previously fitted can now be explicitly calculated. Our aim is to apply this new LE to the fission problem. It is a good case to test the incorporation of dynamical fluctuations into a one-body transport model. The equation contains two-body dissipation mechanism and fluctuations consistent with the "fluctuation-dissipation theorem". The one-body dissipation is not taken into account. The present study will allow us to give a firm ground to the phenomelogical approaches of such phenomena [2,3]. We have to deal with a larger friction coefficient than in references [2,3], which justifies an overdamped limit approximation. Our approach is more sophisticated and realistic because we have kept a mass depending on the collective variable and a second order term consistent with this dependency. Eventually we have some results not far from the ones of reference [2]. Therefore, our study is an a posteriori microscopic justification of part of these early works. Looking more carefully, the larger effective friction decreases the stationary fission rate and increases the transient time which allows more pre-scission neutrons to evaporate.

The paper is organized as follows. In a first part, we briefly recall the derivation of the Langevin equation for a collective variable from the microscopic BLE. In a second part we apply the formalism to fission. After having tested our calculations by comparison with previous results in a third part, some new results are shown in a fourth part. All the results are finally summed up.

## 2 Microscopic derivation of a Langevin equation

The usual Markovian Langevin equation used in the phenomenological studies [3] is, for a collective variable q

$$\ddot{q} = -\frac{\partial V}{\partial q} - \beta M \dot{q} + \sqrt{M\beta T}\, w(t) \qquad (2.1)$$

where w(t) is the stochastic force characterized by its first two moments:

$$\langle w(t) \rangle = 0 \text{ and } \langle w(t)w(t') \rangle = 2\,\delta(t-t') \qquad (2.2)$$

In equation (2.1) M is the collective mass, $\beta$ the reduced friction coefficient, T the nuclear temperature and V(q) the potential. The width of the stochastic force, $\sqrt{M\beta T}$, is given by the "fluctuation-dissipation theorem". If this force is assumed gaussian, the LE is equivalent to a FPE involving q and its conjugate momentum p [3,5]

$$\left(\frac{\partial}{\partial t} + \frac{p}{M}\frac{\partial}{\partial q} - \frac{\partial V}{\partial q}\frac{\partial}{\partial p}\right)P = \frac{\partial}{\partial p}\left(-\beta p + D\frac{\partial}{\partial p}\right)P \qquad (2.3)$$

This latter equation determines the time evolution of the probability distribution P(q,p,t) in terms of the diffusion coefficient, D=M$\beta$T, and the friction. The second order derivative in the r.h.s. of the equation (2.3) is directly linked to the moments of the stochastic force (2.2). A more complicated force than the gaussian one, would a priori involve higher partial derivative in the equivalent FPE.

The LE was originally used to describe brownian motion and one might question the relevance of a diffusion description of fission. Such an application is based on the two different time scales involved in the process, the slow collective elongation with a heavy mass which can be seen as the brownian particle and the fast nucleonic degrees of freedom which represent a heat bath. In many cases the LE is equivalent to a FPE [5], but the first one is far more general and more intuitive. In a LE one studies trajectories, while in a FPE picture one studies the time evolution of a phase space distribution. The first approach is a differential equation with a stochastic component while the second one is a second order partial differential equation which is less tractable in general. With non vanishing higher even order moments of the stochastic force we would have higher order partial derivatives in the equation (2.3). In addition, it is easier to extend the Langevin approach to multidimentional or non-markovian problems. Especially it could allow us to study non-gaussian noise, which is difficult with the FPE.

Equation (2.1) was directly obtained from the picture of a diffusion process for the collective variable q, without any explicit derivation of the parameters entering the equation. A LE, for such a collective operator, can also be derived from a more microscopic approach such as the BLE. This has been done in reference [4]. We will directly use this equation and not its equivalent FPE to study fission. For sake of completness, the derivation is briefly outlined in the following.

An extension of the Boltzmann equation by incorporating dynamical fluctuations into the equation of motion of the single particle density was proposed in the nuclear context in reference [6]

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\nabla\mathbf{r} + \nabla\mathbf{r}Uf\,\nabla\mathbf{p}\right) f(\mathbf{r},\mathbf{p},t) = K(f) + \delta K(f) \qquad (2.5)$$

and named the Boltzmann Langevin Equation (BLE). In equation (2.5), K(f) is the Uehling-Uhlenbeck collision term:

$$K(f_1) = \frac{1}{h^3}\int d\mathbf{p_2}\, d\mathbf{p_3}\, d\mathbf{p_4}\, W(12,34)\, (f_3 f_4 \bar{f}_1 \bar{f}_2 - f_1 f_2 \bar{f}_3 \bar{f}_4) \qquad (2.6)$$

with $f_i = f(\mathbf{r_i},\mathbf{p_i},t)$ and $\bar{f} = 1-f$. The mean spin-isospin averaged transition rates are given by the cross section:

$$W(12,34) = \frac{4}{m}\frac{d\sigma}{d\Omega}\,\delta(\mathbf{p_1}+\mathbf{p_2}-\mathbf{p_3}-\mathbf{p_4})\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \qquad (2.7)$$

In equation (2.5), K(f) only describes the average effect of the two-body collisions, while δK(f) mocks up higher order correlations. The characteristic time scale evolution of δK(f) is of the same order of magnitude as the duration of a two-body collision.

Therefore one can consider it as a stochastic force acting on the one-body density. This force is caracterized by its first two moments:

$$\langle\delta K(\mathbf{r},\mathbf{p},t)\rangle = 0$$
$$\langle\delta K(\mathbf{r},\mathbf{p},t)\, \delta K(\mathbf{r}',\mathbf{p}',t')\rangle = C(\mathbf{p},\mathbf{p}')\, \delta(\mathbf{r}-\mathbf{r}')\, \delta(t-t') \qquad (2.8)$$

The stochastic force is here assumed to be local in space and Markovian, consistenly with the usual form of the collision term K(f) (equation (2.6)).

The BLE can be used to study large amplitude collective motion characterized by one collective variable q(t) [4]. For this purpose, one projects the BLE on q and performs a fluid dynamical reduction of the microscopic equation (2.5). This reduction is therefore valid only for situations not too far from equilibrium. The derivation is made in the diabatic approximation and an irrotationnal and incompressible velocity field is assumed

$$\mathbf{v}(\mathbf{r},t) = \dot{q}(t)\, \nabla\phi(\mathbf{r}) \qquad (2.9)$$

Assuming further that the density is determined in a quasistatic way

$$\rho(\mathbf{r},t) = \rho_0(\mathbf{r},q(t)) \qquad (2.10)$$

one eventually obtains a LE for q which can be written in the form:

$$M\ddot{q} + \frac{1}{2}\frac{\partial M}{\partial q}\dot{q}^2 + \frac{\partial V}{\partial q} = -\int_{-\infty}^{t} dt'\, \gamma(t-t')\, M\dot{q}(t') + \delta F(t) \qquad (2.11)$$

where M is the collective mass associated to q and where the friction kernel is given by

$$\gamma(t-t') = \frac{\Gamma}{M}\exp(-\frac{t-t'}{\tau}) \qquad (2.12)$$

the quantity $\Gamma$ being a restoring force. The random force $\delta F$ has zero mean and its correlation function is given by

$$\langle\delta F(t)\, \delta F(t')\rangle = TM\, \gamma(t-t') \qquad (2.13)$$

again in agreement with the "fluctuation-dissipation theorem".

