Department of PHYSICS

BUP--1992-06.

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University of Bergen, Department of Physics
Scientific/Technical Report No. 1992-06
ISSN 0803-2696

TWO-LOOP LADDER-DIAGRAM CONTRIBUTIONS TO BHABHA SCATTERING
III. THE ϕ^3-LIMIT OF QED

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ABSTRACT

We evaluate, in the high-energy limit, the sum of the Feynman amplitudes corresponding to the six two-loop ladder-like diagrams in Bhabha scattering. This is the limit where $s \to \infty$, while t, the electron mass m and the photon mass λ are all being held fixed. In this limit the sum of the six Feynman amplitudes does not depend on the electron mass. When specialized to the region $s \gg |t| \gg m^2 \gg \lambda^2$ this result complements the one previously obtained. The connection with ϕ^3 theory is also investigated.

1. Introduction

In two-body scattering in QED, with the photon mass λ as an infrared regulator, there are four independent dimensionful quantities, s, t, m^2, and λ^2. Thus, apart from an over-all scale, the scattering amplitude is in general a function of *three* dimensionless quantities. Clearly, such an amplitude may have a very rich structure.

In two previous papers [1], [2], we discussed the contribution to Bhabha scattering from the six two-loop ladder-like diagrams that together constitute a gauge-invariant set. The limit considered in [2], is $\lambda^2 \to 0$, with $s \gg |t| \gg m^2$ being held fixed. In the present paper, we discuss the amplitudes arising from the same set of diagrams but in the limit where λ, m and t are kept fixed, while $s \to \infty$. This limit is easier to handle since only a single Mellin transform is needed. Simultaneously, we get the corresponding results for ϕ^3 theory, basically by taking $\lambda = m$ and removing the complications arising from the spinor couplings of QED.

For the uncrossed ladder diagram, we found the QED amplitude in the large-s limit to be given by the following integral [1],

$$\mathcal{M}_a = \frac{i\alpha^3}{4\pi} \int_0^1 \cdots \int_0^1 d\alpha_1 \dots d\alpha_7 \, \delta(1 - \sum_{j=1}^{7} \alpha_j) \frac{1}{\Lambda^a(\alpha)^3}$$
$$\times \left[\frac{2}{D^a(\alpha)^3} \, N^a_{III} + \frac{1}{2D^a(\alpha)^2} \, N^a_{II} + \frac{1}{4D^a(\alpha)} \, N^a_I \right], \tag{1.1}$$

where

$$\Lambda^a \equiv \Lambda(\alpha) = (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)(\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) - \alpha_6^2, \tag{1.2}$$

$$D^a(\alpha) = D_s s + D_t t + D_m m^2 + D_\lambda \lambda^2 + i\epsilon, \tag{1.3}$$

with

$$\begin{aligned}
D_s &= \alpha_1\alpha_3(\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_2\alpha_4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6(\alpha_1\alpha_4 + \alpha_2\alpha_3), \\
D_t &= \alpha_5\alpha_6\alpha_7, \\
D_m &= -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\Lambda + (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4)(\alpha_5 + \alpha_7) \\
&\quad + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(\alpha_5\alpha_7 + \alpha_5\alpha_6 + \alpha_6\alpha_7), \\
D_\lambda &= -(\alpha_5 + \alpha_6 + \alpha_7)\Lambda.
\end{aligned} \tag{1.4}$$

The numerators N^a_{III}, N^a_{II}, and N^a_I are given by equation (I.6.9) of paper I (we refer to equations from ref. [1] by the prefix 'I.').

Similarly, the amplitude $\mathcal{M}_b$ for the once-crossed diagram is given by an expression identical to (1.1), but with numerators given by eq. (I.6.10), and Λ^b and $D^b(\alpha)$ given by eqs. (I.4.8) and (I.4.9), respectively, as

$$\begin{aligned}\Lambda^b = \Lambda(\alpha) &= (\alpha_1+\alpha_3+\alpha_5)(\alpha_2+\alpha_4+\alpha_6+\alpha_7)+(\alpha_4+\alpha_6)(\alpha_2+\alpha_7)\\ &= (\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_2+\alpha_4+\alpha_6+\alpha_7)-(\alpha_4+\alpha_6)^2, \end{aligned} \tag{1.5}$$

and

$$D^b(\alpha) = D_s s + D_u u + D_t t + D_m m^2 + D_\lambda \lambda^2 + i\epsilon, \tag{1.6}$$

with D_s, D_u, D_t, D_m, and D_λ given by eq. (I.4.10). Unless confusion may arise, we shall leave out the indices a and b used here to distinguish quantities referring to the two different diagrams.

As indicated by eq. (I.7.10), the full amplitude corresponding to the sum of the six diagrams, can be constructed from just these two amplitudes $\mathcal{M}_a$ and $\mathcal{M}_b$, with suitable substitutions among the kinematical variables s, t and u.

In ϕ^3 theory, the amplitudes corresponding to the considered six diagrams have a much simpler structure. There are two essential simplifications: first the numerator simplifies since there are no spinors, and secondly the denominator simplifies since there is only one mass scale. Thus, in ϕ^3 theory, for the a diagram, the corresponding amplitude is

$$\bar{\mathcal{M}}_a = \frac{-ig^6}{(16\pi^2)^2}\int_0^1 \cdots \int_0^1 d\alpha_1 \ldots d\alpha_7\, \delta(1-\sum_{j=1}^{7}\alpha_j)\frac{2\bar{N}^a}{\Lambda^a(\alpha)^3\bar{D}^a(\alpha)^3} \tag{1.7}$$

where

$$\bar{N}^a = [\Lambda^a(\alpha)]^4, \tag{1.8}$$

$$\bar{D}^a(\alpha) = D^a(\alpha)\Big|_{\lambda=m}. \tag{1.9}$$

Hence, the ϕ^3 results are easily extracted from the QED results.

2. The integrals I_{III}, I_{II}, and I_I

We shall not evaluate the complete amplitude $\mathcal{M}_a$ for the uncrossed diagram, but only that part which contributes to the unpolarized differential cross section. In our previous papers I and II this part was denoted $F_{00}^a(s,t)$, and defined in eq. (I.2.4). Its decomposition is identical to that of $\mathcal{M}_a$,

$$F_{00}^a = 2I_{III}^a(s,t) + \tfrac{1}{2}I_{II}^a(s,t) + \tfrac{1}{4}I_I^a(s,t), \tag{2.1}$$

with

$$I^a_{III}(s,t) = \int_0^1 \dots \int_0^1 d\alpha_1 \dots d\alpha_7\, \delta(1-\sum_{j=1}^{7}\alpha_j)\, \frac{\tilde{N}^a_{III}}{\Lambda^a(\alpha)^3 D^a(\alpha)^3},$$

$$I^a_{II}(s,t) = \int_0^1 \dots \int_0^1 d\alpha_1 \dots d\alpha_7\, \delta(1-\sum_{j=1}^{7}\alpha_j)\, \frac{\tilde{N}^a_{II}}{\Lambda^a(\alpha)^3 D^a(\alpha)^2},$$

$$I^a_{I}(s,t) = \int_0^1 \dots \int_0^1 d\alpha_1 \dots d\alpha_7\, \delta(1-\sum_{j=1}^{7}\alpha_j)\, \frac{\tilde{N}^a_{I}}{\Lambda^a(\alpha)^3 D^a(\alpha)}. \tag{2.2}$$

The functions $\tilde{N}$ are obtained by setting $W^{(+)} = W^{(-)} = 1$ in the functions N entering eq. (1.1), and defined in eq. (I.6.9),

$$\begin{aligned}
\tilde{N}^a_{III} &= N^a_{III}\big|_{W^{(+)}=W^{(-)}=1} \\
&= 2s^2[a\tilde{a}c\tilde{c} + (a\tilde{a}+b\tilde{b})(c\tilde{c}+d\tilde{d})], \\
\tilde{N}^a_{II} &= N^a_{II}\big|_{W^{(+)}=W^{(-)}=1}, \\
\tilde{N}^a_{I} &= N^a_{I}\big|_{W^{(+)}=W^{(-)}=1}\,.
\end{aligned} \tag{2.3}$$

Similar expressions hold for the amplitude F^b_{00} arising from the once-crossed diagram.

Among the three terms in eq. (2.1), the first one will dominate. A major part of the present paper is therefore devoted to a study of $I^a_{III}(s,t)$ and $I^b_{III}(s,t)$. The integrals $I_{II}(s,t)$ and $I_I(s,t)$ have been shown [2] to contain additional powers of $1/s$ and are therefore neglected.

3. General structure of I_{III}

As $s \to \infty$, the dominant terms in $I_{III}(s,t)$ behave like $(1/s^2)\log s$ and $1/s^2$. They can be determined by a Mellin transform [3] (see also Appendix A of [2]),

$$I_{III}(s,t) = s^2 I(s,t), \tag{3.1}$$

$$\tilde{I}(\zeta,t) = \int_0^\infty ds\, s^{-\zeta+1}\, I(s,t). \tag{3.2}$$

Observe that we here deviate from the procedure followed in paper II, in that t is kept as an independent variable, which is not Mellin transformed. We write the denominator function $D(\alpha)$ as

$$D(\alpha) = D_s(\alpha)s + \mathcal{D}(\alpha), \tag{3.3}$$

where, for the a-diagram [see eq. (1.3)],

$$\mathcal{D}^a(\alpha) = D_t t + D_m m^2 + D_\lambda \lambda^2 + i\epsilon. \tag{3.4}$$

Furthermore, we extract two powers of s by defining

$$N(\alpha) = \tilde{N}_{III}/s^2. \tag{3.5}$$

Performing the integration over s in eq. (3.2) the Mellin transform becomes

$$\begin{aligned}\tilde{I}(\zeta,t) &\simeq \frac{1-\zeta}{2}\int_0^1 \dots \int_0^1 d\alpha_1 \dots d\alpha_7\, \delta(1-\sum_{j=1}^{7}\alpha_j)\frac{N(\alpha)}{\Lambda(\alpha)^3}\,[D_s(\alpha)]^{-2+\zeta}\,[\mathcal{D}(\alpha)]^{-1-\zeta} \\ &\equiv (1-\zeta)\Big(\frac{A_0}{\zeta^2}+\frac{A_1}{\zeta}\Big) \\ &\simeq \frac{A_0}{\zeta^2}+\frac{A_1-A_0}{\zeta},\end{aligned} \tag{3.6}$$

where we have neglected terms that are finite when $\zeta \to 0$. Inverting the Mellin transform (see Appendix A of [2]), we get the asymptotic form

$$I(s,t) = \frac{1}{s^2}\big[A_0 \log s + (A_1 - A_0)\big], \tag{3.7}$$

in terms of the coefficients A_0 and A_1 defined by eq. (3.6) above.

