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PULSE PILE-UP I: SHORT PULSES

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Abstract

The search for rare large pulses against an intense background of smaller ones involves consideration of pulse pile-up. Approximate methods are presented, based on ruin theory, by which the probability of such pile-up may be estimated for pulses of arbitrary form and of arbitrary pulse-height distribution. These methods are checked against cases for which exact solutions are available. The present paper is concerned chiefly with short pulses of finite total duration.

(submitted to Nuclear Instruments & Methods)

*My consideration of this problem began in the Cavendish Laboratory in 1943 in the context of the measurement of fission in an intensely alpha-particle-active source; it has continued over the years in several institutions and in several contexts, most recently in those institutions appearing in these bylines.

1. Introduction

In particle and nuclear physics one often searches for rare large pulses, perhaps of known amplitude, perhaps not, in the presence of an intense background of smaller pulses, perhaps all of equal amplitude, perhaps not. Since the pulses must be of finite duration in time there is a possibility that, at any moment, the small pulses may pile up on top of each other to the extent that they mimic the magnitude of the rare large pulse that is being sought. This is the pulse pile-up problem; it is important to be able to estimate, in designing an experiment, the likelihood of such pile-up or, *ex post facto*, the likelihood that such pile-up in fact occurred.

Although it is obvious that pile-up is minimized by making the pulses of as short a duration as possible, the nature of the detecting equipment, which is the source of the pulses, imposes its own limitations on this shortening process particularly if good amplitude resolution is required: one is often forced into a trade-off between pile-up and resolution.

The impact of pile-up can also, most importantly, often be mitigated by various forms of pile-up rejection circuitry or procedures involving authentication of the pulse profile by sampling devices such as flash ADCs, which, for example, permit one to compare the maximum peak height of a pulse with the integrated area beneath it or to compare the areas between successive fractions of the maximum peak height or to check for improper changes of slope on the leading or trailing edges of the pulse, either in hardware via dedicated microprocessors or by flexible software interrogation or by visual inspection. However, such stratagems can only diminish the impact of whatever primitive pile-up has, in fact, taken place by, to some greater or lesser degree, recognizing it as such; it remains important to be able accurately to estimate what the probability of that primitive pile-up might be in order to assess the possibility of taking appropriate measures to combat it. In any case such stratagems are of increasingly limited scope the smaller the probability of a feared massive pile-up becomes since to achieve such massive pile-up many small pulses must happen to arrive within a brief interval commensurate with the pulse length so that the pile-up more and more closely resembles a genuine large pulse. The present paper, and its companion, restrict themselves to what has just been called primitive pile-up; the mitigating effects of pulse-form authentication will be considered separately.

It is therefore the object of this paper and of its companion to explore the following question in several contexts and to answer it in general terms: pulses of form $c(t)$ and of (normalized) pulse-height distribution $g(c_{max})$ arise in a random sequence in time at a steady average rate of X per unit time; what is the probability $\Omega(D)$ that, at an arbitrary instant, their superposition is of amplitude D or greater?

Consideration will be here also restricted to unipolar pulses although pile-up is obviously reduced for bipolar pulses: this extension is reserved for later treatment.

No consideration will be given to specific details of electronic processing, neither in the preparation of the pulse form $c(t)$ nor in the sequencing of effective time constants leading to base-line restoration. Note in passing that relative to $X=0$ the base-line for rate X will be at $B = X \int_0^\infty g(c_{max})\, \bar{c}(c_{max})\, dc_{max}$ with $\bar{c}(c_{max}) = c_{max} \int_0^\infty c(t)\, dt$, where here the form $c(t)$ has $c_{max} = 1$. Because of base-line restoration one is, therefore, in practice, often concerned with pile-up to $D = P+B$ to mimic a genuine pulse of height P. Note also in passing that because $\Omega(D)$, as will be exposed in detail, is a rapidly falling function of D, pile-up at a given X is sensitive to "base-line bounce" such as may be produced by fluctuations in X due to whatever cause if those fluctuations occur on a time scale commensurate with internal time constants associated with the definition of the base-line. It is not profitable to attempt to analyse base-line bounce in any detail because it is so strongly dependent upon details of circuitry, but one must be sensitive to its importance in the context of pile-up and take appropriate measures to minimize its effects in relation to the type of fluctuations in X that may be likely to be encountered, and also by explicit measures in the event of such fluctuations, for example by imposing a dead time following recovery of X after a severe downward fluctuation in X.

Frequently, of course, X will be an explicit function of tim.. by virtue of the nature of the ultimate source of the pulse-train, for example an accelerator whose beam has an intrinsic temporal modulation. In such a case the pile-up problem can be again well defined, but consideration is reserved at this time and we here concern ourselves only with the case of constant X; temporal variation of X will be considered separately.

2. Time scales

The present paper concerns its analyses only with short pulses $c(t)$ which are defined as starting at time $t=0$, rising to a maximum value c_{max}, which is referred to as the pulse amplitude or pulse height, and declining to zero at, and after, $t=1$. The pulse length is therefore the unit of time for the measurement of the mean rate, X per unit time, of the arrival of the pulses in a Poisson sequence. The following paper (referred to as II) treats pulses that also start at $t=0$, rise to a maximum value c_{max} according to some prescription and then decline to zero not at a definite time but with, ultimately, a tail exponential in time, *viz.* proportional to e^{-t}. In this case of tailed pulses the logarithmic decrement of the tail therefore defines the unit of time within which pulses arrive at the mean rate X.

3. Illustrations

As a preliminary to a discussion of the calculation of pile-up it is instructive to illustrate three general points concerning the dependence of $\Omega(D)$ upon D; the methods lying behind the generation of the illustrations will be presented later:

(i) pile-up is extremely sensitive to the rate X;

(ii) pile-up is extremely sensitive to the pulse form $c(t)$;

(iii) pile-up is extremely sensitive to the form of the pulse-height distribution $g(c_{max})$, particularly its large-c_{max} tail.

Fig. 1 illustrates points (i) and (ii) by a comparison of $\Omega(D)$ for a variety of forms for short pulses, all of equal height with $c_{max} = 1$, that spans the range likely to be encountered in practice: from square waves, for which pile-up will obviously be greatest, to sawteeth which rise linearly to their maxima and then fall linearly to zero ($\Omega(D)$ is independent of the time between t=0 and t=1 at which the maximum is reached) and for which pile-up will obviously be least, through sugarloaves, *viz.* $c(t) = 4t(1-t)$.

Fig. 2 illustrates points (i) and (iii) for the case of sawteeth: (a) all of equal amplitude $c_{max} = 1$; (b) having an exponential distribution of pulse amplitudes of the same mean height as in (a), *viz.* $g(c_{max}) = e^{-c_{max}}$.

Fig. 3 repeats the points of fig. 2 but for pulses that rise instantaneously to their maximum height and then decay exponentially to zero, *viz.* $c(t) = e^{-t}$ for case (a) and with $g(c_{max}) = e^{-c_{max}}$ for case (b). (Call these pure exponential pulses.)

Fig. 4 illustrates points (i) and (iii) with reference to the extreme sensitivity of $\Omega(D)$ to the tail of $g(c_{max})$. $\Omega(D)$ is shown for square waves with $g(c_{max}) = e^{-c_{max}}$ but with that distribution cut off at $c_{max} = k$. [Note that $e^{-5} = 0.0067\ldots$ so that cutting off less than 1% of the pulse-height spectrum at its upper end reduces pile-up by more than an order of magnitude for the larger values of D considered.]