The parameters entering these coupled equations (2.11,2.12), i.e. the collective mass M, the restoring force Γ due to the deformation of the Fermi surface and the microscopic relaxation time τ, are evaluated from microscopic values (using the notation of reference [4]):

$$M(q) = m \int d\mathbf{r}\, \rho_0(\mathbf{r},q)\, \nabla\phi\nabla\phi \tag{2.14}$$

$$\Gamma = \frac{24}{5} A\, \varepsilon_F\, \xi \tag{2.15}$$

$$\frac{1}{\tau} = 8\, \sigma\, v_F\, \rho \left(\frac{T}{\varepsilon_F}\right)^2 \tag{2.16}$$

where ξ is the mean value over density of $\sum_{ij}(\partial_i\partial_j\phi(\mathbf{r}))(\partial_i\partial_j\phi(\mathbf{r}))\ /\ 6$ and $\varepsilon_F$ and $v_F$ are respectively the Fermi energy and velocity, and T is the temperature. Expression (2.16) for τ is here given for a quadrupole mode at zero frequency. Relaxation times at finite frequencies for different multipolarities are calculated in reference [7]. These parameters are explicitly evaluated from characteristic quantities of the system, T, A, m, σ ($\sigma=\int d\Omega\, d\sigma/d\Omega$), $\varepsilon_F$, ρ and ϕ (equation (2.9)). To get a flavour of the meaning of these parameters, we can apply equation (2.11) without the stochastic force to the Giant Quadrupole Resonnace (GQR). We obtain $\Omega^2=\Gamma/M$ for the frequency and $\hbar/\tau$ for the width coming from the two-body collisions. A numerical estimate leads to $\Omega=66A^{-1/3}$, in agreement with the experimental values of the frequency [4]. However, the width is too small, as compared to the experimental one [7].

In equation (2.11), the friction term as well as the stochastic force depend on the previous history of the evolution. Though the BLE is Markovian, equation (2.11) is a typical Langevin equation with memory effect. It is often called a Generalized Langevin Equation (GLE). If τ vanishes, which is the case when we increase the temperature (eq. 2.16), γ becomes a delta function and the equation becomes Markovian. The velocity field influences two-body collisions which damp the motion and the velocity field. Therefore, a lagged friction appears in the equation of motion. This memory effect affects, in a non trivial way, the results of the various applications of such an equation. As an example, one cannot directly compare the coefficients of both Markovian and non-Markovian equations. The friction term will not have the same effect with and without memory, the Markovian reduction of equation (2.11) does not always give the same results as the equation (2.11).

In this approach (equation (2.11)), the collective mass, M(q), is not constant during the motion. Therefore, in order to be consistent, the second order term, $\frac{1}{2}\frac{\partial M}{\partial q}\dot{q}^2$, is kept in the equation of motion. This term happens to be small but has an important effect. It is always positive and therefore boosts the collective motion on the way to fission. We can expect that it will increase the fission rate. As it is proportional to $\frac{p^2}{2M}$, and therefore proportional to the temperature, its influence will be greater for larger temperatures.

# 3 Application to fission

Fission is a very complicated process, and usual nuclear models have some difficulties to explain the manyfold properties of this collective phenomena. The nucleus, after a large amplitude elongation, breaks into two fragments. Measurement of the properties of the fragments, $\gamma$ rays, fast neutrons and other particles emitted during fission provide most of the information on the process. Theoretically, a complete study of fission requires to understand the dynamics of the penetration of a multidimensional barrier, which is a very difficult problem. We consider here a highly simplified problem in only one dimension, restricting ourselves to symetric fission. It is well known that for heavy nuclei such a one dimension picture makes sense, at least for symetric fission. In this case the collective variable can be the quadrupole moment or any variable closely connected to it. In our calculations, the barrier is taken from a macroscopic Liquid Drop Model calculation without shell effects correction as these effects disappear for temperatures larger than 2-3 MeV [8]. The barrier hence only keeps the dominant competition between surface tension and Coulomb force. The escape over the barrier is due to the temperature which, through fluctuations, allows some nuclei to cross the barrier. Without the Langevin force which contains the effects of the temperature, there would not be any fission for a nucleus initially below the fission barrier. We also consider excited enough nuclei, such as we can safely neglect quantal tunneling.

## 3.1 Ingredients of the calculation

Our aim is to evaluate the fission rate:

$$r(t) = -\frac{1}{Pr(t)}\frac{dPr(t)}{dt} \qquad (3.1)$$

where Pr(t) is the probability that the nucleus is compound, i.e. the number of nuclei with q lower than the saddle point value $q_s$ at time t, over the total number of nuclei.

We have chosen a quadrupole velocity field for describing fission in this work:

$$\phi(\mathbf{r}) = (2z^2-x^2-y^2)/2 \tag{3.2}$$

Therefore the GLE has still the same form as previously and reads:

$$\dot{q} = \frac{p}{M} \tag{3.3a}$$

$$\dot{p} = -\frac{\partial V}{\partial q} + \frac{p^2}{2M^2}\frac{\partial M}{\partial q} - \int_{-\infty}^{t} dt'\, \gamma(t-t')p(t') + \delta F(t) \tag{3.3b}$$

and the coefficients can be explicitely calculated. The friction kernel becomes $\gamma(t-t') = \frac{\beta}{\tau}$ $\exp-(t-t')/\tau$, where $\beta$ is the reduced friction coefficient and $\delta F(t)$ is an Orstein-Uhlenbeck stochastic force with zero mean value and the correlation function of the equation (2.13). The friction parameter is simply

$$\beta = \frac{\tau\Gamma}{M} = \tau\,\Omega^2 \tag{3.4}$$

and $\xi$=1. The collective mass is less easy to calculate. For a quadrupole velocity field it is

$$M(q) = m\int d^3\mathbf{r}\ \rho(\mathbf{r},q)\ (x^2 + y^2 + 4\ z^2) \tag{3.5}$$

therefore $M(0)=2Am\langle r^2\rangle$. To calculate M as a function of q, we performed a Taylor expansion in q. One can evaluate the partial derivatives $\frac{\partial^n M}{\partial q^n}$ as follows. As the conservation equation for $\rho$ reads [4]

$$\frac{\partial \rho}{\partial q} = -\nabla\phi\ \nabla\rho \tag{3.6}$$

one obtains for the first derivative of the collective mass, after having performed a partial integration,

$$\frac{\partial M}{\partial q} = m\int d^3\mathbf{r}\ \rho(\mathbf{r},q)\ (16\ z^2 - 2\ x^2 - 2\ y^2) \tag{3.7}$$

One can easily iterate the procedure to obtain

$$\frac{\partial^n M}{\partial q^n} = m\int d^3\mathbf{r}\ \rho(\mathbf{r},q)\ (4^{n+2}\ z^2 + (-2)^{n+1}\ x^2 + (-2)^{n+1}\ y^2) \tag{3.8}$$

which gives, for q=0:

$$\left.\frac{\partial^n M}{\partial q^n}\right|_{q=0} = \frac{4^{n+1} + 2\,(-2)^n}{6}\, M(0) \tag{3.9}$$

Eventually, we find

$$M(q) = \frac{M(0)}{3}\,(2\exp(4q)+\exp(-2q)) \tag{3.10}$$

### 3.2 Discussion

Equations (3.3) are different from the LE (2.1) used in the phenomenological studies, due to the memory dependence and the non-constant mass. Equation (3.3b) can be replaced by two equations which are easier to manipulate

$$\dot{p} = -\frac{\partial V}{\partial q} + \frac{p^2}{2M^2}\frac{\partial M}{\partial q} + F \tag{3.11a}$$

$$\dot{F} = \frac{1}{\tau}\left(-\,\beta p\ -\ F\ +\ \sqrt{M\beta T}\; w(t)\right) \tag{3.11b}$$

where w(t) is a stochastic term:

$$\langle w(t)\rangle = 0 \text{ and } \langle w(t)w(t')\rangle = 2\,\delta(t-t') \tag{3.12}$$

With this way of writing the equations, the memory effect is transformed into a differential equation for the stochastic force with a Markovian stochastic term which is much more tractable from a numerical point of view.