Substituting for $N^a(\alpha)$ according to eqs. (2.3) and (I.4.6) for the a-diagram, or the appropriate equations for the b-diagram, we see that the integrand is homogeneous in the α-variables, and of degree -7. We may thus use a theorem of Cheng and Wu [3] (see Appendix B of [2]) to transfer the δ-function constraint to a subset of the variables of integration,

$$\begin{aligned}\tilde{I}(\zeta,t) \simeq \frac{1-\zeta}{2}\int_0^\infty \dots \int_0^\infty d\alpha_1 \dots d\alpha_4 \int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7) \\ \times \frac{N(\alpha)}{\Lambda(\alpha)^3}[D_s(\alpha)]^{-2+\zeta}\,[\mathcal{D}(\alpha)]^{-1-\zeta}.\end{aligned} \tag{3.8}$$

The only factor here that depends on the kinematics, is $\mathcal{D}(\alpha)$. We continue to follow the procedure of ref. [3], and rescale the Feynman parameters associated with the fermion lines,

$$\begin{aligned}\alpha_1 &= \rho x, &\quad \alpha_2 &= \rho(1-x), \\ \alpha_3 &= \rho' y, &\quad \alpha_4 &= \rho'(1-y).\end{aligned} \tag{3.9}$$

Substituting into $D_s(\alpha)$, we obtain

$$D_s(\alpha) \equiv \rho\rho' Q(\alpha), \tag{3.10}$$

where for the a-diagram

$$Q^a(\alpha) = \alpha_5(1-x)(1-y) + \alpha_6 + \alpha_7 xy + \rho x(1-x) + \rho' y(1-y). \tag{3.11}$$

In an effort to keep the notation as simple as possible, we have denoted the argument of Q^a by α. In the following, we shall also use α to refer to a subset of the α-variables. Thus, α may refer to the full set, $\alpha_1, \ldots, \alpha_7$, to $\rho, \rho', x, y, \alpha_5, \alpha_6, \alpha_7$, or simply to $\alpha_5, \alpha_6, \alpha_7$.

We now return to eq. (3.6), which can be expressed as

$$\begin{aligned}\frac{A_0}{\zeta^2} + \frac{A_1}{\zeta} \simeq \int_0^1 \cdots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_0^1 dx \int_0^1 dy \\ \times \int_0^\infty d\rho \int_0^\infty d\rho'\, (\rho\rho')^{-1+\zeta}\, \frac{N(\alpha)}{2\Lambda(\alpha)^3}\, [Q(\alpha)]^{-2+\zeta}\, [\mathcal{D}(\alpha)]^{-1-\zeta}.\end{aligned} \tag{3.12}$$

The important point is now that the singularities in ζ can only arise from the regions where $\rho \to 0$ and/or $\rho' \to 0$. In order to extract the coefficients A_0 and A_1, we split the integrations over ρ and ρ' as follows,

$$\int_0^\infty d\rho \int_0^\infty d\rho' = \int_0^1 d\rho \int_0^1 d\rho' + 2\int_1^\infty d\rho \int_0^1 d\rho' + \int_1^\infty d\rho \int_1^\infty d\rho'. \tag{3.13}$$

Since the original integrand is symmetric under the simultaneous interchanges, $\rho \leftrightarrow \rho'$, $x \leftrightarrow y$, there are two identical contributions. This symmetry accounts for the factor of two in front of the second term. It follows that

$$\begin{aligned}\frac{A_0}{\zeta^2} + \frac{A_1}{\zeta} \simeq \int_0^1 \cdots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_0^1 dx \int_0^1 dy \\ \times \left\{ \int_0^1 d\rho \int_0^1 d\rho' + 2\int_1^\infty d\rho \int_0^1 d\rho' + \int_1^\infty d\rho \int_1^\infty d\rho' \right\} (\rho\rho')^{-1+\zeta} \\ \times \frac{N(\alpha)}{2\Lambda(\alpha)^3}\, [Q(\alpha)]^{-2+\zeta}\, [\mathcal{D}(\alpha)]^{-1-\zeta}.\end{aligned} \tag{3.14}$$

The third term of eq. (3.14) does not contribute to A_0 nor to A_1.

3.1 THE DOMINANT TERM

As $\zeta \to 0$ the dominant term of eq. (3.14) comes from the first integral and from the region where both ρ and ρ' are small. Furthermore, we can there replace ζ by zero everywhere except in the powers of ρ and ρ', yielding

$$\begin{aligned} A_0 = & \int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1 - \alpha_5 - \alpha_6 - \alpha_7) \\ & \times \int_0^1 dx \int_0^1 dy \, \frac{1}{\mathcal{Q}(\alpha)^2} \, Z(\alpha) \Big|_{\rho=\rho'=0}, \end{aligned} \tag{3.15}$$

with

$$Z(\alpha) = \frac{N(\alpha)}{2\Lambda(\alpha)^3 \mathcal{D}(\alpha)}. \tag{3.16}$$

3.2 THE SUBDOMINANT TERM

There are three separate contributions to the subdominant term, A_1. The first one, $A_1^{(1)}$, is due to the ρ- and ρ'-dependences of the factors $Z(\alpha)$ and $\mathcal{Q}(\alpha)$ in the integrand of the first integral of eq. (3.14). The second one, $A_1^{(2)}$, is due to the fact that the exponents of $\mathcal{Q}$ and $\mathcal{D}$ deviate from -2 and -1, respectively. Finally, the third one, $A_1^{(3)}$, comes from the second integral of eq. (3.14) and from the region where ρ' is small. With

$$A_1 = A_1^{(1)} + A_1^{(2)} + A_1^{(3)}, \tag{3.17}$$

we have

$$\begin{aligned} A_1^{(1)} = & \int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1 - \alpha_5 - \alpha_6 - \alpha_7) \int_0^1 dx \int_0^1 dy \\ & \times \lim_{\zeta \to 0} \zeta \int_0^1 d\rho \int_0^1 d\rho' \, (\rho\rho')^{-1+\zeta} \left\{ \frac{Z(\alpha)}{\mathcal{Q}(\alpha)^2} - \frac{Z(\alpha)}{\mathcal{Q}(\alpha)^2} \Big|_{\rho=\rho'=0} \right\}, \\ A_1^{(2)} = & \int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1 - \alpha_5 - \alpha_6 - \alpha_7) \int_0^1 dx \int_0^1 dy \\ & \times \lim_{\zeta \to 0} \zeta^2 \int_0^1 d\rho \int_0^1 d\rho' \, (\rho\rho')^{-1+\zeta} \, \frac{Z(\alpha)}{\mathcal{Q}(\alpha)^2} \Big\{ \log \mathcal{Q}(\alpha) - \log \mathcal{D}(\alpha) \Big\}, \\ A_1^{(3)} = & \, 2 \int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1 - \alpha_5 - \alpha_6 - \alpha_7) \int_0^1 dx \int_0^1 dy \\ & \times \lim_{\zeta \to 0} \zeta \int_0^1 d\rho' \int_1^\infty d\rho \, (\rho\rho')^{-1+\zeta} \, \frac{Z(\alpha)}{\mathcal{Q}(\alpha)^2}. \end{aligned} \tag{3.18}$$

Some of the integrations over ρ and ρ' are discussed in Appendix A. Invoking those results, we get

$$A_1^{(1)} = \int_0^1 \ldots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_0^1 dx \int_0^1 dy$$
$$\times \int_0^1 \frac{d\rho}{\rho} \left\{ \frac{Z(\alpha)}{\mathcal{Q}(\alpha)^2}\bigg|_{\rho'=0} - \frac{Z(\alpha)}{\mathcal{Q}(\alpha)^2}\bigg|_{\rho=\rho'=0} \right\},$$
$$A_1^{(2)} = \int_0^1 \ldots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_0^1 dx \int_0^1 dy$$
$$\times \left\{ \frac{Z(\alpha)}{\mathcal{Q}^2(\alpha)} \left[\log \mathcal{Q}(\alpha) - \log \mathcal{D}(\alpha)\right] \right\}_{\rho=\rho'=0},$$
$$A_1^{(3)} = 2 \int_0^1 \ldots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_0^1 dx \int_0^1 dy$$
$$\times \int_1^\infty \frac{d\rho}{\rho} \frac{Z(\alpha)}{\mathcal{Q}^2(\alpha)}\bigg|_{\rho'=0}. \tag{3.19}$$

Since all integrands above are to be evaluated at $\rho' = 0$, it follows from eq. (3.9) that the function $Z(\alpha)|_{\rho'=0}$ is independent of y. However, as seen from eq. (3.11), $\mathcal{Q}(\alpha)|_{\rho'=0}$ still depends on y.

Let [cf. eq. (3.16)]

$$Z_0(\rho) \equiv Z(\alpha)\Big|_{\rho'=0} \equiv \frac{N(\alpha)}{2\Lambda(\alpha)^3 \mathcal{D}(\alpha)}\bigg|_{\alpha_3=\alpha_4=0}, \tag{3.20}$$

where, for brevity, we have exhibited explicitly only the dependence on ρ. As already mentioned, $Z_0(\rho)$ is independent of y.