Figs. 2 and 3 have illustrated the great sensitivity of $\Omega(D)$ to the transition from a delta-function to an exponential form for $g(c_{max})$ while keeping the mean pulse-height constant, while fig. 4 has emphasized the great weight of the higher values of c_{max} in producing this effect. It is to be anticipated that tailed distributions $g(c_{max})$ that fall off to high-c_{max} more rapidly than exponentially will show correspondingly less striking sensitivity to the presence of the tail. This is illustrated for the gamma distribution $g(c_{max}) \sim c_{max}^{\nu-1}\, e^{-(\nu-1)c_{max}}$ of square-wave pulse heights which peaks at $c_{max} = 1$ and is shown in fig. 5 for three values of ν, for purposes of illustration reduced to a common peak value for g(cmax). It is seen in fig. 6 that although the tails of fig. 5 have substantial effect upon $\Omega(D)$ as compared with the delta-function distribution, this effect is nowhere near as dramatic as that seen in figs. 2 and 3 for passing from delta-function to exponential distributions for $g(c_{max})$; this is despite the fact that for the gamma distribution, normalized to peak at $c_{max} = 1$, the mean pulse height increases as the distribution broadens (as $\nu/(\nu-1)$) whereas in figs. 2 and 3 we have maintained the mean pulse-height constant in moving from the delta-function to the exponential $g(c_{max})$. We may illustrate this point about the different forms of the tails of the gamma and exponential distributions by remarking that exponentials fitted approximately to the high-c_{max} side of the gamma distributions of fig. 5 in fact fall much more slowly than the gamma distributions at higher values of c_{max}: for example an exponential that fits the ν=100 distribution at $c_{max} = 1.1$ and 1.2 is approximately 40 times higher than the gamma distribution at $c_{max} = 1.5$.

We may, conversely, note that a Gaussian form for $g(c_{max})$ falls more rapidly than the gamma distribution to which it may be fitted in the region of its peak and that its associated $\Omega(D)$ is correspondingly closer to that for a delta-function for $g(c_{max})$ than is the case for the gamma distribution that it superficially resembles.

An object of these illustrations has been to emphasize the great importance for the pile-up problem of the form of the high-c_{max} tail of the pulse-height distribution and the concomitant necessity of having reliable knowledge of that tail even in regions where $g(c_{max})$ may be very small (cf. fig. 4) and where such information may be difficult to obtain. Conversely, if adequate information as to the actual $g(c_{max})$ cannot be obtained in regions of c_{max} where it may possibly be important (which can be quantified using the methods to be presented), then plausible extreme assumptions must be made as to the behaviour of $g(c_{max})$ in those regions in order to bracket the range within which $\Omega(D)$ may fall.

Similar strictures apply to the need to secure adequate quantitative knowledge of the pulse form $c(t)$, again on account of the extreme sensitivity of pile-up to that form as seen in fig. 1; this should be readily achievable.

The methods to be presented permit the handling of arbitrary pulse forms $c(t)$ and arbitrary pulse-height distributions $g(c_{max})$ with adequate accuracy for practical purposes.

4. Inadequacy of Gaussian approximations

As in most cases relating to random superposition the solution to pile-up problems in the immediate neighbourhood of the most-likely pile-up takes the form of a Gaussian the character of which is easily derived by standard methods. It must, however, be strongly emphasized that such solutions are completely useless for the case that concerns us here, namely $\Omega(D) << 1$. This is illustrated in fig. 7 for the pile-up of square-waves all of the same height, unity, for which the exact solution is:

$$\Omega(D) = \sum_{N=D}^{\infty} \frac{X^N}{N!} e^{-X} \tag{1}$$

and the Gaussian approximation is*:

*Mathematical notations follow those of ref. [1] where instructions as to numerical evaluation are found.

$$\Omega(D) = \tfrac{1}{2}\,\mathrm{erfc}\left[(D-X)/\sqrt{2X}\right] . \qquad (2)$$

A further illustration is for the pile-up of pulses of equal height, unity, and of pure exponential form $c(t) = \mathrm{e}^{-t}$ shown in fig. 8. Here the exact $\Omega(D)$ is given in II: the Gaussian approximation is:

$$\Omega(D) = \tfrac{1}{2}\,\mathrm{erfc}\left[(D-X)/\sqrt{X}\right] \qquad (3)$$

to which, for $h < X^{1/6}$, may be added the correction term:

$$\Delta\,\Omega(D) = \frac{2}{9\sqrt{\pi X}}\,(h^2+4)\,\mathrm{e}^{-h^2} , \qquad (4)$$

where $h = (D-X)/\sqrt{X}$.

Fig. 8 also illustrates the general point that analytical corrections to Gaussian solutions are themselves valid only for values of $\Omega(D)$ that are uninterestingly large in the context of practical pile-up.

5. Ruin theory

For a few particularly simple pulse forms $c(t)$ and simple pulse-height distributions $g(c_{max})$ exact solutions, analytical or numerical, are available for $\Omega(D)$; some of these will be given here and in II. However, these exact solutions are of limited utility in practice, and it is essential to have available a more general method for deriving $\Omega(D)$. This is ruin theory.

Ruin theory [2] arose in the context of the need of insurance companies to know the chance of their becoming bankrupt owing to the incidence of an unusually large number of claims of unusually large magnitude. If claims arise, with a Poisson distribution in time, at the average rate X per the interval in question and if the claims of magnitude c have a probability distribution $f(c)$ such that:

$$\int_0^\infty f(c)dc = 1 , \qquad (5)$$

then the probability $\Omega(D)$ that the total claims during the interval in question will exceed D, measured in their own units, is [2]:

$$\Omega(D) = \frac{\{\exp[X\phi(r)] - 1\}\ \exp(-X - rD)}{r[2\pi X\ \phi''(r)]^{1/2}}\ , \tag{6}$$

where:

$$\phi(r) = \int_0^\infty e^{rc} f(c)\, dc \tag{7}$$

and where r, for insertion into (6), is determined by:

$$\phi'(r) = \frac{D}{X}\ . \tag{8}$$

Ruin theory is tailored to the case $\Omega(D) << 1$ and becomes more accurate the larger X and D; it involves delicate consideration of the behaviour of the tails of Laplace transforms.

Application of ruin theory to the problem of pulse pile-up is immediate in the case of short pulses of duration unity: at the arbitrary instant of time only pulses arising in the previous unit time interval have effect, and since they arrive randomly the probability function that prescribes their superposition, if they are all identical in height as well as in form, is given by:

$$f(c) \sim \left|\frac{dc(t)}{dt}\right|^{-1} \tag{9}$$

so that if $\frac{dc(t)}{dt}$ is an analytical function of c we may proceed to derive $\phi(r)$ by (7) and hence $\Omega(D)$ by (6) following generation of r via (8) either analytically or numerically as appropriate. [It should be remarked that if $\frac{dc(t)}{dt}$ is not an analytical function of c, it is not usually profitable to attempt to generate $f(c)$ numerically from $c(t)$: one should rather synthesize $c(t)$ from analytically tractable segments in the manner to be presented later.] If the pulses are all identical in form but not in height and have a pulse-height distribution $g(c_{\max})$, then that distribution must be integrated over appropriately in the generation of the $f(c)$. [It should also be stressed that if this integration over $g(c_{\max})$ folded in with the form $c(t)$ cannot be carried out analytically then alternative numerical approaches to the generation of the $f(c)$ should not be attempted owing to the extreme sensitivity of $\Omega(D)$ to the high-$c_{\max}$ tail of $g(c_{\max})$ as has been emphasized in the illustrations presented in sect. 3: it is essential that realistic and analytically tractable forms for the tail of $g(c_{\max})$ be adopted and if they cannot be analytically combined with the associated $c(t)$ to generate an analytical $f(c)$, then the methods to be presented later involving the intermediate use of a surrogate square-wave $c(t)$ should be followed.]