To see how the memory effect can affect the final results, we have studied single trajectories. Without the stochastic force and with a constant mass, one can solve analytically the differential equation

$$\dddot{q} + \frac{1}{\tau}\ddot{q} + \left(\frac{\beta}{\tau} \pm \omega_0^2\right)\dot{q} + \frac{\pm\omega_0^2}{\tau}\, q=0 \tag{3.13}$$

which is equivalent to equations (3.3) without the second order term $\frac{p^2}{2M^2}\frac{\partial M}{\partial q}$. We obtain:

$$q(t) = A \exp\left(-\frac{\pm\omega_0^2\, t}{\beta\pm\omega_0^2\tau}\right) + B \exp\left(-\frac{t}{2\tau}\right) \cos(\omega t+\Psi) \tag{3.14}$$

where

$$\frac{1}{M}\frac{\partial V}{\partial q} = \pm\,\omega_0^2\, q \quad \text{and} \quad \omega = \sqrt{\left(\frac{\beta}{\tau}\pm\omega_0^2\right)-\frac{1}{4\tau^2}} \approx \Omega \quad (\Omega >> \omega_0) \tag{3.15}$$

In eq. (3.14), A, B and $\Psi$ depend on initial conditions, and we have assumed that the friction is large as compared to the other frequencies. Some damped oscillations at the frequency $\Omega$ of the giant quadrupole resonance appear in addition to the drift term. Such a behavior has already been noticed in reference [9] but now the stochastic force tends to activate these oscillations, see figure (1). Without memory effect, the nucleus directly goes from the saddle point to the scission point, the Langevin force only inducing diffusion around the mean value. The average value is similar to the calculation without stochastic term. With the memory dependence, the trajectory of the particle seems to oscillate around the average value. One intuitively understands that the memory can modify the final fission rate, but the difference should be small. Over the time course shown on figure (1), q does not change a lot and one can safely consider that $M(q)=M(q_0)$ in equations (3.14) and (3.15). The fact that the mass is not constant will just change the frequency of the oscillations because the drift term of equation (3.14) does not depend on M, which has been checked numerically. The stochastic force is also independant of the collective mass.

A key parameter for the shape and the value of the fission rate is the friction coefficient. Both transient time and Kramers' stationary limit highly depend on it. Our microscopically derived friction coefficient, $\beta$ (equation (3.4)), is two orders of magnitude larger than the one used in phenomelogical studies. But we cannot directly compare the Markovian and the non-Markovian cases. The memory term tends to decrease the effect of the viscosity and one has hence to consider an effective friction. This property can be seen on the stationary limit fission rate calculated analytically for a large friction [10,11]

$$k = \frac{\lambda}{\omega_b}\,\frac{\omega_0}{2\pi}\,\exp -\frac{V_b}{T} \tag{3.16}$$

where $\lambda$ is the largest real root of the Laplace transform of equation (3.14), $\omega_0$ and $\omega_b$ are respectively the frequencies of the potential in the well and at the barrier top. $V_b$ is the height of the barrier. The root $\lambda$ can be easily evaluated analytically if the differential equation is linear. A numerical approach is needed in the other cases. This means that we do not know how to evaluate analytically the influence of the mass as a function of q on the stationary limit. For the Markovian case and with a constant mass, one eventually recovers Kramers' formula [1]

$$k = \frac{\omega_0}{2\pi\omega_b}\left(\sqrt{\omega_b^2+(\beta/2)^2} - (\beta/2)\right) \exp - \frac{V_b}{T} \tag{3.17}$$

For the non-Markovian case, when $\tau$ is large enough to have oscillations in the single trajectories and if we still consider the mass as a constant, one can calculate analytically $\lambda$, but the expression is not very easy to manipulate. For very large friction one can give a simple approximate expression

$$k = \frac{\omega_0\, \omega_b}{2\pi\beta - \omega_b^2\tau} \exp - \frac{V_b}{T} \tag{3.18}$$

which shows how the memory affects the fission rate. One finds back the Markovian formula of the overdamped limit given by Kramers [1] for $\tau$=0. It means that the friction coefficient do not have the same effects with and without memory. A numerical stimulate in the cases studied in the next section shows that this correction is very small.

One does not know how to evaluate analytically the transient time. It is a very important quantity because the number of evaporated pre-scission neutrons is very sensitive to it. A numerical approach is then necessary. It is a difficult task to solve numerically a Langevin Equation, but these previous analytical formulas, (3.16), (3.17) and (3.18), will allow us to test our numerical scheme. We do not know the limit with the mass depending on q, so that our calculations have to be tested with a linear equation.

## 4 Numerical simulation of a LE. Comparison between the FPE and the LE approaches.

The FPE used in reference [2] (equation (2.2)) is equivalent to a Markovian Langevin equation with a gaussian noise. Because of the stochastic force, it is tricky to solve numerically a LE, hence, before solving the GLE of the preceding section we have tested our numerical resolution scheme on the results of references [2] and [3]. We solved numerically this equation following the usual algorithm [3,5]. We calculate trajectories for a large number of events and then evaluate the ensemble average value of relevant observables.

We have applied this algorithm to the case of reference [2], namely a FPE description of fission of a nucleus of mass 248 at two different temperatures and with two values of the friction coefficient. We took exactly the same phenomelogical conditions as in reference [2] and solved equation (2.1). In this case, the collective variable is an elongation and the potential is given by:

$$V(q) = Mg\, q^2 (q-c)(q+b) \tag{4.1}$$

with $M=940$ A MeV/$c^2$, $g=1.3287\ 10^{-40}$ fm$^{-2}$ s$^{-2}$, b=5 fm, c=19.688 fm. The barrier is 3.67 MeV high. The initial conditions are randomly picked according to the distribution:

$$d(q,p) = D \exp - \frac{1}{2T_0}\left(\frac{p^2}{M} + M\ \omega_0^2\ (q-q_1)^2\right) \tag{4.2}$$

which is a "cold" gaussian, with $T_0=0.3$ MeV and where $\omega_0$ and $q_1$ are the parameters of the osculatory parabola at the bottom of the potential well. In equation (4.2), D is the normalization constant. We show, in figure (2), a comparison of our calculation with the one of reference [2] for two different temperatures and frictions. As expected, our results are similar to the FPE calculations of reference [2]. They exactly reproduce the stationary Kramers' limit calculated analytically, for both temperatures. The discrepancies may be due to the fact that the reference [2] gives approximate solutions of FPE, not exact ones.