Then

$$A_1^{(1)} = \int_0^1 \ldots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_0^1 \frac{d\rho}{\rho} \int_0^1 dx$$
$$\times \left[Z_0(\rho) \int_0^1 \frac{dy}{\mathcal{Q}^2(\rho)} - Z_0(\rho=0) \int_0^1 \frac{dy}{\mathcal{Q}^2(\rho=0)} \right]_{\rho'=0},$$
$$A_1^{(2)} = \int_0^1 \ldots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_0^1 dx \, Z_0(\rho=0)$$
$$\times \left\{ \int_0^1 \frac{dy}{\mathcal{Q}^2(\rho=0)} \left[\log \mathcal{Q}(\rho=0) - \log \mathcal{D}(\rho=0)\right] \right\}_{\rho'=0},$$
$$A_1^{(3)} = 2 \int_0^1 \ldots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_1^\infty \frac{d\rho}{\rho} \int_0^1 dx \; Z_0(\rho)$$
$$\times \int_0^1 \frac{dy}{\mathcal{Q}^2(\rho)}\bigg|_{\rho'=0}. \tag{3.21}$$

4. Cancellation of dominant terms

For both diagram a and b we have (see appendix B)

$$Z_0(\rho = 0) = \left.\frac{N(\alpha)}{2\Lambda(\alpha)^3\mathcal{D}(\alpha)}\right|_{\rho=\rho'=0} = \frac{-2\Lambda_0(\alpha)}{\alpha_5\alpha_6\alpha_7|t| + \Lambda_0(\alpha)\lambda^2}, \tag{4.1}$$

with

$$\Lambda_0(\alpha) \equiv \alpha_5\alpha_6 + \alpha_6\alpha_7 + \alpha_7\alpha_5, \tag{4.2}$$

i.e., $Z_0(\rho = 0)$ is independent of the electron mass.

We shall find it useful to introduce the following three functions,

$$V(\alpha) \equiv \int_0^1 dx \int_0^1 dy \left.\frac{1}{Q(\alpha)^2}\right|_{\rho=\rho'=0}, \tag{4.3}$$

$$W(\rho, x, \alpha) \equiv \int_0^1 \left.\frac{dy}{Q(\alpha)^2}\right|_{\rho'=0}, \tag{4.4}$$

$$X(\alpha) \equiv \int_0^1 dx \int_0^1 \left.\frac{dy}{Q(\alpha)^2} \log Q(\alpha)\right|_{\rho=\rho'=0}, \tag{4.5}$$

where on the left-hand sides the argument α now refers to the set (α_5, α_6, α_7). The first of these integrals is evaluated below, while the other two are discussed in the following section.

4.1 THE INTEGRAL V

For A_0 (and also for $A_1^{(1)}$) we need the integral

$$V(\alpha) = \int_0^1 dx \int_0^1 dy \left.\frac{1}{Q(\alpha)^2}\right|_{\rho=\rho'=0}, \tag{4.6}$$

for both the a and b diagrams. We start with the a diagram. Taking $Q^a(\alpha)$ from Appendix B, we have

$$\begin{aligned} V^a(\alpha) &= \int_0^1 dx \int_0^1 dy \frac{1}{(\xi y + \eta_0 + i\epsilon)^2}, \\ &= \int_0^1 dx \frac{1}{\eta_0(\eta_0 + \xi)}, \end{aligned} \tag{4.7}$$

with

$$\begin{aligned} \xi &= \alpha_7 x - \alpha_5(1 - x), \\ \eta_0 = \eta|_{\rho=0} &= \alpha_5(1 - x) + \alpha_6 \geq 0. \end{aligned} \tag{4.8}$$

The denominator in eq. (4.7) is always non-negative, so the integral is real. We get

$$V^a(\alpha) = \frac{1}{\Lambda_0(\alpha)}\left[\log\frac{\alpha_5+\alpha_6}{\alpha_6} + \log\frac{\alpha_6+\alpha_7}{\alpha_6}\right]. \tag{4.9}$$

Since the remaining integrand in A_0, eq. (3.15), is symmetric in α_5, α_6, and α_7, we may replace $V^a(\alpha)$ by the more compact expression

$$V^a(\alpha)\big|_{\rm symm} = \frac{2}{\Lambda_0(\alpha)}\,\log\frac{\alpha_5+\alpha_6}{\alpha_6}. \tag{4.10}$$

For the b diagram, the quantities ξ and η_0 in $Q^b(\alpha)$ are interchanged, as compared with $Q^a(\alpha)$. Since the denominator may vanish, some care is required in carrying out the integration. The y-integration yields

$$V^b(\alpha) = \int_0^1 dx\left(\frac{1}{\xi+i\epsilon} - \frac{1}{\xi+\eta_0}\right)\frac{1}{\eta_0}. \tag{4.11}$$

Here, ξ may pass through zero, so the first integral is complex. For the part which is symmetric in the variables α_5, α_6, and α_7, we obtain the result

$$\begin{aligned} V^b(\alpha)\big|_{\rm symm} &= -\frac{1}{\Lambda_0(\alpha)}\left[\log\frac{\alpha_5+\alpha_6}{\alpha_6} + i\pi\right] \\ &= -\tfrac{1}{2}V^a(\alpha)\big|_{\rm symm} - \frac{i\pi}{\Lambda_0(\alpha)}. \end{aligned} \tag{4.12}$$

4.2 THE CANCELLATION

The dominant term, as $s\to\infty$, arising from one particular diagram, is given by eq. (3.15), where the only dependence on x and y is through $Q(\alpha)$. Using the abbreviations (4.1) and (4.3), we find

$$A_0 = \int_0^1 \cdots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7)\, Z_0(\rho=0)\, V(\alpha), \tag{4.13}$$

where the result of the integrations over x and y are given by the integral $V(\alpha)$ of eq. (4.3). Invoking the results (4.10) and (4.12), we can write the contributions to A_0^a and A_0^b as

$$\begin{aligned} A_0^a &= -4C_{31}(t), \\ A_0^b &= 2[C_{31}(t) + iC_{30}(t)], \end{aligned} \tag{4.14}$$

where we have introduced the functions [3]:

$$C_{30}(t) = \int_0^1 \cdots \int_0^1 \frac{d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7)}{\alpha_5\alpha_6\alpha_7|t| + \Lambda_0(\alpha)\lambda^2}\, \pi,$$

$$C_{31}(t) = \int_0^1 \cdots \int_0^1 \frac{d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7)}{\alpha_5\alpha_6\alpha_7|t| + \Lambda_0(\alpha)\lambda^2} \log\frac{\alpha_6+\alpha_7}{\alpha_7}. \tag{4.15}$$

Furthermore, by the substitutions of section 7 of paper I [1],

$$\begin{aligned} F_{00}^{(d)}(s,t) &= F_{00}^{(a)}(u,t), \\ F_{00}^{(c)}(s,t) &= F_{00}^{(b)}(s,t), \\ F_{00}^{(e)}(s,t) &= F_{00}^{(b)}(u,t), \\ F_{00}^{(f)}(s,t) &= F_{00}^{(b)}(u,t), \end{aligned} \tag{4.16}$$

we get

$$\begin{aligned} A_0^c &= 2[C_{31}(t) + iC_{30}(t)], \\ A_0^d &= -4C_{31}(t), \\ A_0^e &= 2[C_{31}(t) - iC_{30}(t)], \\ A_0^f &= 2[C_{31}(t) - iC_{30}(t)]. \end{aligned} \tag{4.17}$$

Thus, *the sum of the contributions from all six diagrams vanishes,*

$$A_0 = A_0^a + A_0^b + A_0^c + A_0^d + A_0^e + A_0^f = 0. \tag{4.18}$$

This implies that in eq. (3.7) the terms proportional to $\log s$ cancel, when summed over all six diagrams.

5. Integrating over x and y in the subdominant terms

In the integrals (3.21), defining the subdominant terms, the dependence on the kinematical variables enters through $Z_0(\rho)$. It follows from eq. (4.1) that the term $A_1^{(2)}$, which only depends on $Z_0(\rho=0)$, is independent of the electron mass, m. The terms $A_1^{(1)}$ and $A_1^{(3)}$, on the other hand, do depend on the electron mass.

With the functions V, W and X defined by eqs. (4.3), (4.4), and (4.5), the three terms of eq. (3.21) can be written as

$$A_1^{(1)} = \int_0^1 \cdots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7)$$

$$\times \int_0^1 \frac{d\rho}{\rho} \int_0^1 dx \Big[Z_0(\rho) W(\rho) - Z_0(\rho=0) W(\rho=0) \Big], \tag{5.1}$$

$$A_1^{(2)} = \int_0^1 \ldots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1-\alpha_5-\alpha_6-\alpha_7)\, Z_0(\rho=0)$$
$$\times \Big[X(\alpha) - V(\alpha) \log \mathcal{D}(\rho=0) \Big], \tag{5.2}$$

$$A_1^{(3)} = 2 \int_0^1 \ldots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7 \, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_1^\infty \frac{d\rho}{\rho}$$
$$\times \int_0^1 dx \, Z_0(\rho) W(\rho). \tag{5.3}$$

As several times before, we use a compact notation where $W(\rho)$ refers to the function W that depends on all three arguments ρ, x and $\alpha = (\alpha_5, \alpha_6, \alpha_7)$, while $W(\rho = 0)$ refers to the function W with $\rho = 0$ (and thus x irrelevant) and similarly for the function $Z_0(\rho)$. The function $V(\alpha)$ was evaluated in the previous section. We shall first evaluate the remaining two functions $W(\rho)$ of eq. (4.4) and $X(\alpha)$ of eq. (4.5).