If the pulses $c(t)$ are not short but are exponentially tailed then $f(c)$ cannot be generated in the simple way described above since, in the tail, $f(c) \sim \frac{1}{c}$ and the normalization (5) diverges. If, however, we convert the infinitely tailed, exponentially decaying pulses effectively into short ones by cutting off the exponential tail at some long time T (in units of the time constant of the exponential decay) corresponding to the pulses' having fallen to some small fraction ϵ of their maximum value, so that the X-value, X_s, a function of ϵ, for the effective short pulse is given by $X_s = TX$, then we find that ϵ cancels in the numerator of (6) as $\epsilon \rightarrow 0$ and that $\Omega(D)$ becomes just a well-behaved function of X, D. The acceptability of this procedure will be demonstrated in II.

The expression (6) is the lowest-order result of ruin theory [2] which has been considerably elaborated and extended and provided with higher-order corrections [3]. These extensions tend to be tedious to apply in practice; it is unlikely that in the context of pulse pile-up their evaluation will be worth while since they amount typically to corrections to $\Omega(D)$ of only some 20% or less.

As remarked above, ruin theory is tailored to just the circumstance of our present concern, namely small values of $\Omega(D)$; but it is not expected to be applicable for X below certain values that depend upon $f(c)$, *viz.* upon the forms of $c(t)$, $g(c_{max})$ in our context. We now explore the reliability of ruin theory by comparing its previsions as to $\Omega(D)$ with the exact solutions that are available for a range of cases for both $c(t)$ and $g(c_{max})$. This range of exact solutions covers short pulses from the broadest (square-waves) to the sharpest (sawteeth) and also exponential pulses and pulse-height distributions from the sharpest (delta-function) to the broadest (exponential); acceptable agreement between ruin theory and the exact solutions would give considerable confidence in the applicability of ruin theory for all practical $c(t)$, $g(c_{max})$; the comparison should also illuminate the lower values of X to which the method might be safely applied.

6. Tests of ruin theory

Exact solutions are available for $\Omega(D)$ for the following five cases against which we now test ruin theory:

(1) Square waves all of the same height;

(2) Square waves with exponential height distribution;

(3) Sawteeth all of the same height;

(4) Pure exponential pulses all of the same height;

(5) Pure exponential pulses with exponential height distribution.

6.1 Square waves all of the same height

The exact solution has been given in (1). Ruin theory here gives a closed form for

$\Omega(D)$:

$$\Omega(D) = \frac{(e^D - 1)\, e^{-X} (D/X)^{-D}}{(2\pi D)^{1/2} \ln(D/X)} . \tag{10}$$

In this case (only) we encounter the obvious problem that whereas ruin theory gives an expression for $\Omega(D)$ continuous as a function of D, the exact $\Omega(D)$ is defined only for discrete integral values of D. We should therefore expect that ruin theory might fit the exact $\Omega(D)$ not for the D-value of the exact expression but rather for some value roughly midway between D and $D-1$. This is indeed the case: fig. 9 compares the exact $\Omega(D)$ at integral values of D with the $\Omega(D)$ of (10) evaluated at $D-0.4$ where the constant offset is purely empirical. It is seen that there is excellent agreement between the exact $\Omega(D)$ and ruin theory for values of X as small as 0.1 or less although this agreement is dependent upon the empirical offset in D of 0.4.

6.2 Square waves with an exponential height distribution

The exact solution is:

$$\Omega(D) = e^{-(X+D)} \sum_{N=1}^{\infty} \frac{X^N}{N!} \sum_{m=0}^{N-1} \frac{D^m}{m!} \tag{11}$$

$$= X\, e^{-X} \sum_{N=1}^{\infty} \frac{X^{N-1}}{N!(N-1)!} \Gamma(D,X) \tag{12}$$

to which an analytical approximation is available:

$$\Omega(D) = \left[\frac{X}{D}\right]^{1/4} \frac{1}{2\sqrt{\pi}} \frac{1}{\omega} e^{-\omega^2} \left[1 - \frac{1}{4\omega\sqrt{D}} + \ldots\right] , \tag{13}$$

where $\omega = \sqrt{D} - \sqrt{X}$.

Ruin theory again gives $\Omega(D)$ in closed form:

$$\Omega(D) = \frac{(\exp\sqrt{DX} - 1)\, \exp(-X - D + \sqrt{DX})}{2\sqrt{\pi X}\,([D/X]^{3/4} - [D/X]^{1/4})} . \tag{14}$$

Fig. 10 compares the exact $\Omega(D)$ of (11) or (12) with the ruin theory $\Omega(D)$ of (14) and also with the approximate analytical (13); comparison between the exact $\Omega(D)$

and ruin theory is extended to lower values of X in fig. 11. As for the delta-function distribution of pulse height we see excellent agreement between ruin theory and the exact solution down to X=0.1 or less (but in the present case without any empirical adjustment constant being needed). (The analytical approximation (13) to the exact solution is precisely equal to the ruin theory expression (14) if the small correction term in the square brackets is dropped from the former and -1 in the first term of the numerator from the latter.) Fig. 10 also shows some (exact) values of $\Omega(D)$ for the case of the delta-function distribution to emphasize, analogously to figs. 2 and 3, the tremendous effect on $\Omega(D)$ of moving from the delta-function to the exponential $g(c_{max})$.

6.3 Sawteeth all of the same height

It is a standard result [4] that the probability $p_N(\ell)\,d\ell$ that N randomly superposed sawteeth of equal magnitude unity add up to between ℓ and $\ell+d\ell$ is given by:

$$p_N(\ell) = \frac{1}{(N-1)!}\sum_{n=0}^{N}(-)^n\binom{N}{n}(\ell-n)_+^{N-1}\,, \tag{15}$$

where $x_+ = \frac{1}{2}(x+\mid x\mid)$. [This result is independent of the time between t=0 and t=1 at which the maximum height is reached.]

Now recognize that the N sawteeth derive from a Poisson distribution of mean X and write:

$$P_X(\ell) = \sum_{N=\ell_i+1}^{\infty}\frac{X^N}{N!}\,\mathrm{e}^{-X}\,p_N(\ell)\,, \tag{16}$$

where ℓ_i is the integral part of ℓ. Then the exact

$$\Omega(D) = \int_D^{\infty} P_X(\ell)d\ell\,. \tag{17}$$

The $\Omega(D)$ of ruin theory derives from:

$$\phi(r) = (\mathrm{e}^r - 1)/r\,. \tag{18}$$

Fig. 12 compares the exact (17) with the $\Omega(D)$ of ruin theory based on (18): again agreement is excellent down to $X = 0.1$ below which it falls off rather more than in the case of square-waves displayed in figs. 9, 10 and 11.

[Although no exact solution is available for the case of sawteeth having an exponential height distribution, it may be noted that in this case ruin theory derives from:

$$\phi(r) = -[\ln(1-r)]/r \ . \tag{19}$$

(19) was used in the preparation of fig. 2 and will be used again later.]

6.4 Pure exponential pulses all of the same height

The exact $\Omega(D)$ is presented in II. Ruin theory gives:

$$\Omega(D) = \frac{\exp\{X[Ei(r) - \gamma - \ln r] - rD\}}{[2\pi r(Xe^r - D)]^{1/2}} \ , \tag{20}$$

where r derives from:

$$\frac{D}{X} = (e^r - 1)/r \ . \tag{21}$$

The comparison between the exact $\Omega(D)$ and ruin theory is made in fig. 13 where again excellent agreement is seen down to $X = 0.1$ below which divergences become substantial.