We have also checked our code on the case of reference [3] and we have found the same results. The main difference with the previous studies is the initial condition. In order to reduce the CPU time in the numerical approach the authors of reference [3] consider a hot initial condition. The distribution is not confined in the potential well anymore, therefore one can see a flow over the saddle point during the first time steps which has no physical meaning. This flow is compensated by a reverse flow due to some initial conditions taken beyond the saddle point. The initial distribution is:

$$d(q,p) = \begin{cases} D' \exp(-\frac{p^2/2M+V(q)}{T}) & \text{for } q<q_s=1.8 \\ d(q=q_s) & \text{for } q>q_s \end{cases} \tag{4.3}$$

where D' is the normalization, $M=44.55\ \hbar^2$/MeV and the potential is

$$V(q) = \begin{cases} 37.46\ (q-1)^2 & \text{for } 0<q<1.27 \\ 8.0 - 18.73\ (q-1.8)^2 & \text{for } q>1.27 \end{cases} \tag{4.4}$$

The fission rate for the $^{205}$At at T=2 MeV and for $\beta=1.5\ 10^{21}s^{-1}$ is calculated, the barrier height being in this case 8 MeV. On figure (3) we compare both results. The small disagreements may be due to the low statistics of our calculation. After $5.10^{-21}$ seconds, These two examples show the importance of the initial condition on the transient time. To compare the numerical results to the experimental data, one has to be cautious with the definition of the life time of the nucleus, which depends on the chosen initial condition.

## 5 Results

Now that we are confident in our numerical scheme, we can solve the GLE. It is numerically more demanding because, due to the memory kernel, one has to take a much smaller time step. Because of the oscillations in the single trajectories, one event will pass several times over the barrier top. Therefore, we need a larger statistics to have a smooth enough result. A more performant algorithm than in the Markovian case has been used [12]. In spite of these facts, it is easier than solving a generalized FPE, which would be a partial differential equation with an order higher than two.

We parametrize the shape radius of the nucleus in the following way

$$R = R_0 \left(1 + \alpha_2 P_2(\cos\theta)\right) \qquad (5.1)$$

where $P_2$ is the second Legendre polynomial, and the collective variable is $q=\alpha_2/2$ in order to match with the condition (2.9) over q [13]. The position of the saddle point is known [14] and we can build a potential with two parabolas. There is still a free parameter, the width of the barrier. We have arbitrarily chosen the same frequencies for the barrier top and for the potential well. The barrier is 4 MeV high, therefore the potential is, in MeV:

$$V(q) = \begin{cases} 135.5\, q^2 & \text{for } q<0.122 \\ -135.5\,(q-0.24)^2 + 4 & \text{for } q>0.122 \end{cases} \qquad (5.2)$$

The random force is chosen gaussian. A more complicated one could have been chosen, but the influence of the nature of the noise is not the purpose of this study. To calculate the coefficients entering the GLE, we have used the usual microscopic values, $\sigma=5\text{fm}^2$, $\varepsilon_F=37\text{MeV}$, $\rho=0.16\text{fm}^{-3}$. For $^{248}$Cf, these coefficients give $M(0)=376.3\ \hbar^2/\text{MeV}$, $\tau=4/T^2$ [$\hbar$/MeV], $\beta=464.4/T^2$ [MeV/$\hbar$] (T is given in MeV). The initial condition distribution for $q_0$ and $p_0$ is the same as the one of reference [2], with the new potential, see equation (4.2), and with the mass taken equal to M(0). For F, the initial condition has been chosen equal to zero, which is equivalent to choose a spherical distribution for the microscopic moments. Such a choice is in agreement with previous studies [9].

(4.2), and with the mass taken equal to M(0). For F, the initial condition has been chosen equal to zero, which is equivalent to choose a spherical distribution for the microscopic moments. Such a choice is in agreement with previous studies [9].

The large value of the friction parameter suggests the study of the overdamped limit. In this case one sets $\dot{p}=0$ in equation (3.11); the Langevin equation is then reduced to:

$$\dot{q} = -\frac{1}{M\beta}\frac{\partial V}{\partial q} + \sqrt{\frac{T}{M\beta}}\, w(t) \qquad (5.3)$$

This means just keeping the drift term of (3.15), plus the stochastic force. As $\beta \propto 1/M$, this equation does not depend on the mass.

### 5.1 Calculations with a fixed potential

We did calculations at different temperatures and with different levels of approximation in the GLE, in order to understand the influence of the various terms entering this equation. The results are shown on figure (4). The stationary limit can only be calculated analytically for a linear GLE, so we performed a calculation with a constant mass. We see a good agreement of Kramers' limit with the numerical calculations. For temperatures larger than the barrier height (4MeV) there is an overshoot and the convergence is slower. For each temperature, the fission rate calculated with a mass depending on q is larger than with a constant mass. Such a behavior may appear surprising because we saw that the drift term of the trajectories does not depend on the mass. So we did a calculation with the second order term but a constant mass: M(q)=M(0) and $\frac{\partial M}{\partial q} = \left.\frac{\partial M}{\partial q}\right|_{q=0}$. Of course this is not consistent but it gives a result similar to the case with a non-constant mass. Therefore the fact that the stationary limit is larger is due to this second order term that boosts the particles to fission. Though it is small, its effect is not negligible.

For each temperature, one may compare our results to the ones of the reference [2] with the friction coefficient given by the reference [15]. But, we do not have the same definition of the temperature as in the phenomenological studies. Our level density parameter is $a=\pi^2 A/4\varepsilon_F$, while a=A/10 in reference [2]. The barrier heights are not exactly the sames, neither. These point have to be taken into account in the comparison.

We have first studied the fission rate at T=4.88 MeV which can be compared to the case T=4MeV of the reference [2], shown in the figure (2). For such a temperature, the relaxation time $\tau$ is small enough ($\tau$=33fm/c=1.1x$10^{-22}$s) to neglect the memory effect. This can be checked on the single trajectories: without the Langevin force, there are no oscillations anymore. Our friction coefficient highly depends on temperature and at T=4.88MeV, $\beta$=0.1c/fm=30x$10^{+21}s^{-1}$. The friction coefficient of reference [2] could be as large as 5x$10^{+21}s^{-1}$ in order to match with the experimental data [15] which means 6 times smaller than ours. As we do not have the same equations one cannot directly compare the friction coefficients, but the previous differences are significant enough to have a rough idea. If we compare the stationary fission rates, ours is about 5 times smaller than theirs, with this strongest friction and our transient time, 5 times longer.

For the temperature T=4 MeV we did the calculations in the overdamped limit too. The result is quite similar to the one obtained with the linear GLE. Though the memory effect strongly affects the behavior of the single trajectories, it does not modify much the final fission rate. The stationary limits are exactly the same, the transient time is slightly smaller. This means that for further studies one can safely use this approximation which is more tractable numerically. In the case of this nucleus the memory term has a small effect on the stationary limit; analytically there is only a very small relative difference of $10^{-3}$ between the friction $\beta$ and the effective friction $\beta-\tau\omega^2/2\pi$ of equation (3.18).

At lower temperature, e.g. T=3MeV, $\tau$=88fm/c=3x$10^{-22}$s, we cannot do the Markovian approximation anymore. The effective friction coefficient is larger than at 4.88 MeV, $\beta$=0.27c/fm=77x$10^{+21}s^{-1}$, this means a longer transient time and a smaller stationary rate. For comparison, we also made a calculation (see reference [16]) in the phenomenological case of reference [2] at a temperature of 2.46MeV, corresponding to the same excitation energy as our T=3MeV case and with a friction coefficient of 5x$10^{+21}s^{-1}$, i.e. 15 times smaller than the present one. Now the stationary fission rate is about ten times smaller than in the phenomenological approach, and the transient time, ten times longer.

Similar results are shown for a temperature T=2 MeV. Though the total number of events is very large, the number of nuclei that undergo fission is too small to have smooth enough curves. One can nevertheless have an idea of the characteritics of the fission rate. When we decrease the temperature, we increase the friction coefficient and the motion becomes slower. This can be seen on both stationary limit and transient time, the first one decreases and the second one increases for cooler nuclei. The evolution is faster than in the phenomenological cases of references [2] and [3], due to the temperature dependant friction parameter. The evolution of this parameter, $\beta \propto 1/T^2$, is in agreement with calculations of the viscosity made from a Boltzmann Equation [17]. For low excitation energies, the transient time becomes so long that the fission process becomes scarce.