5.1 THE INTEGRAL W

Consider the integral (remember that the arguments x and α are not written out)

$$W(\rho) = \int_0^1 \frac{dy}{Q(\alpha)^2}\bigg|_{\rho'=0}, \tag{5.4}$$

first introduced in eq. (4.4). For the a diagram, taking $Q^a(\alpha)$ from Appendix B, we get

$$W^a(\rho) = \frac{1}{\xi + i\epsilon}\left(\frac{1}{\eta} - \frac{1}{\eta+\xi}\right)$$
$$= \frac{1}{\alpha_7 x - \alpha_5(1-x) + i\epsilon}\left[\frac{1}{\alpha_5(1-x) + \alpha_6 + \rho x(1-x)} - \frac{1}{\alpha_7 x + \alpha_6 + \rho x(1-x)}\right]. \tag{5.5}$$

Here, since ξ can vanish, the $i\epsilon$ prescription must be retained. However, the remaining factors in the integrands of eqs. (5.1) and (5.3) are symmetric under the simultaneous interchanges $\alpha_1 \leftrightarrow \alpha_2$, $\alpha_5 \leftrightarrow \alpha_7$. Thus, after integration over α_5, α_6 and α_7, the two terms in the square bracket of eq. (5.5) give identical contributions. In particular, the δ-function part from the $i\epsilon$ prescription drops out, leaving only the principal value part. Thus, we may replace $W^a(\rho)$ by

$$W^a(\rho)\big|_{\text{symm}} = 2 \frac{\mathcal{P}}{\alpha_7 x - \alpha_5(1-x)} \frac{1}{\alpha_5(1-x) + \alpha_6 + \rho x(1-x)}. \tag{5.6}$$

For the b diagram, taking $Q^b(\alpha)$ from Appendix B, we get

$$W^b(\rho) = \frac{1}{\eta}\left(\frac{1}{\xi + i\epsilon} - \frac{1}{\eta + \xi}\right)$$
$$= -\frac{1}{\alpha_7 x - \alpha_5(1-x) - i\epsilon}\,\frac{1}{\alpha_5(1-x) + \alpha_6 + \rho x(1-x)}, \tag{5.7}$$

which leads to

$$W^b(\rho)\Big|_{\rm symm} = -\frac{\mathcal{P}}{\alpha_7 x - \alpha_5(1-x)}\,\frac{1}{\alpha_5(1-x) + \alpha_6 + \rho x(1-x)}$$
$$- i\pi\delta\left(x - \frac{\alpha_5}{\alpha_5 + \alpha_7}\right)\frac{1}{\Lambda_0(\alpha) + \rho\frac{\alpha_5\alpha_7}{\alpha_5+\alpha_7}}$$
$$= -\tfrac{1}{2}W^a(\rho)\Big|_{\rm symm} - i\pi\delta\left(x - \frac{\alpha_5}{\alpha_5 + \alpha_7}\right)\frac{1}{\Lambda_0(\alpha) + \rho\frac{\alpha_5\alpha_7}{\alpha_5+\alpha_7}}. \tag{5.8}$$

5.2 THE INTEGRAL X

For $A_1^{(2)}$ we shall also need the integral $X(\alpha)$ of eq. (4.5). For the a diagram, this integral takes the form

$$X^a(\alpha) = \int_0^1 dx \int_0^1 dy\, \frac{1}{(\xi y + \eta_0 + i\epsilon)^2}\, \log(\xi y + \eta_0 + i\epsilon), \tag{5.9}$$

with ξ and η_0 as defined in eq. (4.8).

Integrating by parts, we obtain

$$X^a(\alpha) = \int_0^1 dx\, \frac{1}{\xi}\left[-\frac{1}{\xi y + \eta_0}\,\log(\xi y + \eta_0) - \frac{1}{\xi y + \eta_0}\right]_0^1$$
$$= \int_0^1 dx\, \frac{1}{\alpha_7 x - \alpha_5(1-x) + i\epsilon}\left\{-\frac{1}{\alpha_6 + \alpha_7 x}[\log(\alpha_6 + \alpha_7 x) + 1]\right.$$
$$\left. + \frac{1}{\alpha_6 + \alpha_5(1-x)}[\log(\alpha_6 + \alpha_5(1-x)) + 1]\right\}. \tag{5.10}$$

Since the remaining factors of the integrand of eq. (5.2), defining $A_1^{(2)}$, are symmetric under the interchange $\alpha_5 \leftrightarrow \alpha_7$, we may replace $X^a(\alpha)$ by

$$X^a(\alpha)\Big|_{symm} = \int_0^1 dx\left[-\frac{1}{\alpha_7 x - \alpha_5(1-x) + i\epsilon} + \frac{1}{\alpha_5(1-x) - \alpha_7 x + i\epsilon}\right]$$
$$\times \frac{1}{\alpha_6 + \alpha_7 x}\left[\log(\alpha_6 + \alpha_7 x) + 1\right]$$

$$= 2\int_0^1 dx \, \frac{\mathcal{P}}{\alpha_5(1-x) - \alpha_7 x} \, \frac{1}{\alpha_6 + \alpha_7 x} \left[\log(\alpha_6 + \alpha_7 x) + 1\right]$$

$$= -\frac{2}{\Lambda_0(\alpha)} \int_0^1 dx \left(\frac{1}{x + \frac{\alpha_6}{\alpha_7}} - \frac{\mathcal{P}}{x - \frac{\alpha_5}{\alpha_5+\alpha_7}} \right) \left[\log\left(x + \frac{\alpha_6}{\alpha_7} \right) + \log \alpha_7 + 1 \right]. \tag{5.11}$$

In order to proceed, we need the dilogarithmic function, for which we use the notation [4], [5]

$$\begin{aligned} \mathrm{Li}_2(z) &= -\int_0^1 \frac{dt}{t} \log(1 - zt) \\ &= -\int_0^z \frac{dt}{t} \log(1-t). \end{aligned} \tag{5.12}$$

The second term above may then be integrated to give

$$\int_0^1 dx \frac{\mathcal{P}}{x - a} \log(c + x) = \log(c + a) \log \frac{a-1}{a} - \mathrm{Li}_2\left(\frac{a-1}{a+c}\right) + \mathrm{Li}_2\left(\frac{a}{a+c}\right). \tag{5.13}$$

Again, exploiting the symmetry of the remaining part of the integrand of eq. (5.2) under the interchange $\alpha_5 \leftrightarrow \alpha_7$, we arrive at

$$\begin{aligned} X^a(\alpha)\big|_{\mathrm{symm}} = \frac{2}{\Lambda_0(\alpha)} \Bigg[&(1 + \log \alpha_5) \log \frac{\alpha_5 + \alpha_7}{\alpha_5} + \frac{1}{2} \log^2 \frac{\alpha_5 + \alpha_7}{\alpha_5} \\ &+ \mathrm{Li}_2\left(-\frac{\alpha_5^2}{\Lambda_0(\alpha)} \right) - \mathrm{Li}_2\left(\frac{\alpha_5 \alpha_7}{\Lambda_0(\alpha)} \right) \Bigg]. \end{aligned} \tag{5.14}$$

Finally, for the b diagram, we have

$$\begin{aligned} X^b(\alpha) = \int_0^1 dx \, \frac{1}{\eta_0} \Bigg[&-\frac{1}{\xi + \eta_0} \log(\xi + \eta_0) \\ &+ \frac{1}{\xi + i\epsilon} \log(\xi + i\epsilon) - \frac{1}{\xi + \eta_0} + \frac{1}{\xi + i\epsilon} \Bigg]. \end{aligned} \tag{5.15}$$

This function is evaluated in Appendix C. We only need that part which is symmetric under interchange of $\alpha_5 \leftrightarrow \alpha_7$. It is given by

$$\begin{aligned} X^b(\alpha)\big|_{\mathrm{symm}} &= \frac{1}{\Lambda_0(\alpha)} \Bigg\{ -(1 + \log \alpha_5) \log \frac{\alpha_5 + \alpha_7}{\alpha_5} - \frac{1}{2} \log^2 \frac{\alpha_5 + \alpha_7}{\alpha_5} \\ &\qquad - \mathrm{Li}_2\left(-\frac{\alpha_5^2}{\Lambda_0(\alpha)} \right) + \mathrm{Li}_2\left(\frac{\alpha_5 \alpha_7}{\Lambda_0(\alpha)} \right) + R(\alpha) \\ &\qquad - i\pi \left[1 - \log \frac{(\alpha_5 + \alpha_7)^2}{\alpha_5 \Lambda_0(\alpha)} \right] \Bigg\} \\ &= -\tfrac{1}{2} X^a(\alpha)\big|_{\mathrm{symm}} + \frac{1}{\Lambda_0(\alpha)} \left\{ R(\alpha) - i\pi \left[1 - \log \frac{(\alpha_5 + \alpha_7)^2}{\alpha_5 \Lambda_0(\alpha)} \right] \right\}, \end{aligned} \tag{5.16}$$

with

$$R(\alpha) = \frac{\pi^2}{6} + 2\left\{\mathrm{Li}_2\left(1 - \frac{\alpha_5\alpha_7}{\Lambda_0(\alpha)}\right) - \mathrm{Li}_2\left(\frac{\alpha_5\alpha_7}{\Lambda_0(\alpha)}\right)\right\}$$
$$+ \mathrm{Li}_2\left(\frac{\Lambda_0(\alpha)}{(\alpha_5+\alpha_6)(\alpha_6+\alpha_7)}\right) - \mathrm{Li}_2\left(1 - \frac{\Lambda_0(\alpha)}{(\alpha_5+\alpha_6)(\alpha_6+\alpha_7)}\right). \quad (5.17)$$