6.5 Pure exponential pulses with an exponential height distribution

Here the exact $\Omega(D)$ is remarkably simple [5]:

$$\Omega(D) = \frac{\Gamma(X, D)}{\Gamma(X)} \tag{22}$$

while that for ruin theory is also in simple closed form:

$$\Omega(D) = \frac{\exp\left\{X\left(\ln\frac{D}{X} + 1\right) - D\right\}}{\left(\frac{D}{X} - 1\right)(2\pi X)^{1/2}} \ . \tag{23}$$

Fig. 14 makes the comparison between the exact $\Omega(D)$ and ruin theory: agreement is again excellent for larger values of X, but now already for $X = 0.1$ the discrepancy is about a factor 2 while for lower values of X ruin theory is evidently quite unreliable.

6.6 Summary of tests

Ruin theory is not designed to apply to small values of X but we have seen, in all five cases considered, which span a wide range of $c(t)$ and $g(c_{max})$, that in no case is it in error by more than about 30% at $X = 0.5$ or by a factor of 2 even at $X = 0.1$. We have also seen that different cases behave differently at low X-values in respect of the reliability of ruin theory; this is only to be expected: for example the ratio of the ruin theory to the exact values of $\Omega(D)$ for the cases that we have treated in sects. 6.1, 6.2 and 6.5, and for which general analytical forms are available for both the exact and the ruin theory solutions, go as $\ln^{-1}(D/X)$, $(DX)^{-1/4}$ and $X^{-1/2}$, respectively.

7. Synthesis of pulse form $c(t)$

Ruin theory may be applied to short pulses of arbitrary form $c(t)$ and all of the same height by synthesizing $c(t)$ out of segments each of which has known $f(c)$, $\phi(r)$ and grafting them together appropriately at various values of t. Such tractable segments are: constant, linear, quadratic, exponential; they will now be individually listed with their $f(c)$ individually normalized so that the resultant $\phi(r)$ would be appropriate were that segment of $c(t)$ the entire pulse: in practice the several $f(c)$ must be added together to derive an overall normalization for the total $f(c)$ of the full $c(t)$, this overall normalization then carrying through to the individual $\phi(r)$ that will be added together to give the final overall $\phi(r)$ for the full $c(t)$.

7.1 Constant

$c(t) = a$

$f(c) = \delta(a)$

$\phi(r) = e^{ra}$

7.2 Linear

$$c(t) = a + bt \qquad c_1 < c < c_2$$

$$f(c) = \frac{1}{c_2 - c_1}$$

$$\phi(r) = \frac{1}{c_2 - c_1}\,\frac{1}{r}[e^{rc_2} - e^{rc_1}]$$

7.3 Quadratic

$$c(t) = a + bt + dt^2 \qquad c_1 < c < c_2$$

$$f(c) = \frac{2d}{y_2 - y_1}$$

$$\phi(r) = \frac{\sqrt{\pi}}{2}\,\frac{1}{x_2 - x_1} e^{rL/4d}\,\{\mathrm{erf}\,x_2 - \mathrm{erf}\,x_1\}$$

where: $y = (L + 4dc)^{1/2}$; $L = b^2 - 4ad$; $x = \frac{1}{2}\sqrt{-r/d}\;\; y$.

7.4 Exponential

$$c(t) = a + be^{\pm dt} \qquad c_1 < c < c_2$$

$$f(c) = A\frac{1}{c - a}$$

$$\phi(r) = A\,e^{ra}\{Ei(z_2) - Ei(z_1)\}$$

where $z = r(c-a)$; $A = \ln^{-1}\left[\frac{c_2 - a}{c_1 - a}\right]$.

7.5 Illustration of synthesis

As an illustration of the synthesis of $c(t)$ consider the sugarloaf, *viz.* itself a pure quadratic for which, with height unity:

$$\phi(r) = \frac{1}{2}\left(\frac{\pi}{r}\right)^{1/2} e^{r}\,\mathrm{erf}\sqrt{r}\ .$$

Now synthesize the sugarloaf out of n straight line segments that touch the sugarloaf, that is itself of duration unity, at equal time increments of $\frac{1}{n}$ so that for n=2 we have the sawtooth, and as n (even) increases we approximate more and more closely to the sugarloaf. Fig. 15 shows the result for n=2,4,6 where the convergence towards the sugarloaf's $\Omega(D)$ is well seen.

With the above list of analytically tractable segments we can approximate the desired arbitrary $c(t)$ for short pulses as closely as wished and can verify the convergence at successive stages of elaboration. For exponentially tailed pulses we encounter the normalization problem referred to above which is solved, as there mentioned, by cutting off the infinite exponential tail at a very small value which may be taken to zero in the final formula for $\Omega(D)$. Examples of this will be given in II.

8. Examples of useful pulse forms

The pulse forms so far considered have illustrated the tremendous sensitivity of $\Omega(D)$ to that form. We should wish to have available estimates of $\Omega(D)$ for pulse forms more closely resembling those encountered in practice. This can be done, as explained in sect. 7, by appropriate grafting, but it is desirable to have available some semi-realistic pulse forms against which actual pulse forms may be quickly checked as an orientation prior to their more realistic simulation by grafting. Two such pulse forms are the double sugarloaf and the exponential/sugarloaf.

8.1 Double sugarloaf

This pulse form consists of a sugarloaf centre with a portico also of quadratic form and a (finite) tail of the same form as illustrated in fig. 16; the portico and tail have zero slope at t=0 and 1, respectively; the junctions at which the smooth connections are effected are at $c(t) = f$ in terms of the maximum pulse height. (All pulses are of equal height unity).

We have:

$$\frac{D}{X} = \frac{1}{2r}(\mathrm{e}^{rf} - \phi(r)) + M$$

$$\phi(r) = \sqrt{\frac{f}{r}}\mathrm{daw}\sqrt{rf} + M \ ,$$

where:

$$M = \frac{1}{2}\sqrt{1-f}\, e^r \sqrt{\frac{\pi}{r}}\, \mathrm{erf}\sqrt{r(1-f)}$$

$$\phi''(r) = \frac{1}{2r}\left\{\left(1-\frac{1}{r}\right)e^{rf} - \frac{D}{X} + \frac{1}{r}\phi(r)\right\} + \left(1-\frac{1}{2r}\right)M$$

and daw is the Dawson integral:

$$\mathrm{daw}\, x = \int_0^x e^{t^2}\, dt\ .$$

Fig. 17 illustrates the dependence of $\Omega(D)$ upon f.

8.2 Sugarloaf/exponential

This pulse form also has a sugarloaf centre but now the portico and tail are exponentials: the portico is $T(e^{kt}-1)$ and the tail its mirror. The smooth junctions between exponential and sugarloaf are at the fraction f of the maximum height (unity) of the (all equal) pulses and at times t_1 and $1-t_1$. This pulse form is illustrated in fig. 18 for the choice $f = 0.2$. For a given choice of f and t_1, T is determined by:

$$(f+T)\ln\left(\frac{f}{T}+1\right) = \frac{4t_1(1-f)}{1-2t_1}$$

and then k by:

$$k\, e^{kt_1} = \frac{4(1-f)}{T(1-2t_1)}\ .$$

Then define:

$$A = \left[\frac{1}{k}\ln\left(\frac{f}{T}+1\right) + \frac{1}{2} - t_1\right]^{-1}$$

when r is determined by:

$$\frac{D}{X} = A\left\{\frac{1-2t_1}{4}\,\frac{1}{r}\,e^{rf} + H\left(1-\frac{1}{2r}\right) + \frac{1}{kr}(e^{rf}-1) - \frac{T}{k}E\right\}\ ,$$

where:

$$
\begin{aligned}
H &= \frac{1-2t_1}{4\sqrt{1-f}}\,\mathrm{e}^{r}\sqrt{\frac{\pi}{r}}\,\mathrm{erf}\sqrt{r(1-f)} \\
E &= Ei(r[f+T]) - Ei(rT) \\
\phi(r) &= A\left\{\frac{E}{k} + H\right\} \\
\phi''(r) &= A\left\{\frac{\mathrm{e}^{rf}}{r}\left[\frac{1}{k}\left(f - T - \frac{1}{r}\right) + \frac{1-2t_1}{4}\left(1 + f - \frac{3}{2r}\right)\right]\right. \\
&\quad \left. + \frac{1}{kr}\left(\frac{1}{r} + T\right) + H\left(1 - \frac{1}{r} + \frac{3}{4r^2}\right) + \frac{T^2}{k}E\right\} .
\end{aligned}
$$

Fig. 19 illustrates the dependence of $\Omega(D)$ upon t_1 for the fixed value $f = 0.2$.