A summary of the complete GLE is shown on figure (5). For high temperature (T=4.88MeV), the fission rate is quite close to the T=4MeV, $\beta$=5 $10^{21}$ $s^{-1}$ case of the reference [2]. The discrepancy between both approaches becomes larger at lower temperature because our friction coefficient depends on temperature and the influence of non-linearities decreases.

### 5.2 Calculations with a potential barrier depending on temperature

The fission rates calculated previously are not very realistic, especially at low temperature. The fission process is too slow to reproduce the experimental data. To have more realistic calculations we have considered a fission barrier depending on the temperature. It is well known that for heavy nuclei, there is no barrier anymore for temperatures higher than 4 MeV. The fission process then becomes a free diffusive process and its time scale only depends on the diffusion coefficient or friction coefficient.

For the dependence of the barrier on the temperature, we simply took a cubic form [18]:

$$V_b(T) = \frac{V_b(0)}{(1-x(0))^3}\,(1-x(T))^3 \qquad (5.4)$$

where x(T) is the temperature dependant fissibility parameter:

$$x(T) = x(0)\,(1 + 7.1\ 10^{-3}\,T^2) \qquad (5.5)$$

and x(0) is the usual fissibility parameter [13]. To deduce the potential, we still arbitrarily choose the same frequencies at the barrier top and the potential well. All the fission rate calculations, for the same temperatures as previously, are shown on figure (6). The calculations were performed with the complete GLE.

As expected, the stationary limit is higher for every cases. The transient time is vanishing when we increase the temperature and is quite small at T=4MeV. When the potential barrier decreases, the potential becomes almost flat and therefore the initial condition distribution becomes wider and wider. For temperature higher than 4MeV, there is no barrier, our initial condition is not valid anymore. Even with this correction on the barrier, our fission rate, at T=3MeV, is lower by about a factor of eight with the one of the phenomenological approach [16]. This is due to the large friction coefficient which dominates all the process.

### 5.3 Summary

We present on figure (7) a summary of all the transient time and stationary limit obtained in phenomenological cases and in our cases for the $^{248}$Cf at different temperatures. We extracted the transient times from the probability of the nucleus to be compound as a function of time, which is a smoother curve than the fission rate. Nevertheless, the accuracy on the plotted quantities is not very good and one can only draw some general behaviors. There is a large difference between our results and the one from the reference [2] due to a larger friction coefficient and the corrections on the mass depending on the collective variable, the memory effects or a temperature-dependent barrier do not change dramatically the final observables. For such an approach, the dynamics of fission is dominated by the friction and stochastic terms.

## 6 Conclusion

We investigated thermal fission process using a memory dependant Langevin equation for a single collective variable, which is taken as the quadrupole moment of the density distribution. This one-dimensional collective transport equation and the associated transport coefficients are deduced from the microscopic BLE for the single density in a diabatic approximation. In diabatic limit, the ordinary one-body friction does not appear and the damping properties of the collective motion is governed by the two-body dissipation mechanism. This situation is in contrast to the adiabatic-linear response treatment of reference [19], in which the damping is described by the one-body mechanism only.

In the diabatic limit, the two-body friction depends on the size and the temperature of the system, and its magnitude comes out to be very large. For medium weight nuclei at moderate temperatures, the magnitude of the two-body friction is one order of magnitude larger than the friction coefficient extracted from the pre-scission data of fission of Er nucleus in reference [15]. As a result in the diabatic picture, the shape diffusion proceeds about ten times slower than the experimental information seems to indicate. However, this result is consistent with the conclusions reached in a more detailed investigation of the shape evolution in fission on the basis of the two-body dissipation mechanism [20]. Some new experimental results [21] seems to indicate that the friction coefficient may be higher than the one of the reference [15], but still lower than ours.

These observations indicate that the diabatic limit with highly symmetric strong constraints on the mean-field may not provide a realistic description of the optimal fission path. In order to have a better understanding of the fission dynamics, further studies are needed by allowing unconstrained shape evolution. Some recent calculations with a phenomenological one-body friction term reproduce well neutron multiplicities and the kinetic energy distributions are not far from experiment [22]. Our work contains other simplifications. The potential has a very simple shape, but we believe that a more sophisticated one would not change drastically the fission rate, mostly dominated by the friction term. We have considered a one dimension approach. Some calculations, with a two-dimension LE have already been performed in a phenomenological case [23] and the two-dimension stationary fission rate is only 15% larger than the one-dimension case.

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## Figure captions

**figure 1:** Effect of the Langevin force on a single trajectory, in the Markovian and the non-Markovian case. The initial condition has been chosen far beyond the saddle point. The oscillations are a signature of the memory effect.

**figure 2:** Fission rate as a function of time for a nucleus of mass 248 at two different temperatures and frictions. Comparison is made between the FPE calculation of reference [2] and this work. The initial condition distribution is "cold", $T_0$=0.3MeV. The stationary limit, calculated analytically is also indicated. N is the number of calculated events.

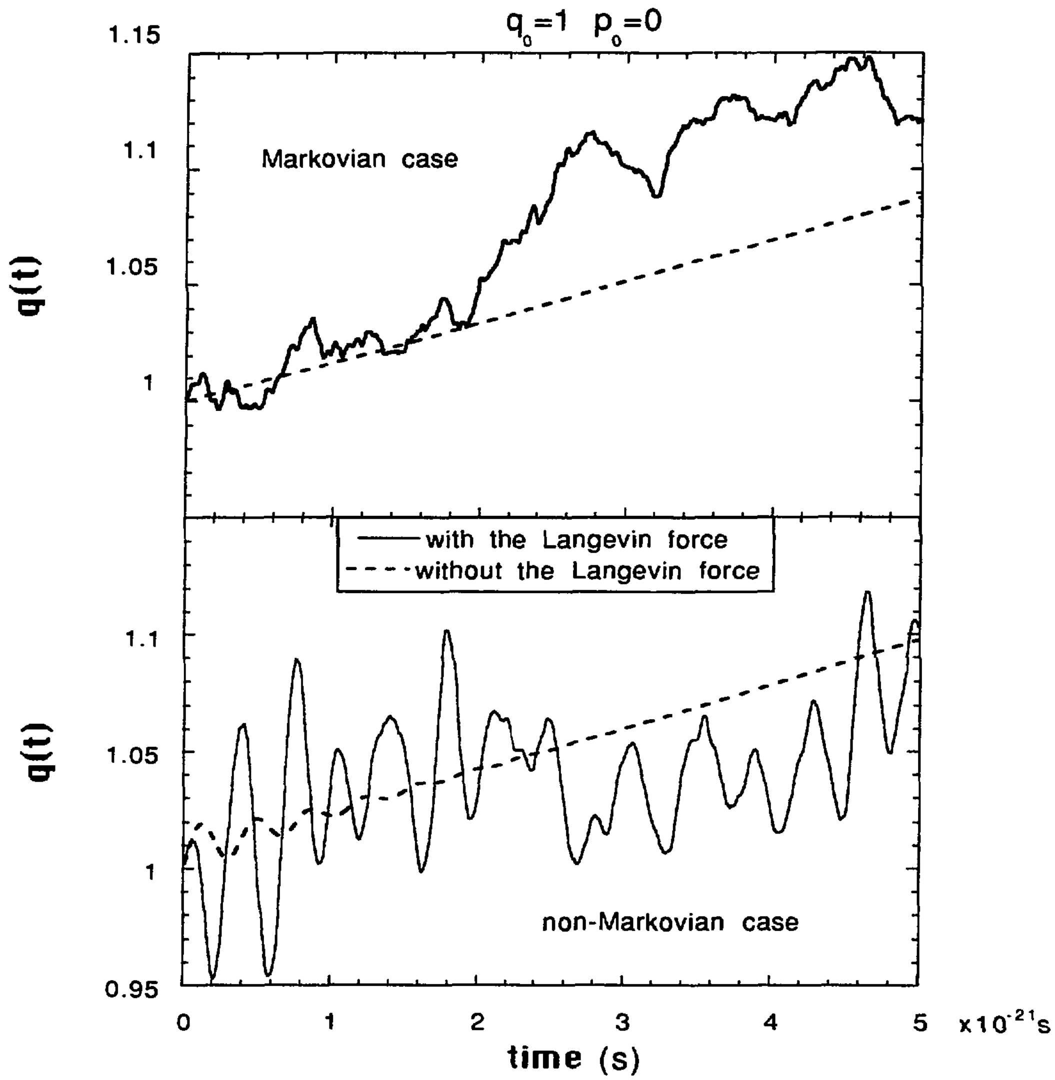
**figure 3:** Fission rate as a function of time for the $^{205}$At. Comparison is performed between the calculation of the reference [3] and ours, both being made with a LE. The initial distribution is hot, $T_0$=2MeV. The stationary limit, calculated analytically is also indicated. N is the number of calculated events.