However, $X(\alpha)$ enters the integrand of eq. (5.2) with factors which are completely symmetric in α_5, α_6 and α_7. Therefore, $R(\alpha)$ can be further symmetrized, replacing it by

$$\begin{aligned} R(\alpha)\Big|_{\mathrm{symm}} = \frac{\pi^2}{6} - \frac{2}{3}\Big[&\mathrm{Li}_2\left(\frac{\alpha_6\alpha_7}{\Lambda_0(\alpha)}\right) - \mathrm{Li}_2\left(1 - \frac{\alpha_6\alpha_7}{\Lambda_0(\alpha)}\right) + \mathrm{Li}_2\left(\frac{\alpha_7\alpha_5}{\Lambda_0(\alpha)}\right) \\ &-\mathrm{Li}_2\left(1 - \frac{\alpha_7\alpha_5}{\Lambda_0(\alpha)}\right) + \mathrm{Li}_2\left(\frac{\alpha_5\alpha_6}{\Lambda_0(\alpha)}\right) - \mathrm{Li}_2\left(1 - \frac{\alpha_5\alpha_6}{\Lambda_0(\alpha)}\right)\Big] \\ -\frac{1}{3}\Big[&\mathrm{Li}_2\left(\frac{\alpha_5^2}{(\alpha_5+\alpha_6)(\alpha_5+\alpha_7)}\right) - \mathrm{Li}_2\left(1 - \frac{\alpha_5^2}{(\alpha_5+\alpha_6)(\alpha_5+\alpha_7)}\right) \\ &+\mathrm{Li}_2\left(\frac{\alpha_6^2}{(\alpha_6+\alpha_5)(\alpha_6+\alpha_7)}\right) - \mathrm{Li}_2\left(1 - \frac{\alpha_6^2}{(\alpha_6+\alpha_5)(\alpha_6+\alpha_7)}\right) \\ &+\mathrm{Li}_2\left(\frac{\alpha_7^2}{(\alpha_7+\alpha_5)(\alpha_7+\alpha_6)}\right) - \mathrm{Li}_2\left(1 - \frac{\alpha_7^2}{(\alpha_7+\alpha_5)(\alpha_7+\alpha_6)}\right)\Big]. \end{aligned} \quad (5.18)$$

It was pointed out in paper II that this expression is in fact independent of the variables α_5, α_6 and α_7 and that furthermore

$$R(\alpha)\Big|_{\mathrm{symm}} = \frac{\pi^2}{3}. \quad (5.19)$$

6. Summing contributions to the subdominant term A_1

The subdominant term A_1 consists of three parts, $A_1^{(1)}$, $A_1^{(2)}$ and $A_1^{(3)}$, which are evaluated for the a and b diagrams in the following three subsections.

6.1 THE COEFFICIENT $A_1^{(1)}$

We define the two functions

$$D_{30}(t) = -\frac{\pi}{2}\int_0^1 \cdots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7)\int_0^1 \frac{d\rho}{\rho}$$
$$\times\left[Z_0(\rho)\Big|_{x=\frac{\alpha_5}{\alpha_5+\alpha_7}} \frac{1}{\Lambda_0(\alpha)+\rho\frac{\alpha_5\alpha_7}{\alpha_5+\alpha_7}} - Z_0(\rho=0)\frac{1}{\Lambda_0(\alpha)}\right],$$

$$D_{31}(t) = -\frac{1}{2}\int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_0^1 \frac{d\rho}{\rho} \int_0^1 dx \frac{\mathcal{P}}{\alpha_7 x - \alpha_5(1-x)}$$
$$\times \left[Z_0(\rho)\, \frac{1}{\alpha_5(1-x)+\alpha_6+\rho x(1-x)} - Z_0(\rho=0)\, \frac{1}{\alpha_5(1-x)+\alpha_6} \right].$$
(6.1)

It then follows from eqs. (5.1), (5.6). and (5.8) that

$$A_1^{(1)a} = -4D_{31}(t),$$
$$A_1^{(1)b} = 2[D_{31}(t) + iD_{30}(t)]. \tag{6.2}$$

6.2 THE COEFFICIENT $A_1^{(2)}$

We first note that (see appendix B)

$$\log \mathcal{D}(\alpha)\big|_{\rho=\rho'=0} = \log(\alpha_5\alpha_6\alpha_7|t| + \Lambda_0(\alpha)\lambda^2) + i\pi. \tag{6.3}$$

By eqs. (4.1), (5.2) and (6.3), it follows that

$$A_1^{(2)} = -i\pi A_0$$
$$-2\int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7) \frac{\Lambda_0(\alpha)}{\alpha_5\alpha_6\alpha_7|t| + \Lambda_0(\alpha)\lambda^2}$$
$$\times \left[X(\alpha) - V(\alpha)\log|\mathcal{D}(\rho=0)|\right]. \tag{6.4}$$

It is convenient to express the coefficient $A_1^{(2)}$ in terms of the functions

$$E_{30}(t) = \int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7) \frac{\pi}{\alpha_5\alpha_6\alpha_7|t| + \Lambda_0(\alpha)\lambda^2}$$
$$\times \left[1 - \log\frac{(\alpha_5+\alpha_7)^2}{\alpha_5\Lambda_0(\alpha)}\right],$$
$$E_{31}(t) = \int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7) \frac{-1}{\alpha_5\alpha_6\alpha_7|t| + \Lambda_0(\alpha)\lambda^2}$$
$$\times \left[(1+\log\alpha_5)\log\frac{\alpha_5+\alpha_7}{\alpha_5} + \frac{1}{2}\log^2\frac{\alpha_5+\alpha_7}{\alpha_5} + \mathrm{Li}_2\left(-\frac{\alpha_5^2}{\Lambda_0(\alpha)}\right) - \mathrm{Li}_2\left(\frac{\alpha_5\alpha_7}{\Lambda_0(\alpha)}\right)\right], \tag{6.5}$$

$$F_{30}(t) = \int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7) \frac{1}{\alpha_5\alpha_6\alpha_7|t| + \Lambda_0(\alpha)\lambda^2}$$

$$\times \log(\alpha_5\alpha_6\alpha_7|t| + \Lambda_0(\alpha)\lambda^2)\,\pi,$$

$$F_{31}(t) = \int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7)\, \frac{1}{\alpha_5\alpha_6\alpha_7|t| + \Lambda_0(\alpha)\lambda^2}$$
$$\times \log(\alpha_5\alpha_6\alpha_7|t| + \Lambda_0(\alpha)\lambda^2)\, \log\frac{\alpha_5+\alpha_7}{\alpha_7}. \tag{6.6}$$

Invoking now the results (4.10), (4.12), (5.14), and (5.16), as well as the definitions (6.5) and (6.6), we find for the a and b diagrams, respectively,

$$A_1^{(2)a} = -i\pi A_0^a(t) - 4E_{31}(t) + 4F_{31}(t),$$
$$A_1^{(2)b} = -i\pi A_0^b(t) + 2[E_{31}(t) + iE_{30}(t)] - \frac{2\pi}{3}C_{30}(t) - 2[F_{31}(t) + iF_{30}(t)]. \tag{6.7}$$

The C_{30}-term comes from the $R(\alpha)$-part of eq. (5.16), where $R(\alpha)$ is replaced by its value eq. (5.19).

6.3 THE COEFFICIENT $A_1^{(3)}$

Next, in order to write out $A_1^{(3)}$, we define the functions

$$\tilde{D}_{30}(t) = -\pi \int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_6 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_1^\infty \frac{d\rho}{\rho}$$
$$\times Z_0(\rho)\Big|_{x=\frac{\alpha_5}{\alpha_5+\alpha_7}} \frac{1}{\Lambda_0(\alpha) + \rho\frac{\alpha_5\alpha_7}{\alpha_5+\alpha_7}},$$

$$\tilde{D}_{31}(t) = -\int_0^1 \dots \int_0^1 d\alpha_5 d\alpha_5 d\alpha_7\, \delta(1-\alpha_5-\alpha_6-\alpha_7) \int_1^\infty \frac{d\rho}{\rho} \int_0^1 dx$$
$$\times Z_0(\rho)\, \frac{\mathcal{P}}{\alpha_7 x - \alpha_5(1-x)}\, \frac{1}{\alpha_5(1-x) + \alpha_6 + \rho x(1-x)}. \tag{6.8}$$

Invoking now eqs. (5.3), (5.6), and (5.8), we find for the a and b diagrams, respectively,

$$A_1^{(3)a} = -4\tilde{D}_{31}(t),$$
$$A_1^{(3)b} = 2[\tilde{D}_{31}(t) + i\tilde{D}_{30}(t)]. \tag{6.9}$$

6.4 ADDING THE CONTRIBUTIONS FROM ALL SIX DIAGRAMS

Before adding the contributions from all six diagrams, we add the three terms that make up A_1, eq. (3.17). We obtain for the a and b diagrams,

$$A_1^a = -i\pi A_0^a(t) - 4D_{31}(t) - 4E_{31}(t) + 4F_{31}(t) - 4\tilde{D}_{31}(t),$$
$$A_1^b = -i\pi A_0^b(t) + 2[D_{31}(t) + iD_{30}(t)] + 2[E_{31}(t) + iE_{30}(t)]$$
$$-\frac{2\pi}{3}C_{30}(t) - 2[F_{31}(t) + iF_{30}(t)] + 2[\tilde{D}_{31}(t) + i\tilde{D}_{30}(t)]. \tag{6.10}$$

With the substitutions of section 7 of paper I, we find for the other diagrams (see eq. (4.16) and appendix D of paper II)

$$\begin{aligned}
Q^c &= Q^b, \\
Q^d &= -Q^{a*} = -Q^a, \\
Q^e &= Q^f = -Q^{b*}.
\end{aligned} \tag{6.11}$$

Thus,

$$\begin{aligned}
\log(Q^d + i\epsilon) &= \log Q^a + i\pi, \\
\log(Q^e + i\epsilon) &= (\log Q^b)^* + i\pi,
\end{aligned} \tag{6.12}$$

and the coefficients for the remaining four diagrams become

$$\begin{aligned}
A_1^c &= A_1^b, \\
A_1^d &= -4D_{31}(t) - 4E_{31}(t) + 4F_{31}(t) - 4\tilde{D}_{31}(t), \\
A_1^e &= 2[D_{31}(t) - iD_{30}(t)] + 2[E_{31}(t) - iE_{30}(t)] \\
&\quad - \frac{2\pi}{3}C_{30}(t) - 2[F_{31}(t) - iF_{30}(t)] + 2[\tilde{D}_{31}(t) - i\tilde{D}_{30}(t)], \\
A_1^f &= A_1^e.
\end{aligned} \tag{6.13}$$

Summing the contributions from all six diagrams, we obtain

$$\begin{aligned}
A_1 &= -i\pi[A_0^a(t) + 2A_0^b(t)] - \frac{8\pi}{3}C_{30}(t) \\
&= \frac{4\pi}{3}C_{30}(t).
\end{aligned} \tag{6.14}$$

Thus, A_1 is completely determined by the function $C_{30}(t)$.