9. Synthesis of pulse-height distribution $g(c_{max})$: Square-waves

We have remarked the extreme sensitivity of $\Omega(D)$ to $g(c_{max})$, in particular to the form and extent of the high-c_{max} tail. Appropriate incorporation of the best information as to $g(c_{max})$ is therefore critical to a realistic assessment of $\Omega(D)$. This may readily be done for the case of square-waves by methods similar to those already employed for the handling of arbitrary $c(t)$, *viz.* by a synthesis of $g(c_{max})$ from forms for which the transform to $\phi(r)$ is readily available since, for square waves, we have simply:

$$
\begin{aligned}
\phi(r) &= \int_0^\infty \mathrm{e}^{rc}\, f(c)\, dc \\
\text{where } f(c) &= g(c_{max}) \, .
\end{aligned}
$$

In the present case, as opposed to that for synthesizing $c(t)$, we may synthesize $g(c_{max})$ either by a grafting of analytically tractable elements, *viz.* the use of different elements for different ranges of c_{max}, or by their superposition, *viz.* the addition of two or more elements for the same range of c_{max}, or by any suitable combination. Useful elements for the synthesis of $g(c_{max})$ are polynomials and exponentials, for which the $\phi(r)$ transforms are elementary and also the Gaussian and gamma distributions which will now be tested.

9.1 Gaussian

$$g(c_{max}) \sim \exp[-(c_{max} - c_0)^2/A]$$

$$\phi(r) = e^B(1 + \mathrm{erf}\, E)/(1 + \mathrm{erf}[c_0/\sqrt{A}])$$

$$\text{where } B = r(Ar/4 + c_0)$$

$$E = \frac{\sqrt{A}}{2}(r + 2c_0/A) \; .$$

[Note that if $\mathrm{erfc}[c_0/\sqrt{A}] << 1$ then $\phi(r)$ reduces to e^B.]

9.2 Gamma distribution

$$g(c_{max}) \sim c_{max}^{\nu-1}\, e^{-c_{max}/\mu} \; .$$

Writing:

$$\theta = D/(X\nu\mu)$$

$$r = (1 - \theta^{-1/(\nu+1)})/\mu$$

$$\phi(r) = \theta^{\nu/(\nu+1)}$$

$$\phi''(r) = \nu(\nu+1)\mu^2\, \theta^{(\nu+2)/(\nu+1)} \; .$$

9.3 Illustrations of synthesis

The gamma distribution, for which the solution has been given in 9.2, displays a usefully wide variety of shapes from a hollow form as $\nu \to 0$ through the exponential for ν=1 to a sharply peaked form, that we have already displayed in fig. 5, as $\nu \to \infty$. We will now use it to illustrate synthesis of $g(c_{max})$.

Fig. 20 shows the result of synthesizing the gamma distribution of $\nu = 0.5$, *viz.* a "hollowed exponential" by three superposed exponentials for which the fit to the "exact" (i.e. full ruin) result is seen to be very close. Also shown is the best that can be done with a single exponential, adjusted to give exact fit for $X = 0.5$ at $D = 11.5$.

Fig. 21 shows the result of synthesizing the gamma distribution of $\nu = 10$, *viz.* curve A of fig. 5, from three superposed Gaussians; the fit is seen to be excellent. Also shown is the best that can be done with a single Gaussian, adjusted to give exact fit for $X = 0.5$ at $D = 8$.

It is important to establish that such synthesis of $g(c_{max})$ is valid even when the $g(c_{max})$ may not at all resemble any single simple form. To illustrate this consider again square-waves where $g(c_{max})$ is that of a gamma distribution of $\nu = 10$ (curve A of fig. 5) accompanied by N times as many pulses belonging to an exponential distribution of mean height 5 times smaller than that corresponding to the maximum of the gamma distribution. In this case the total $g(c_{max})$ is simply the appropriately normalized sum of the gamma and exponential distributions giving the pile-up shown in fig. 22. [Note that X in this figure refers to the rate of arrival of the pulses belonging to the gamma distribution only so that the total pulse rate is $X(1+N)$.] It is seen that the small pulses have a considerable effect on $\Omega(D)$ even though by themselves their pile-up, as shown by the dashed lines for a rate of $10X$ per unit time, is very small; they achieve this effect by "riding on the backs" of the larger gamma-distribution pulses. The test that we now seek for the reliability of ruin theory for a mixture of very different pulse-height distributions $g(c_{max})$ is provided by the circles in fig. 22 which have been computed, for $N=6$, by numerically convoluting the exact differential pile-up function for the exponential distribution (*viz.* the differential of (11) or (12) above) for pile-up into the range D_e to $D_e + dD_e$ with the $\Omega(D-D_e)$ of ruin theory for the gamma distribution: the fit is excellent.

10. Synthesis of pulse-height distribution $g(c_{max})$: Arbitrary pulse form

We now have confidence that we can handle the pile-up of arbitrary pulse forms $c(t)$ when the pulses are all of the same height and that we can handle square waves of arbitrary pulse-height distribution $g(c_{max})$. But how can we handle pulses of arbitrary form and of arbitrary distributions of height?

There is no general solution to this problem but we may gain useful insight into it by comparing the pile-up of pulses of very different form and of very different pulse-height distribution. Specifically, compare the pile-up of the "bluntest" pulses possible, namely square-waves, with the "sharpest" that it is reasonable to consider, namely sawteeth for short pulses and pure exponentials for tailed pulses, both pulse forms having pulse-height distributions of the narrowest possible namely the delta-function and also the broadest possible namely the exponential distribution. In other words, we ask if the effect on the pile-up of going from a delta-function to an exponential height distribution for sharp pulses is similar to the effect for square-waves; if it is then we may be reasonably confident that the effect of going from a delta-function height distribution to the arbitrary $g(c_{max})$ for the arbitrary pulse form $c(t)$ will be similar to the effect of going from a delta-function height distribution to the same $g(c_{max})$ for square-waves. Since we know the pile-up for the arbitrary $c(t)$ with the delta-function height distribution and also the pile-up for square-waves with the arbitrary $g(c_{max})$, the above comparison would yield an estimate of the desired pile-up for the arbitrary $c(t)$ with the arbitrary $g(c_{max})$.

We now need to define a procedure for making the comparisons. This we illustrate for definiteness in terms of the comparison between square-waves (SW) and sawteeth (ST). $\Omega(D)_{\mathrm{SWD}}$ stands for the pile-up of square-waves with a delta-function height distribution and $\Omega(D)_{\mathrm{SWE}}$ for that of square-waves with the exponential distribution $g(c_{max}) = e^{-c_{max}}$. $\Omega(D)_{\mathrm{STD}}$ and $\Omega(D)_{\mathrm{STE}}$ stand similarly for the sawteeth. The procedure is now as follows:

(i) For a given X and D for the sawteeth define

$\mathrm{ST} = \Omega(D)_{\mathrm{STE}}/\Omega(D)_{\mathrm{STD}}$

(ii) For the same X-value find the D-value, namely D^* that gives

$\Omega(D^*)_{\mathrm{SWD}} = \Omega(D)_{\mathrm{STD}}$

(iii) Define $\mathrm{SW} = \Omega(D^*)_{\mathrm{SWE}}/\Omega(D^*)_{\mathrm{SWD}}$

(iv) Define $R = \mathrm{ST}/\mathrm{SW}$.