**figure 4:** Fission rate as a function of time for $^{248}$Cf at various temperatures. Calculations were made with a linear GLE, with the complete GLE and with a constant mass but with a second order term. The stationary limit, calculated analytically in the linear case is also indicated. N is the number of calculated events. See text for details.

**figure 5:** Fission rate as a function of time for $^{248}$Cf at various temperatures. Calculations were made with the complete GLE.

**figure 6**: Fission rate as a function of time for $^{248}$Cf at various temperatures. The potential barrier depends on the temperature. N is the number of calculated events.

**figure 7**: Transient times and stationary fission rates for every approaches developed in the present paper. For the phenomenological points the temperature has been corrected and the reduced friction coefficient is $\beta$=5 $10^{21}s^{-1}$.



$q_0=1$  $p_0=0$
1.15
1.1
1.05
1
Markovian case
q(t)
with the Langevin force
without the Langevin force
1.1
1.05
1
0.95
q(t)
non-Markovian case
0
1
2
3
4
5
x10$^{-21}$s
time (s)


figure 1

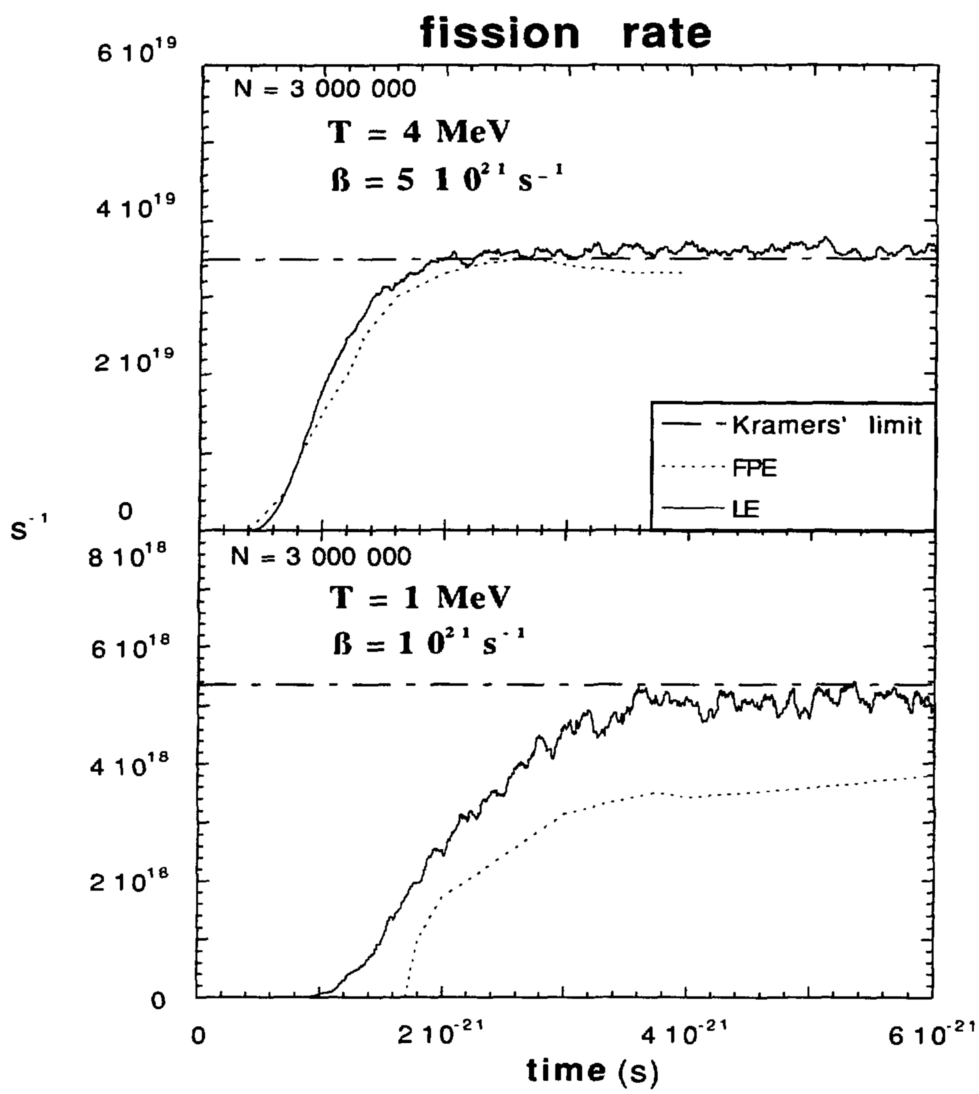
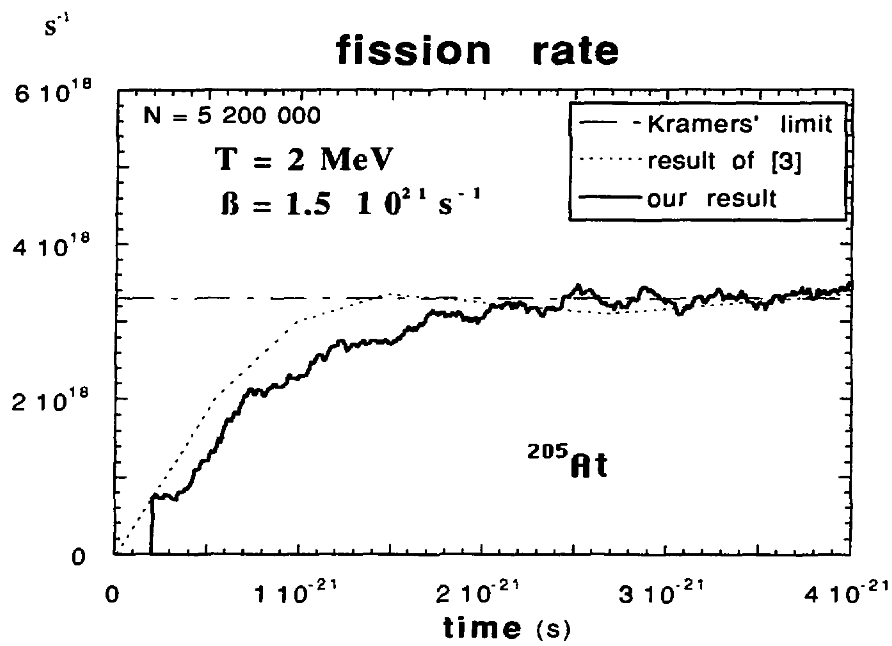
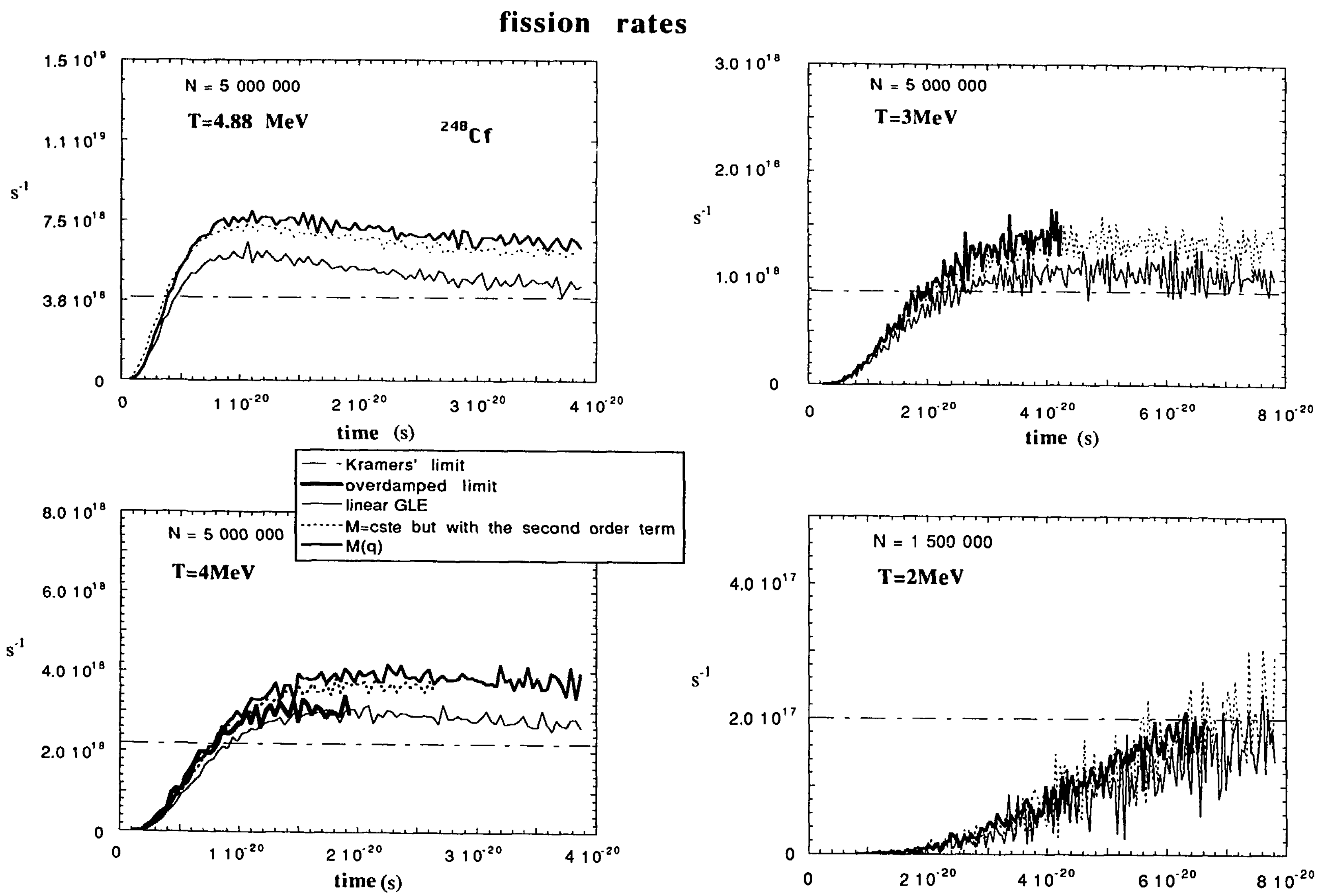

