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DEFORMATION OF THE EXTERIOR ALGEBRA AND THE GL_q (r, $\mathbb{C}$) ALGEBRA

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ABSTRACT

The deformation of the associative algebra of exterior forms is performed. This operation leads to a Y.B. equation. Its relation with the braid group B_{n-1} is analyzed. The correspondence of this deformation with the $GL_q(r,\mathbb{C})$ algebra is developed.

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Recently, much interest has been given to the understanding of quantum groups known also as q-Lie groups. They appear in several directions namely in two dimensional conformal field theories [1], integrable models and statistical phenomena. Moreover, they have the structure of bialgebras and were introduced as deformation [2,3] of certain Hopf algebras [4] of classical Lie groups. The properties of these bialgebras depend explicitly on what is called a quantum R-matrix. The latter is subject to satisfy the Yang-Baxter (Y-B) equation [5]. An explicit expression of this matrix is known from statistical mechanics.

In this letter, it is shown by explicit computation that the deformation of the associative algebra of exterior forms $\Omega(V)$ on a vector space V, leads to the Y-B equation. This fact is related to the associativity of the deformed algebra $\Omega_q(V)$. This property is then motivated by some braid relations; the deformation matrix extends to a representation of the braid group B_{n-1}[6].

Finally, we show that the quantum group associated with the mentioned deformation is of type $GL_q(r, \mathbb{C})$. These results agree with those obtained, in another way, in Refs. [7,8].

We start with the study of q-deformation of the associative algebra of differential forms $\Omega(V)$ defined on a vector space V. The obtained algebra will be denoted by $\Omega_q(V)$ where q is the deformation parameter. We will see that the Yang-Baxter equation [5] appears as a consequence of the associativity of the new wedge product on $\Omega_q(V)$.

To start, let V be a vector space over the field $\mathbb{C}$ such that $\dim_{\mathbb{C}} V = r$ and $(e_1,\ldots, e_r)$ an arbitrary basis. Let then $(\sigma^1,\ldots,\sigma^r)$ the basis of the dual space V^* such that :

$$\sigma^i(e_j) = \delta^i{}_j \tag{1}$$

Recall that $\{\sigma^i\ ;\ i=1, \ldots, r\}$ are nothing but one differential forms on V; then their exterior product is given by :

$$\sigma^i \wedge \sigma^j = \sigma^i \otimes \sigma^j - \sigma^j \otimes \sigma^i \tag{2}$$

Here we are interested in the study of a deformation of $\Omega_1(V)$ through a natural generalization of Eq.(2), that is

$$\sigma^i \tilde{\wedge} \sigma^j = \sigma^i \otimes \sigma^j - \Lambda^{ij}{}_{kl}\, \sigma^k \otimes \sigma^l \tag{3}$$

The elements $\Lambda^{ij}{}_{kl}$ are entries of some matrix $\Lambda \in \mathrm{End}_{\mathbb{C}}(V^* \otimes V^*)$

$$\Lambda(\sigma^i \otimes \sigma^j) = \Lambda^{ij}{}_{kl}\, \sigma^k \otimes \sigma^l \tag{4}$$

and thus

$$\sigma^i \tilde{\wedge} \sigma^j = \sigma^i \otimes \sigma^j - \Lambda(\sigma^i \otimes \sigma^j) \tag{5}$$

We demand that all $\Lambda^{ij}{}_{;kl}$ to be scalars depending on an arbitrary complex number q referred to as the deformation parameter. Moreover, we suppose that in the limit q =1, we have

$$\Lambda^{ij}{}_{;kl}\,(q = 1) = \delta^{i;l}\,\delta^{j;k} \tag{6}$$

Or equivalently $\Lambda = P$, where P is the permutation matrix. In this situation, the deformed product coincides with the classical one given by Eq.(2).

We now proceed to the construction of the algebra of exterior forms built out of the deformed wedge product Eq,(3). The first remark to make is that the skewsymmetry of the classical product given by Eq.(2) is broken. Indeed $\sigma^i \wedge;\tilde{}\; \sigma^j \neq - \sigma^j \wedge;\tilde{}\; \sigma^i$. Thus, elements of the type $\sigma^i \wedge;\tilde{}\; \sigma^j$ and $\sigma^j \wedge;\tilde{}\; \sigma^i$; $i \neq j$ will be viewed as linearly independent. By $\Omega^{(2);q}(V)$ we denote the vector space generated by the set $\{ \sigma^i \wedge;\tilde{}\; \sigma^j ; i,j=1 , ..., r\}$. It follows that a general q-two-form reads as :

$$\omega^{(2);q} = \omega_{ij}\, \sigma^i \wedge;\tilde{}\; \sigma^j \quad , \tag{7}$$

where ω_{ij} are complex numbers.