The procedure for comparing pure exponentials and square-waves is exactly the same.

Figs. 23 and 24 show the results for R for the comparisons with square-waves of sawteeth and pure exponentials, respectively (in the latter case using the methods of II). It is seen that the ratios R indeed vary very little over the whole range of $\Omega(D)$ of interest in both cases and are also quite similar for the two cases. Since, as remarked, their comparison has involved the extremes of pulse form and pulse-height distribution, we may be reasonably confident that the comparison between the arbitrary $c(t)$, $g(c_{\mathrm{max}})$ and the square-waves will lie within the ranges of figs. 23 and 24 and will involve an uncertainty of no more than a factor of 2 or so in our estimate of R, hence of $\Omega(D)$, even for the smallest values of $\Omega(D)$ that we have been considering here.

For definiteness, the procedure for the arbitrary $c(t)$, $g(c_{\mathrm{max}})$ is as follows in which c and g are used in an obvious notation. For the given X and D:

(i) Find $\Omega(D)_{\mathrm{cD}}$;

(ii) Find D^* such that $\Omega(D^*)_{\mathrm{SWD}} = \Omega(D)_{\mathrm{cD}}$ for the same X-value;

(iii) Find $\mathrm{SW} = \Omega(D^*)_{\mathrm{SW}g}/\Omega(D^*)_{\mathrm{SWD}}$;

(iv) Estimate the likely R-value from figs. 23 and 24;

(v) Then $\Omega(D)_{cg} \simeq R \times \mathrm{SW} \times \Omega(D)_{c\mathrm{D}}$;

(vi) If necessary adjust the guess as to R in the light of the $\Omega(D)_{cg}$-value deduced in (v).

[In these procedures $\Omega(D)_{\mathrm{SWD}}$ is taken as that of (10) above, *viz.* ruin theory without the empirical trimming of 0.4 for D.]

References

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[3] P. Embrechts et al., Adv. Appl. Prob. 17 (1985) 623.

[4] W. Feller, Introduction to probability theory and its applications, Vol. II (Wiley, New York, 1971).

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Figure captions

1. Comparison of pile-up for square-waves (circles), sugarloaves (dash-dot) and sawteeth (full lines) all of $c_{max} = 1$. [For square-waves $\Omega(D)$ is defined only at integral values of D.]

2. Comparison of pile-up for sawteeth: (a) all of equal height $c_{max} = 1$; (b) of an exponential distribution of pulse height $g(c_{max}) = e^{-c_{max}}$. Case (a): solid lines; case(b): dashed lines. X-values given on the curves.

3. As for fig. 2 but for pulses of pure exponential form $c(t) = e^{-t}$.

4. Pile-up of square waves with the distribution $g(c_{max}) = e^{-c_{max}}$, that distribution being cut off at $c_{max} = k$. k-values given on the curves.

5. Gamma distributions $g(c_{max}) \sim c_{max}^{\nu-1} e^{-(\nu-1)c_{max}}$ reduced to the same maximum value of $g(c_{max})$. Curves A, B and C are for $\nu = 10$, 18.5 and 100, respectively.

6. Pulse pile-up for square-waves having the gamma-distribution forms for $g(c_{max})$ as shown in fig. 5. A, B and C have here the same connotation as in fig. 5 while the curve labelled δ is for a delta-function distribution of pulse heights ($\nu \to \infty$).

7. Exact solution to the pile-up of square-waves of equal height unity (circles) compared with the Gaussian approximation (full line). X-values given on the curves.

8. Exact solution to the pile-up of pulses of the pure exponential form $c(t) = e^{-t}$ of equal height unity (full line) compared with the Gaussian approximation (dashed line). For values of D below the vertical line of dots the dashed line includes the correction term given in the text. [The dashed lines are at the

integral value of D below $(D-X)/\sqrt{X} = X^{1/6}$ for which the correction term becomes valid.] X-values given on the curves.

9. Comparison between the exact $\Omega(D)$ for square-waves all of the same height (circles) and ruin theory evaluated at $D-0.4$ (full curves). X-values given on the curves.

10. Comparison between the exact $\Omega(D)$ for square-waves with an exponential height distribution (full curves) and ruin theory (circles). Also shown as the dashed curves is the analytical approximation (13) to the exact $\Omega(D)$. The triangles show the exact $\Omega(D)$ for the corresponding delta-function distribution of pulse heights. X-values given on the curves.

11. As for fig. 10 for smaller X-values as given on the curves.

12. Comparison between the exact $\Omega(D)$ for sawteeth all of the same height (full curves) and ruin theory (circles). X-values given on the curves.

13. Comparison between the exact $\Omega(D)$ for pure exponential pulses all of the same height (full curves) and ruin theory (circles). X-values given on the curves.

14. Comparison between the exact $\Omega(D)$ for pure exponential pulses with an exponential distribution of pulse height (full curves) and ruin theory (circles). X-values given on the curves.

15. $\Omega(D)$ following synthesis of sugarloaf pulse form by straight lines. The dashed line is for the full sugarloaf; the full lines result from synthesizing the sugarloaf from the number of straight-line segments given on the curves.

16. Double sugarloaf pulse form. The sugarloaf centre is joined smoothly to quadratic

portico and tail at the fractions f of the maximum pulse height given on the curves.

17. Pulse pile-up for the double sugarloaf pulse form of fig. 16. The fractions f of the maximum pulse height at which the smooth junctions are effected are given on the curves.

18. Sugarloaf/exponential pulse form. The sugarloaf centre is joined smoothly to exponential portico and tail at a fraction f of the maximum pulse height ($f = 0.2$ in the figure) and at the t_1-values given on the curves. The dashed line is the pure sugarloaf.

19. Pulse pile-up for the sugarloaf/exponential pulse form of fig. 18. The smooth junctions are effected at $f = 0.2$ and at the t_1-values given on the curves. The dashed curve is for the pure sugarloaf.

20. Pulse pile-up for square-waves having the gamma distribution $g(c_{\max}) \sim c_{\max}^{-0.5} e^{-c_{\max}}$ (full line) compared with that for a superposition of three exponentials simulating its form (circles). The dashed line shows the best that can be done using a single exponential adjusted to give exact fit for $X = 0.5$ at $D = 11.5$. X-values given on the curves.

21. Pulse pile-up for square-waves having the gamma distribution $g(c_{\max}) \sim c_{\max}^{9} e^{-9c_{\max}}$ (full line) compared with that for a superposition of three Gaussians simulating its form (circles). The dashed line shows the best that can be done using a single Gaussian adjusted to give exact fit for $X = 0.5$ at $D = 8$. X-values given on the curves.

22. $\Omega(D)$ given by ruin theory for a combination of X pulses per unit time of square-

waves belonging to the gamma distribution illustrated as curve A of fig. 5 plus N times as many pulses of an exponential distribution of mean height 5 times less than the height corresponding to the maximum of the gamma distribution. N-values are given on the curves. The dashed lines show the pile-up that would result from the small exponentially distributed pulses alone for a rate $10X$ per unit time. The circles show, for $N=6$, the result of numerically convoluting the exact (differential) result for the exponential distribution with ruin theory for the gamma distribution.

23. The effect on pile-up of passing from a delta-function distribution of pulse heights to an exponential distribution of pulse heights for sawteeth compared with the similar effect for square-waves. R measures the ratio of the effects as described in the text. $\Omega(D)_{\mathrm{STE}}$ refers to sawteeth with an exponential height distribution.