fission rate
N = 3 000 000
T = 4 MeV
ß = 5 10²¹ s⁻¹
6 10¹⁹
4 10¹⁹
2 10¹⁹
0
Kramers' limit
FPE
LE
s⁻¹
N = 3 000 000
T = 1 MeV
ß = 1 0²¹ s⁻¹
8 10¹⁸
6 10¹⁸
4 10¹⁸
2 10¹⁸
0
0
2 10⁻²¹
4 10⁻²¹
6 10⁻²¹
time (s)

figure 2



s-1
fission rate
6 10^18
4 10^18
2 10^18
0
N = 5 200 000
T = 2 MeV
ß = 1.5 10^21 s^-1
Kramers' limit
result of [3]
our result
205At
0
1 10^-21
2 10^-21
3 10^-21
4 10^-21
time (s)


figure 3



fission rates
N = 5 000 000
T=4.88 MeV
248Cf
s-1
time (s)
N = 5 000 000
T=3MeV
Kramers' limit
overdamped limit
linear GLE
M=cste but with the second order term
M(q)
N = 5 000 000
T=4MeV
N = 1 500 000
T=2MeV


figure 4

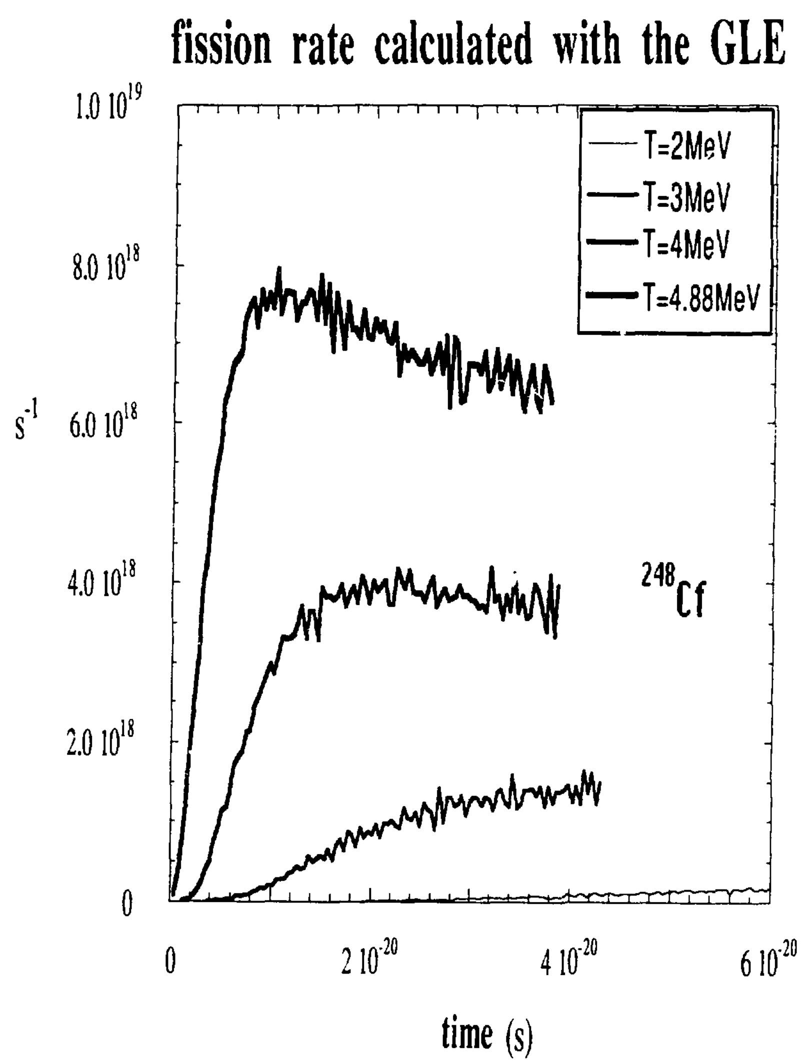