7. Summary of high-energy results

Combining eqs. (6.14) and (3.7) with eqs. (2.1) and (3.1), we get in the high-energy limit for the sum of the six ladder-like diagrams,

$$F_{00}(s,t) = \frac{8\pi}{3}C_{30}(t). \tag{7.1}$$

The function $C_{30}(t)$ depends on $|t|$ and λ^2, but not on m^2. This result complements the one obtained in paper II [2].

In Bhabha scattering at LEP the interesting limit is $s \gg |t| \gg m^2 \gg \lambda^2$. The asymptotic value of the function $C_{30}(t, \lambda^2)$ in the region $|t| \gg \lambda^2$ is easily determined. The mathematical details are given in appendices D and E, where two different methods of evaluation are presented. They give the result

$$F_{00}(s,t) = \frac{8\pi^2}{|t|} \log^2\left(\frac{|t|}{\lambda^2}\right), \tag{7.2}$$

for $s \gg |t| \gg \lambda^2$. There are no linear logarithms nor constants.

The expression quoted in eq. (7.2) is identical to the one given in paper *II* for leading logarithms. In that paper, though, no attempt was made to determine the subleading terms. In spite of the agreement, there is an important difference between the two calculations. The present one has been performed in the limit $s \to \infty$ with t, m^2 and λ^2 fixed. It is not à priori obvious that this result will hold also in the QED limit $\lambda^2 \to 0$, with s, t and m^2 fixed. Our guess is that this will be the case, but to be certain one must also carry out the calculation of the subleading terms along the lines of paper II. There are many instances in which the two limits do not commute [6].

The coefficients A_0 for each of the six diagrams has previously been calculated by Cheng and Wu [3] in ϕ^3 theory [cf. eq. (1.7)]. Our result agrees with theirs. This is an important check, since for the determination of A_0 we can set $\rho = \rho' = 0$ in the numerator function, and since $N_{III}(\rho = \rho' = 0) = 4[\Lambda(\rho = \rho' = 0)]^4$, it follows that the coefficients A_0 must be the same in QED and ϕ^3 theory.

Cheng and Wu [3] have also shown that, for the sum of the six diagrams, part of the term independent of $\log s$, is simply related by analytic continuation to the $\log s$ terms of the individual diagrams. We have calculated all terms, also those that are not related in this way. Consequently, our result for F_{00}, eq. (7.1), being more complete, differs from theirs by a factor of $\frac{2}{3}$.

It might also be of some interest to display the leading $\log s$ contribution to each of the six diagrams. In section 4.2 they are shown to be determined by the functions $C_{30}(t)$ and $C_{31}(t)$. The asymptotic values of both functions in the region $|t| \gg \lambda^2$ are determined in Appendices D and E. The results obtained there give for the uncrossed a diagram and the once-crossed b diagram

$$F_{00}^{a} = -\frac{16}{3|t|} \log\left(\frac{s}{|t|}\right)\left[\log^3\left(\frac{|t|}{\lambda^2}\right) + \pi^2 \log\left(\frac{|t|}{\lambda^2}\right) - 3\zeta(3)\right] + \cdots, \tag{7.3}$$

$$F_{00}^{b} = \frac{8}{3|t|} \log\left(\frac{s}{|t|}\right)\left[\log^3\left(\frac{|t|}{\lambda^2}\right) + \pi^2 \log\left(\frac{|t|}{\lambda^2}\right) - 3\zeta(3) + \frac{9\pi i}{2} \log^2\left(\frac{|t|}{\lambda^2}\right)\right] + \cdots, \tag{7.4}$$

where the dots denote terms independent of $\log s$. They remain to be determined. To do so, all the functions of section 6 must be calculated. Some of them might depend on the electron mass m.

Acknowledgments

It is a pleasure to thank professor Tai Tsun Wu for helpful discussions. This research has been supported by the Norwegian Research Council for Science and the Humanities and by the Swedish National Research Council.

Appendix A. Integrations over ρ and ρ'

Consider first an integral appearing in eq. (3.18),

$$f_1[F] \equiv \lim_{\zeta \to 0} \zeta \int_0^1 \rho^{-1+\zeta} F(\rho) d\rho. \tag{A.1}$$

Integrating by parts, we find

$$\begin{aligned} f_1[F] &= \lim_{\zeta \to 0} \left\{ \rho^\zeta F(\rho) \Big|_0^1 - \int_0^1 d\rho\, \rho^\zeta \frac{d}{d\rho} F(\rho) \right\} \\ &= F(1) - \int_0^1 d\rho \frac{d}{d\rho} F(\rho) \\ &= F(0). \end{aligned} \tag{A.2}$$

Next, consider

$$\begin{aligned} f_2[F] &\equiv \lim_{\zeta \to 0} \zeta^2 \int_0^1 d\rho \int_0^1 d\rho' (\rho\rho')^{-1+\zeta} F(\rho, \rho') \\ &= \lim_{\zeta \to 0} \zeta \int_0^1 d\rho\, \rho^{-1+\zeta} \left[\rho'^\zeta F(\rho, \rho') \Big|_{\rho'=0}^1 - \int_0^1 d\rho' \rho'^\zeta \frac{d}{d\rho'} F(\rho, \rho') \right] \\ &= \lim_{\zeta \to 0} \zeta \int_0^1 d\rho\, \rho^{-1+\zeta} \left[F(\rho, 1) - \int_0^1 d\rho' \rho'^\zeta \frac{d}{d\rho'} F(\rho, \rho') \right] \\ &= \lim_{\zeta \to 0} \left\{ \left[\rho^\zeta F(\rho, 1) - \int_0^1 d\rho' \rho'^\zeta \frac{d}{d\rho'} F(\rho, \rho') \right]_{\rho=0}^1 \right. \\ &\qquad \left. - \int_0^1 d\rho\, \rho^\zeta \left[\frac{d}{d\rho} F(\rho, 1) - \int_0^1 d\rho' \rho'^\zeta \frac{d^2}{d\rho d\rho'} F(\rho, \rho') \right] \right\} \\ &= F(1,1) - \int_0^1 d\rho' \frac{d}{d\rho'} F(1, \rho') - \int_0^1 d\rho \frac{d}{d\rho} F(\rho, 1) + \int_0^1 d\rho \int_0^1 d\rho' \frac{d^2}{d\rho d\rho'} F(\rho, \rho') \\ &= F(0,0). \end{aligned} \tag{A.3}$$

Both results (A.2) and (A.3) are needed for the evaluation of eq. (3.18).

Appendix B. Diagrams a and b for $\rho' = 0$

All the functions pertaining to the uncrossed diagram a are defined in section 4.1 of paper I. With the abbreviation

$$\Lambda_0(\alpha) \equiv \alpha_5\alpha_6 + \alpha_6\alpha_7 + \alpha_7\alpha_5, \tag{B.1}$$

which is symmetric under any interchange among α_5, α_6, and α_7, we find

$$\Lambda^a(\alpha)\Big|_{\rho'=0} = \Lambda_0(\alpha) + \alpha_1(\alpha_6 + \alpha_7) + \alpha_2(\alpha_5 + \alpha_6) + \alpha_1\alpha_2. \tag{B.2}$$

Furthermore, according to eq. (3.11),

$$\begin{aligned} Q^a(\alpha)\Big|_{\rho'=0} &= \alpha_5(1-x)(1-y) + \alpha_6 + \alpha_7 xy + \rho x(1-x) + i\epsilon \\ &\equiv \xi y + \eta + i\epsilon, \end{aligned} \tag{B.3}$$

with

$$\begin{aligned} \xi &= \alpha_7 x - \alpha_5(1-x), \\ \eta &= \alpha_5(1-x) + \alpha_6 + \rho x(1-x). \end{aligned} \tag{B.4}$$

We note that $Q^a(\alpha)\Big|_{\rho'=0}$ is non-negative.