24. As for fig. 23 but comparing pure exponential pulses with square-waves, $\Omega(D)_{\mathrm{EE}}$ now referring to pure exponential pulses with an exponential height distribution.

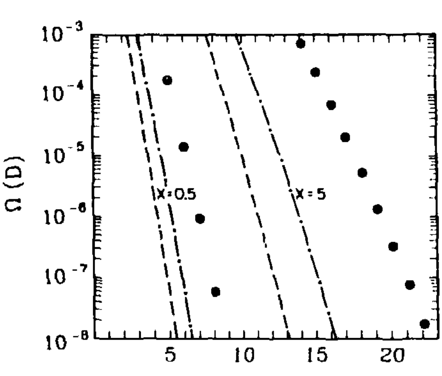

Fig. 1

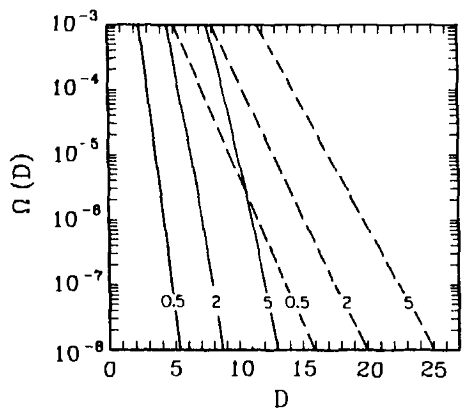

Fig. 2

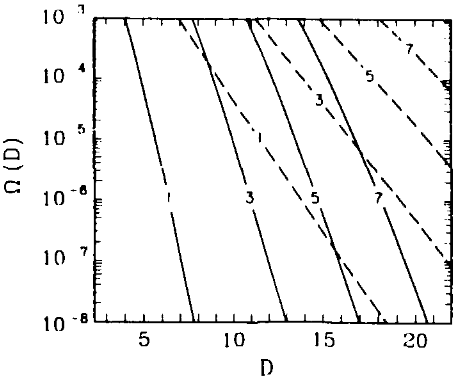

Fig. 3

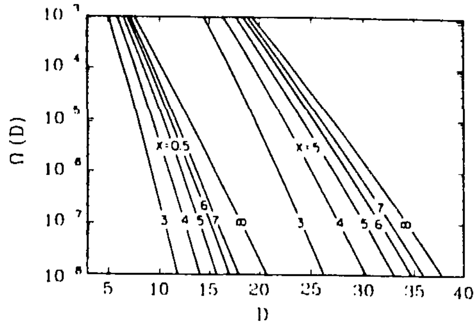

Fig. 4

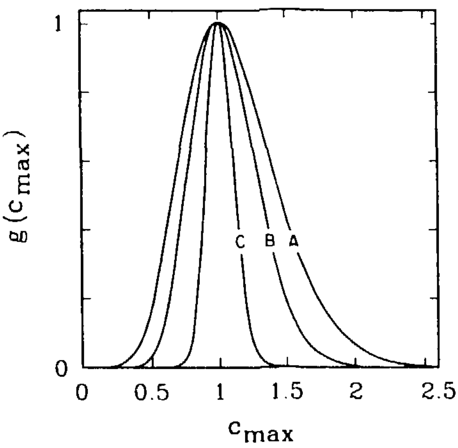

Fig. 5

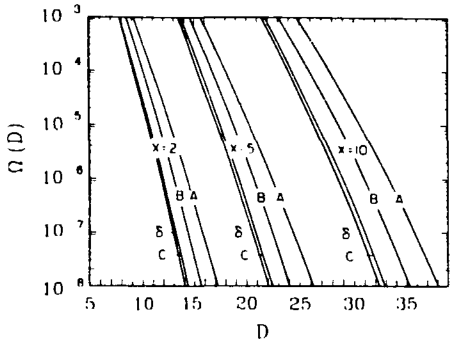

Fig. 6

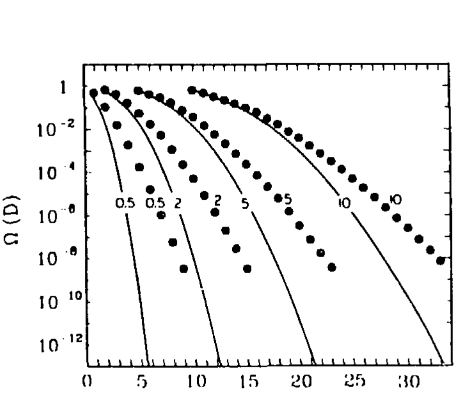

Fig. 7

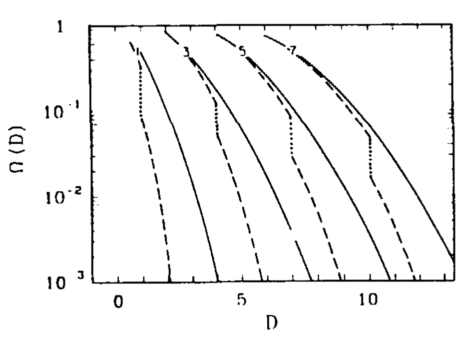

Fig. 8

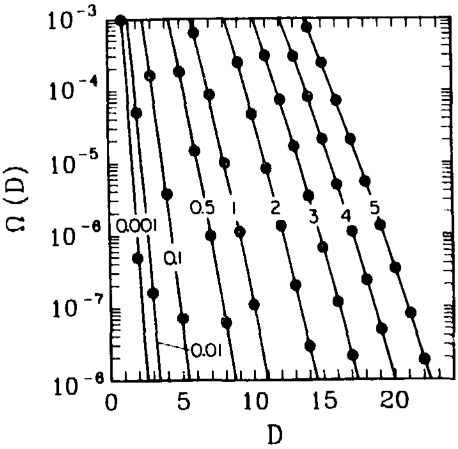

Fig. 9

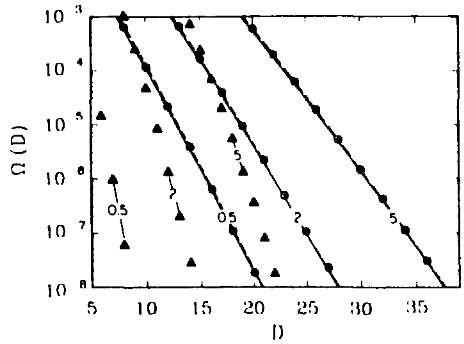

Fig. 10