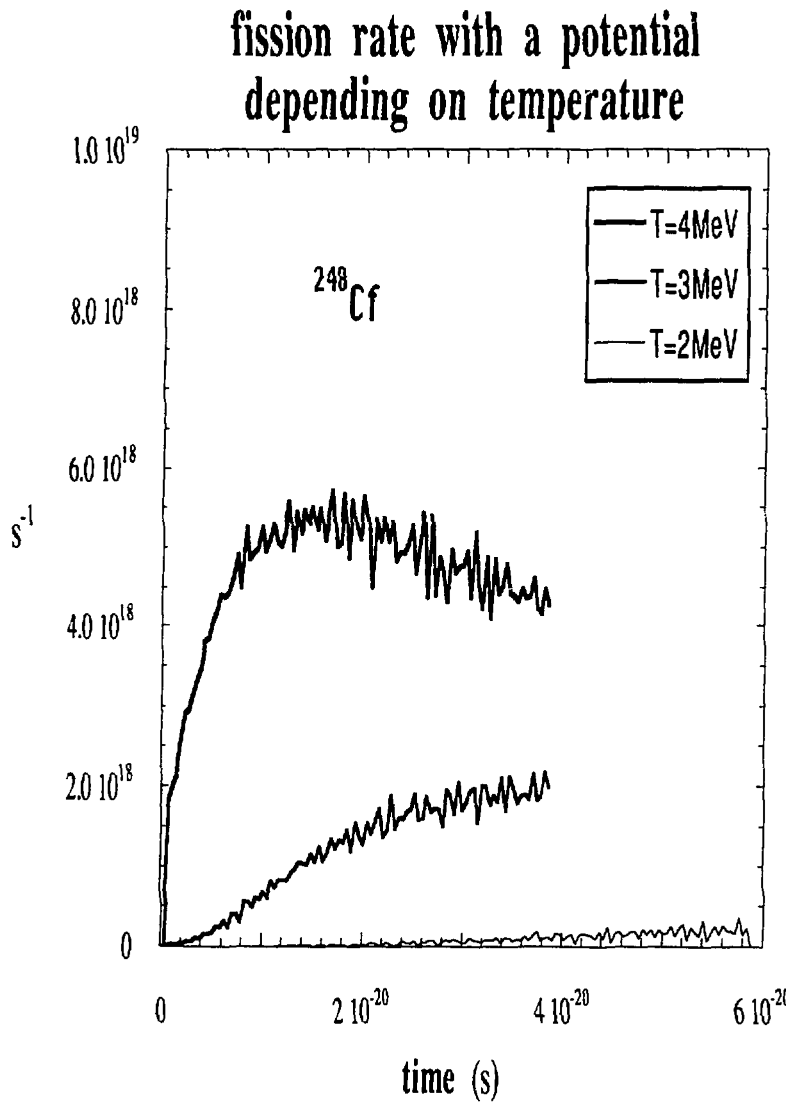

fission rate calculated with the GLE
1.0 10^19
8.0 10^18
6.0 10^18
4.0 10^18
2.0 10^18
0
s^-1
T=2MeV
T=3MeV
T=4MeV
T=4.88MeV
248Cf
0
2 10^-20
4 10^-20
6 10^-20
time (s)

figure 5



fission rate with a potential
depending on temperature
1.0 10^19
8.0 10^18
6.0 10^18
4.0 10^18
2.0 10^18
0
s^-1
248Cf
T=4MeV
T=3MeV
T=2MeV
0
2 10^-20
4 10^-20
6 10^-20
time (s)


figure 6

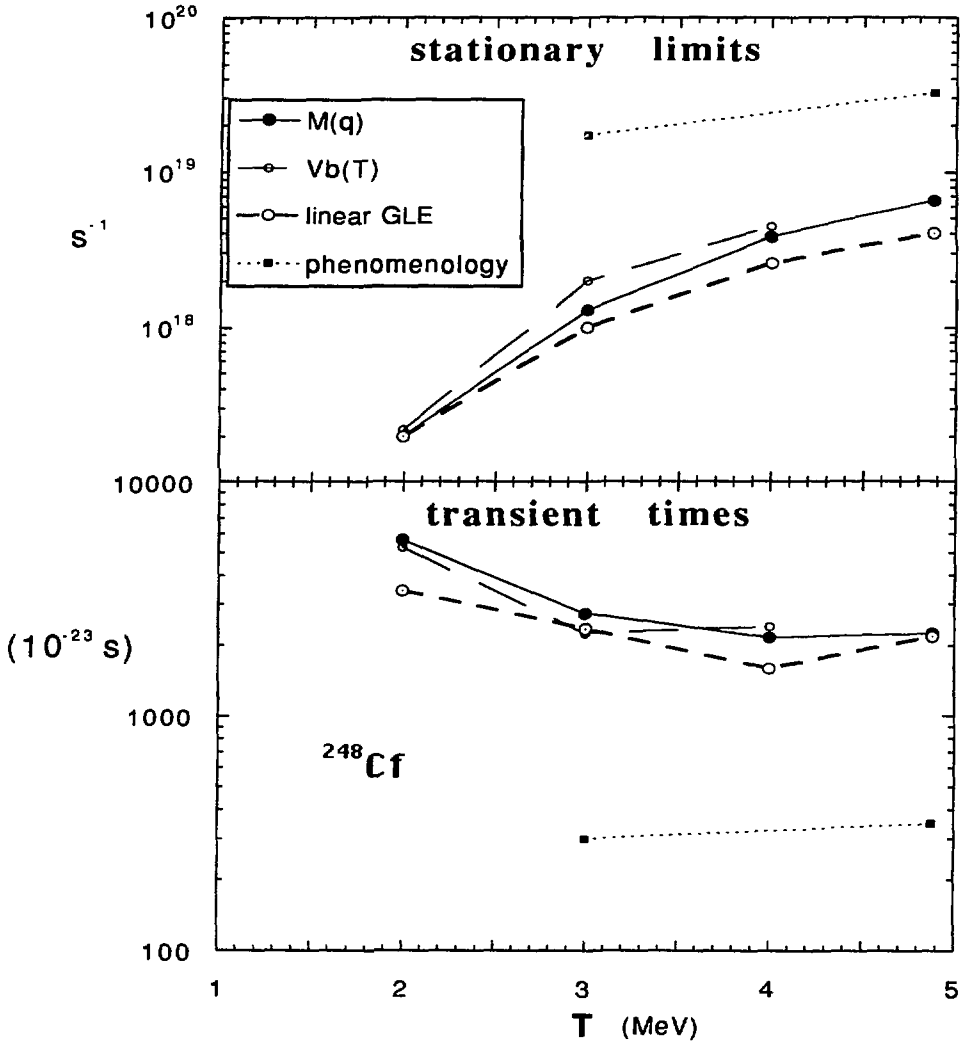

stationary limits
M(q)
Vb(T)
linear GLE
phenomenology
$10^{20}$
$10^{19}$
$10^{18}$
$s^{-1}$
10000
transient times
$(10^{-23}$ s)
1000
100
$^{248}$Cf
1
2
3
4
5
T (MeV)

figure 7