With $\rho' = 0$, or equivalently $\alpha_3 = \alpha_4 = 0$, we have the following equalities [for definition of the variables, see eq. (I.4.6)]

$$\begin{aligned} b &= d = 0, \\ a &= \Lambda_0(\alpha) + \alpha_2\alpha_5, \\ \tilde{a} &= \Lambda_0(\alpha) + \alpha_1(\alpha_6 + \alpha_7) + \alpha_2(\alpha_5 + \alpha_6) + \alpha_1\alpha_2 = \Lambda(\alpha)|_{\rho'=0}, \\ c &= \Lambda_0(\alpha) + \alpha_1\alpha_7, \\ \tilde{c} &= \tilde{a}. \end{aligned} \tag{B.5}$$

Thus, the numerator function of eq. (2.3) becomes

$$\begin{aligned} N^a(\alpha)\big|_{\rho'=0} &= 4a\tilde{a}c\tilde{c} \\ &= 4[\Lambda_0(\alpha) + \alpha_1(\alpha_6 + \alpha_7) + \alpha_2(\alpha_5 + \alpha_6) + \alpha_1\alpha_2]^2 \\ &\quad \times (\Lambda_0(\alpha) + \alpha_2\alpha_5)(\Lambda_0(\alpha) + \alpha_1\alpha_7). \end{aligned} \tag{B.6}$$

Finally, for $\mathcal{D}^a(\alpha)\big|_{\rho'=0}$ we need

$$\begin{aligned} D_t &= \alpha_5\alpha_6\alpha_7, \\ D_m\big|_{\rho'=0} &= -\alpha_1\alpha_2(\alpha_1 + \alpha_2) - (\alpha_1 + \alpha_2)^2\alpha_6 - \alpha_2^2\alpha_5 - \alpha_1^2\alpha_7, \\ D_\lambda\big|_{\rho'=0} &= -(\alpha_5 + \alpha_6 + \alpha_7)\Lambda(\alpha)\big|_{\rho'=0}. \end{aligned} \tag{B.7}$$

For the special case where also $\rho = 0$, we have

$$\mathcal{D}^a(\alpha)\Big|_{\rho=\rho'=0} = \alpha_5\alpha_6\alpha_7 t - (\alpha_5 + \alpha_6 + \alpha_7)\Lambda_0(\alpha)\lambda^2. \tag{B.8}$$

All functions pertaining to diagram b are defined in section 4.2 of paper I. With $\Lambda_0(\alpha)$ as defined by eq. (B.1),

$$\Lambda^b(\alpha)\Big|_{\rho'=0} = \Lambda_0(\alpha) + \alpha_1(\alpha_6 + \alpha_7) + \alpha_2(\alpha_5 + \alpha_6) + \alpha_1\alpha_2. \tag{B.9}$$

Furthermore,

$$\begin{aligned} Q^b(\alpha)\Big|_{\rho'=0} &= -\alpha_5(1-x)(1-y) + \alpha_6 y + \alpha_7 x + \rho x(1-x)y + i\epsilon \\ &= \eta y + \xi + i\epsilon, \end{aligned} \tag{B.10}$$

with η and ξ as defined in eq. (B.4). We note that this function changes sign in the domain of integration.

For the particular case $\rho' = 0$, or equivalently $\alpha_3 = \alpha_4 = 0$, the variables b, d, a, $\tilde{a}$, c and $\tilde{c}$ become the same for diagram b and diagram a, eq. (B.5). This is also the case for the numerator function, and for the remainder of the denominator function,

$$N^b(\alpha)\big|_{\rho'=0} = 4a\tilde{a}c\tilde{c}, \tag{B.11}$$

$$\mathcal{D}^b(\alpha)\big|_{\rho'=0} = \mathcal{D}^a(\alpha)\big|_{\rho'=0}. \tag{B.12}$$

Appendix C. The integral $X^b(\alpha)$

The integral $X^b(\alpha)$ is defined by eq. (5.15). We split it into three terms, F_1, F_2 and F_3,

$$\begin{aligned} X^b(\alpha) &= \int_0^1 dx\, \frac{1}{\eta_0}\Big[-\frac{1}{\xi+\eta_0}\log(\xi+\eta_0) \\ &\quad + \frac{1}{\xi+i\epsilon}\log(\xi+i\epsilon) - \frac{1}{\xi+\eta_0} + \frac{1}{\xi+i\epsilon}\Big] \\ &= F_1 + F_2 + F_3, \end{aligned} \tag{C.1}$$

with

$$F_1 \equiv -\int_0^1 \frac{dx}{\eta_0(\xi+\eta_0)}\log(\xi+\eta_0), \tag{C.2}$$

$$F_2 \equiv \int_0^1 \frac{dx}{\eta_0(\xi+i\epsilon)}\log(\xi+i\epsilon), \tag{C.3}$$

$$F_3 \equiv \int_0^1 \frac{dx}{(\xi+i\epsilon)(\xi+\eta_0)}. \tag{C.4}$$

The first integral F_1, can by partial fractioning be written as

$$F_1 = -\frac{1}{\Lambda_0(\alpha)}\int_0^1 dx\left(-\frac{1}{x-\frac{\alpha_5+\alpha_6}{\alpha_6}}+\frac{1}{x+\frac{\alpha_6}{\alpha_7}}\right)\left[\log\left(x+\frac{\alpha_6}{\alpha_7}\right)+\log\alpha_7\right]$$

$$= -\frac{1}{\Lambda_0(\alpha)}\left[\log\Lambda_0(\alpha)\,\log\frac{\alpha_5+\alpha_6}{\alpha_6}-\frac{1}{2}\log^2\frac{\alpha_6}{\alpha_7}\right.$$

$$\left.+\frac{1}{2}\log^2\frac{\alpha_5+\alpha_6}{\alpha_5}+\mathrm{Li}_2\left(\frac{\alpha_6\alpha_7}{\Lambda_0(\alpha)}\right)-\mathrm{Li}_2\left(\frac{(\alpha_5+\alpha_6)\alpha_7}{\Lambda_0(\alpha)}\right)\right]. \tag{C.5}$$

For the second integral F_2, we obtain upon substitution for η_0 and ξ,

$$F_2 = \int_0^1 \frac{dx}{\eta_0(\xi+i\epsilon)}\log(\xi+i\epsilon)$$

$$= \frac{1}{\Lambda_0(\alpha)}\int_0^1 dx\left(\frac{1}{x-\frac{\alpha_5}{\alpha_5+\alpha_7}+i\epsilon}-\frac{1}{x-\frac{\alpha_5+\alpha_6}{\alpha_6}}\right)$$

$$\times\left[\log\left(x-\frac{\alpha_5}{\alpha_5+\alpha_7}+i\epsilon\right)+\log(\alpha_5+\alpha_7)\right]. \tag{C.6}$$

We here invoke eq. (5.13), and project out the part that is symmetric under $\alpha_5 \leftrightarrow \alpha_7$,

$$F_2\Big|_{symm} = \frac{1}{\Lambda_0(\alpha)}\left[-i\pi\log\alpha_5+\frac{\pi^2}{2}-\log\frac{\Lambda_0(\alpha)}{\alpha_5}\log\frac{\alpha_6}{\alpha_5+\alpha_6}\right.$$

$$\left.+\mathrm{Li}_2\left(1-\frac{\alpha_5\alpha_7}{\Lambda_0(\alpha)}\right)-\mathrm{Li}_2\left(1+\frac{\alpha_5^2}{\Lambda_0(\alpha)}-i\epsilon\right)\right]. \tag{C.7}$$

The dilogarithms can be rearranged making use of the formulas (see ref. [4])

$$\mathrm{Li}_2\left(1+\frac{\alpha_5^2}{\Lambda_0(\alpha)}-i\epsilon\right) = -\mathrm{Li}_2\left(-\frac{\alpha_5^2}{\Lambda_0(\alpha)}\right)-\log\frac{(\alpha_5+\alpha_6)(\alpha_5+\alpha_7)}{\Lambda_0(\alpha)}\log\frac{\alpha_5^2}{\Lambda_0(\alpha)}$$

$$+\frac{\pi^2}{6}-i\pi\log\frac{(\alpha_5+\alpha_6)(\alpha_5+\alpha_7)}{\Lambda_0(\alpha)},$$

$$\mathrm{Li}_2\left(1-\frac{\alpha_5\alpha_7}{\Lambda_0(\alpha)}\right) = -\mathrm{Li}_2\left(\frac{\alpha_5\alpha_7}{\Lambda_0(\alpha)}\right)-\log\frac{\alpha_5\alpha_7}{\Lambda_0(\alpha)}\log\frac{(\alpha_5+\alpha_7)\alpha_6}{\Lambda_0(\alpha)}+\frac{\pi^2}{6}, \tag{C.8}$$

giving

$$F_2\Big|_{symm} = \frac{1}{\Lambda_0(\alpha)}\left[i\pi\log\frac{(\alpha_5+\alpha_6)(\alpha_5+\alpha_7)}{\alpha_5\Lambda_0(\alpha)}+\frac{\pi^2}{3}-\log\alpha_5\log\frac{\alpha_6}{\alpha_5+\alpha_6}\right.$$

$$\left.+\log\frac{\Lambda_0(\alpha)}{\alpha_5^2}\log\frac{\Lambda_0(\alpha)}{\alpha_6(\alpha_5+\alpha_7)}+\mathrm{Li}_2\left(\frac{\alpha_6(\alpha_5+\alpha_7)}{\Lambda_0(\alpha)}\right)+\mathrm{Li}_2\left(-\frac{\alpha_5^2}{\Lambda_0(\alpha)}\right)\right]. \tag{C.9}$$

For the last integral F_3, we get

$$F_3 = -\frac{1}{\Lambda_0(\alpha)}\left[\log\frac{\alpha_6+\alpha_7}{\alpha_6}+i\pi\right]. \tag{C.10}$$

The symmetrized sum, $F_1+F_2+F_3$, is given by eq. (5.16).