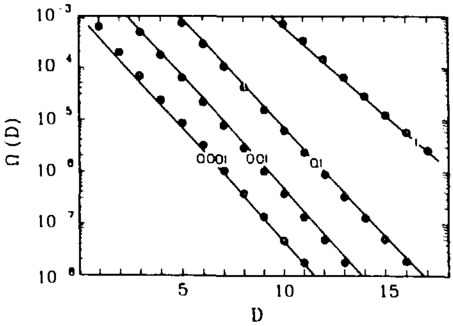

Fig. 11

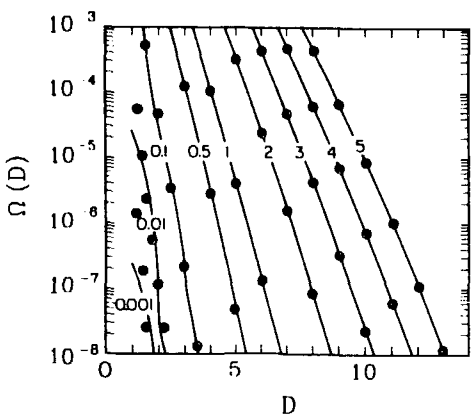

Fig. 12

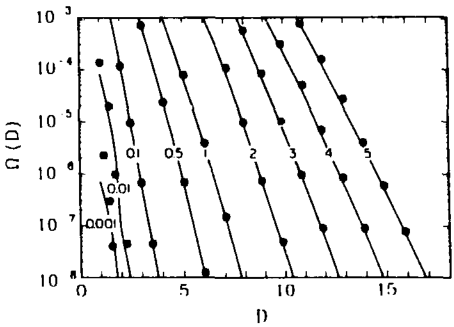

Fig. 13

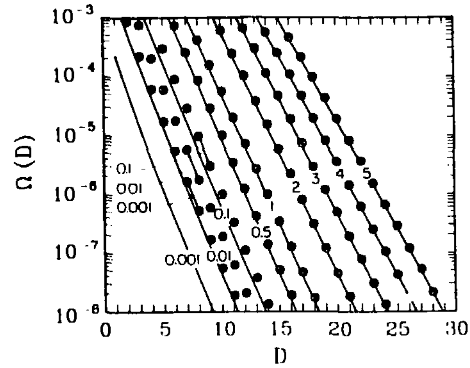

Fig. 14

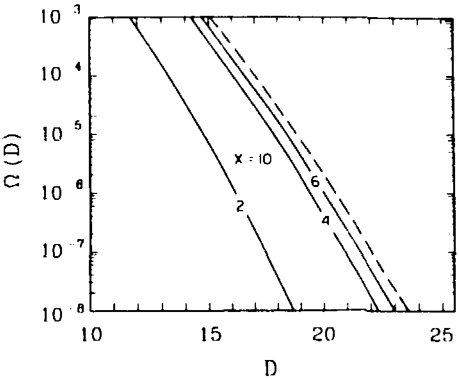

Fig. 15

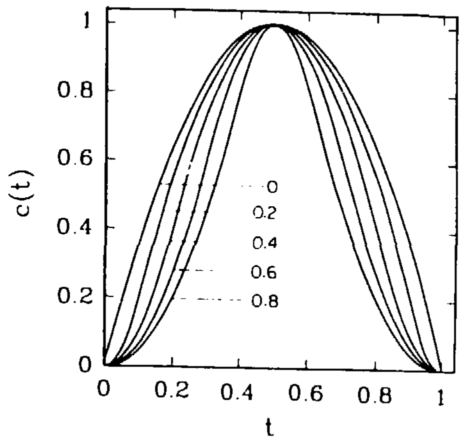

Fig. 16

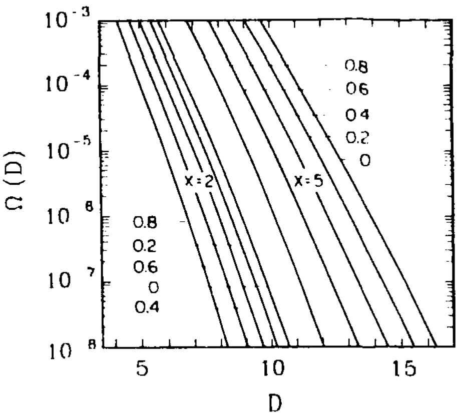

Fig. 17

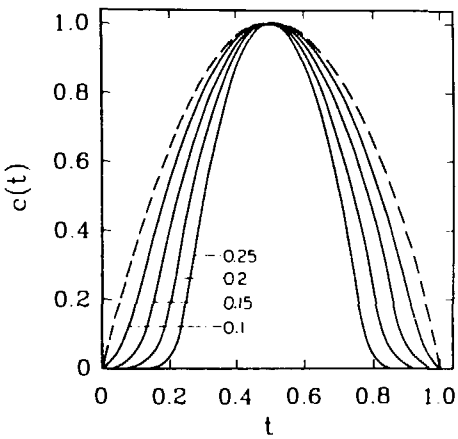

Fig. 18

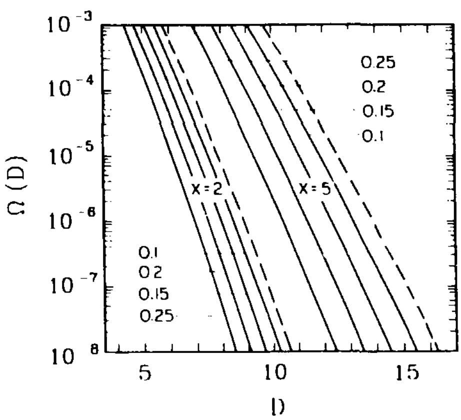

Fig. 19

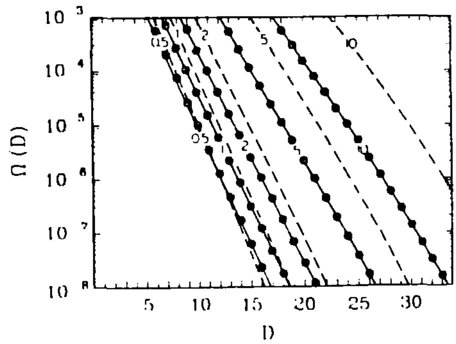

Fig. 20

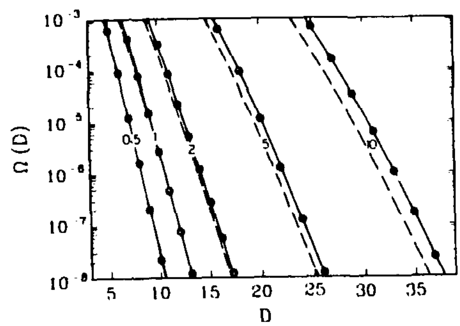

Fig. 21

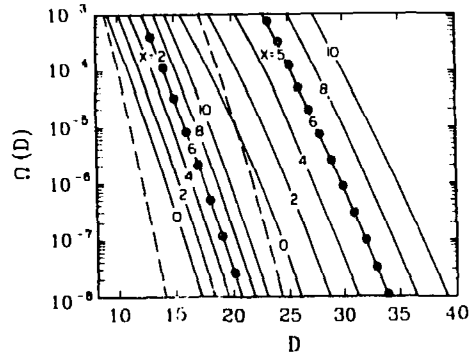

Fig. 22

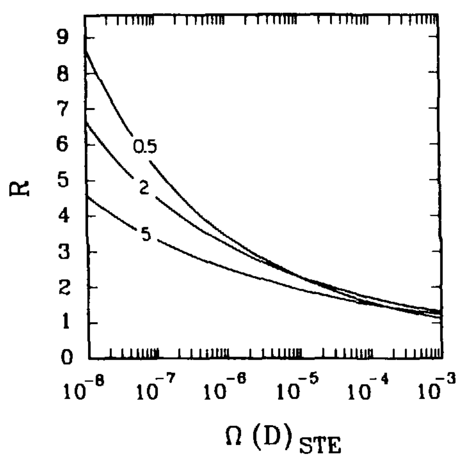

Fig. 23

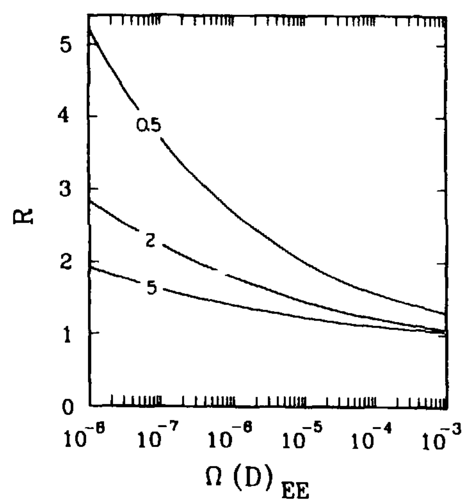

Fig. 24