Appendix D. Evaluation of $C_{30}(t,\lambda^2)$ and $C_{31}(t,\lambda^2)$

In order to evaluate $C_{30}(t,\lambda^2)$ of eq. (4.15), we first homogenize the integrand in the α variables and then apply the theorem of Cheng and Wu [3], to obtain

$$C_{30}(t,\lambda^2) = \frac{\pi}{|t|}\int_0^\infty d\alpha_6 \int_0^1 d\alpha_5 \int_0^1 d\alpha_7 \, \frac{\delta(1-\alpha_5-\alpha_7)}{\alpha_5\alpha_6\alpha_7 + \delta^2(1+\alpha_6)(\alpha_6+\alpha_5\alpha_7)}, \tag{D.1}$$

where

$$\delta^2 \equiv \frac{\lambda^2}{|t|} \ll 1. \tag{D.2}$$

Next, we write the integral as

$$C_{30}(t,\lambda^2) = \frac{\pi}{|t|}\int_0^\infty dz \int_0^1 dx \, \frac{1}{D(z,x)}, \tag{D.3}$$

with

$$\begin{aligned} D(z,x) &= x(1-x)[z+\delta^2(1+z)] + \delta^2 z(1+z) \\ &\equiv \delta^2(z-z_1)(z-z_2). \end{aligned} \tag{D.4}$$

We are interested in the structure of $C_{30}(t,\lambda^2)$ when δ^2 becomes small, but we are not interested in terms that vanish in this limit. For this application we may approximate the roots z_1 and z_2 by

$$z_1 \simeq -\delta^2 \frac{x(1-x)}{(x+\delta^2)(1+\delta^2-x)}, \qquad z_2 \simeq -\frac{1}{\delta^2}(x+\delta^2)(1+\delta^2-x). \tag{D.5}$$

Since furthermore, in the limit of small δ^2,

$$\delta^2(z_1-z_2) \simeq (x+\delta^2)(1+\delta^2-x), \tag{D.6}$$

we can write

$$\frac{1}{D(z,x)} \simeq \left[\frac{1}{x+\delta^2} + \frac{1}{(1-x)+\delta^2}\right]\left[\frac{1}{z-z_1} - \frac{1}{z-z_2}\right]. \tag{D.7}$$

The integration over z is straightforward, and for small δ^2 we get the integral representation

$$C_{30}(t,\lambda^2) = \frac{2\pi}{|t|}\int_0^1 \frac{dx}{x+\delta^2} \log\left[\frac{(x+\delta^2)^2(1+\delta^2-x)^2}{\delta^4 x(1-x)}\right]. \tag{D.8}$$

The integration over x is done by the formula

$$L_{n,p} = \int_0^1 \frac{dx}{x+\delta^2} \log^n \frac{x}{\delta^2} \log^p \frac{x+\delta^2}{\delta^2}, \qquad n,p \geq 0,$$

$$= \frac{1}{n+p+1} \log^{n+p+1}\left(\frac{1}{\delta^2}\right) + \int_0^{1/\delta^2} \frac{dt}{1+t} \left[\log^n t - \log^n(1+t)\right] \log^p(1+t)$$

$$\simeq \frac{1}{n+p+1} \log^{n+p+1}\left(\frac{1}{\delta^2}\right) + n!p!(-1)^n C_{n,p+1}, \tag{D.9}$$

where in the last step we have neglected terms that vanish when $\delta^2 \to 0$. The constants $C_{n,p+1}$ are defined by the formula

$$C_{n,p+1} = \frac{(-1)^n}{n!p!} \int_0^\infty \frac{dt}{1+t} \left[\log^n t - \log^n(1+t)\right] \log^p(1+t), \tag{D.10}$$

and identical to the corresponding constants in ref. [5]. The first few values of $C_{n,p+1}$ are

$$C_{0,1} = C_{0,2} = C_{0,3} = C_{2,1} = 0$$
$$C_{1,1} = \zeta(2), \qquad C_{1,2} = \zeta(3), \tag{D.11}$$

where $\zeta(n)$ is the Riemann zeta function.

Neglecting contributions which vanish when $\delta^2 \to 0$, we get

$$C_{30}(t,\lambda^2) = \frac{3\pi}{|t|} \log^2 \delta^2$$
$$= \frac{3\pi}{|t|} \log^2 \frac{|t|}{\lambda^2}. \tag{D.12}$$

The integrand of $C_{31}(t,\lambda^2)$ has an additional factor $\log[(\alpha_5+\alpha_7)/\alpha_5]$. In the integration variables of eq. (D.3) this is simply $\log x$. Going through the same steps as for $C_{30}(t,\lambda^2)$ we get for small δ^2, the representation

$$C_{31}(t,\lambda^2) = \frac{-1}{|t|} \int_0^1 \frac{dx}{x+\delta^2} \log\left[\frac{(x+\delta^2)^2(1+\delta^2-x)^2}{\delta^4 x(1-x)}\right] \log[x(1-x)]. \tag{D.13}$$

Applying formulas (D.9) and retaining only terms that survive in the limit $\delta \to 0$, we have

$$C_{31}(t,\lambda^2) = \frac{1}{|t|}\left[\frac{2}{3}\log^3\left(\frac{|t|}{\lambda^2}\right) + \frac{2\pi^2}{3}\log\left(\frac{|t|}{\lambda^2}\right) - 2\zeta(3)\right]. \tag{D.14}$$

If we are only interested in the leading logarithms, the detailed expansions given above are not required. An alternative, and often simpler method is given in appendix E. However, that method may possibly fail for some of the other integrals of section 6.

Appendix E. Evaluation of $C_{30}(t,\lambda^2)$ and $C_{31}(t,\lambda^2)$ by contour integration

We introduce variables similar to those used in appendix G of paper II,

$$\begin{aligned}\alpha_5 &= 1-\rho_1,\\ \alpha_6 &= \rho_1(1-\rho_2),\\ \alpha_7 &= \rho_1\rho_2.\end{aligned} \tag{E.1}$$

The function $C_{30}(t,\lambda^2)$ of eq. (4.15) can then be written as

$$C_{30}(t,\lambda^2)=\frac{\pi}{|t|}\int_0^1 d\rho_1\int_0^1 d\rho_2\,\frac{\rho_1}{\rho_1^2\rho_2(1-\rho_1)(1-\rho_2)+\rho_1[(1-\rho_1)+\rho_1\rho_2(1-\rho_2)]\delta^2}, \tag{E.2}$$

with $\delta^2=\lambda^2/|t|$. Using (II.A.5), we transform $C_{30}(t,\lambda^2)$ into

$$\begin{aligned}C_{30}(t,\lambda^2)&=\frac{\pi}{|t|}\int_0^1 d\rho_1\int_0^1 d\rho_2\,\frac{1}{(2\pi i)^2}\int_{C_1}dz_1\int_{C_2}dz_2\,\Gamma(z_1)\Gamma(z_2)\Gamma(1-z_1-z_2)\\&\quad\times\rho_1[\rho_1^2\rho_2(1-\rho_1)(1-\rho_2)]^{-1+z_1+z_2}[\rho_1(1-\rho_1)]^{-z_1}[\rho_1^2\rho_2(1-\rho_2)]^{-z_2}\\&\quad\times(\delta^2)^{-z_1-z_2}\\&=\frac{\pi}{|t|}\frac{1}{(2\pi i)^2}\int_{C_1}dz_1\int_{C_2}dz_2\,\Gamma(z_1)\Gamma(z_2)\Gamma(1-z_1-z_2)\\&\quad\times B(z_1,z_2)B(z_1,z_1)(\delta^2)^{-z_1-z_2}\\&=\frac{\pi}{|t|}\frac{1}{(2\pi i)^2}\int_{C_1}dz_1\int_{C_2}dz_2\,\Gamma(z_1)\Gamma(z_2)\Gamma(1-z_1-z_2)\\&\quad\times\frac{\Gamma^3(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)\Gamma(2z_1)}(\delta^2)^{-z_1-z_2}.\end{aligned} \tag{E.3}$$

We do the z_2-integration first. Since $\delta^2=\lambda^2/|t|\ll 1$, we may close the contour in the left half-plane. We get contributions from poles at $z=-n$, $n=0,1,2,\ldots$, but it turns out that contributions from $n=1,2,\ldots$ are of order δ^2 and may thus be neglected. We obtain

$$\begin{aligned}C_{30}(t,\lambda^2)&=\frac{\pi}{|t|}\frac{1}{2\pi i}\int_{C_1}dz_1\Bigg[-\frac{\Gamma^3(z_1)\Gamma(1-z_1)}{\Gamma(2z_1)}(\delta^2)^{-z_1}\log(\delta^2)\\&\quad+\frac{\partial}{\partial z_2}\left(\frac{\Gamma(1-z_1-z_2)\Gamma^2(1+z_2)}{\Gamma(z_1+z_2)}\right)\Bigg|_{z_2=0}\frac{\Gamma^4(z_1)}{\Gamma(2z_1)}(\delta^2)^{-z_1}\Bigg]+\mathcal{O}(\delta^2).\end{aligned} \tag{E.4}$$

Similarly, the leading contribution to the z_1 integral comes from the pole at $z_1=0$. It turns out that linear logarithms and constants cancel, and we are left with the result (D.12).

With the same variables, we have

$$
\begin{aligned}
C_{31}(t,\lambda^2) &= -\frac{1}{|t|}\int_0^1 d\rho_1 \int_0^1 d\rho_2 \frac{\rho_1 \log\rho_2}{\rho_1^2\rho_2(1-\rho_1)(1-\rho_2)+\rho_1[(1-\rho_1)+\rho_1\rho_2(1-\rho_2)]\delta^2} \\
&= -\frac{1}{|t|}\lim_{x\to 0}\frac{\partial}{\partial x}\int_0^1 d\rho_1 \int_0^1 d\rho_2 \frac{\rho_1 \rho_2^x}{\rho_1^2\rho_2(1-\rho_1)(1-\rho_2)+\rho_1[(1-\rho_1)+\rho_1\rho_2(1-\rho_2)]\delta^2} \\
&= -\frac{1}{|t|}\frac{1}{(2\pi i)^2}\lim_{x\to 0}\frac{\partial}{\partial x}\int_{C_1} dz_1 \int_{C_2} dz_2\, \Gamma(z_1)\Gamma(z_2)\Gamma(1-z_1-z_2) \\
&\quad \times B(z_1,z_2)B(z_1+x,z_1)(\delta^2)^{-z_1-z_2}. \qquad \text{(E.5)}
\end{aligned}
$$

We proceed in the same way as for $C_{30}(t,\lambda^2)$, arriving at (D.14).

For further applications of this technique, which is well suited for computer treatment, we refer to ref. [6].

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