Before proceeding to construct q-form of higher order, notice that as $\sigma^i \wedge;\tilde{}\; \sigma^j$ lies in T(V) (The tensor algebra of V), we are led to define the overlapping between the two operations $\otimes$ and $\wedge;\tilde{}$. For an element of $V^*\otimes V^*$ and V^*, we define their exterior deformed product by generalizing Eq.(3), that is

$$(\sigma^i \otimes \sigma^j) \wedge;\tilde{}\; \sigma^k = \sigma^i \otimes \sigma^j \otimes \sigma^k - \Lambda^{jk}{}_{;lm}\, \sigma^i \otimes \sigma^l \otimes \sigma^m + \Lambda^{jk}{}_{;lm}\, \Lambda^{il}{}_{;np}\, \sigma^n \otimes \sigma^p \otimes \sigma^m \tag{8.a}$$

and

$$\sigma^i \wedge;\tilde{}\; (\sigma^j \otimes \sigma^k) = \sigma^i \otimes \sigma^j \otimes \sigma^k - \Lambda^{ij}{}_{;lm}\, \sigma^l \otimes \sigma^m \otimes \sigma^k + \Lambda^{ij}{}_{;lm}\, \Lambda^{mk}{}_{;np}\, \sigma^l \otimes \sigma^n \otimes \sigma^p \tag{8.b}$$

The signs + or - appearing in these relations are related to the number of Λ . Furthermore, the consistency of Eqs.(8) will be given in terms of braiding group operations.
We rewrite the above formulae in a compact simple form [9]

$$(\sigma^1 \otimes \sigma^2) \wedge;\tilde{}\; \sigma^3 = (E - \Lambda_{23} + \Lambda_{23}\Lambda_{12})\, \sigma^1 \otimes \sigma^2 \otimes \sigma^3 \tag{9.a}$$

$$\sigma^1 \wedge;\tilde{}\; (\sigma^2 \otimes \sigma^3) = (E - \Lambda_{12} + \Lambda_{12}\Lambda_{23})\, \sigma^1 \otimes \sigma^2 \otimes \sigma^3 \tag{9.b}$$

Now, we are in position to write the deformed product of three one forms. Some calculations lead to the following expressions:

$$(\sigma^1 \wedge;^{\sim} \sigma^2) \wedge;^{\sim} \sigma^3 = (E - \Lambda_{23} + \Lambda_{23}\Lambda_{12} - \Lambda_{12} + \Lambda_{12}\,\Lambda_{23} - \Lambda_{12}\,\Lambda_{23}\,\Lambda_{12})\, \sigma^1 \otimes \sigma^2 \otimes \sigma^3 \tag{10.a}$$

$$\sigma^1 \wedge;^{\sim} (\sigma^2 \wedge;^{\sim} \sigma^3) = (E - \Lambda_{12} + \Lambda_{12}\Lambda_{23} - \Lambda_{23} + \Lambda_{23}\,\Lambda_{12} - \Lambda_{23}\,\Lambda_{12}\,\Lambda_{23})\, \sigma^1 \otimes \sigma^2 \otimes \sigma^3 \tag{10.b}$$

By requiring then the associativity of the new product $\wedge;^{\sim}$, we see that the matrix deformation Λ is subject to satisfy the constraint

$$\Lambda_{12}\,\Lambda_{23}\,\Lambda_{12} = \Lambda_{23}\,\Lambda_{12}\,\Lambda_{23} \tag{11}$$

This relation is a representation of the braid group B_{n-1} [6] generated by $\{\varepsilon_i,\ i=1,..,n-1\}$ with

$$\varepsilon_i\,\varepsilon_{i+1}\,\varepsilon_i = \varepsilon_{i+1}\,\varepsilon_i\,\varepsilon_{i+1} \quad \& \quad \varepsilon_i\,\varepsilon_j = \varepsilon_j\,\varepsilon_i \ \text{for} \ |i-j| \geq 2 \tag{12}$$

Notice that the last equality is consistent with Eqs.(9). Indeed, all permutations of type $i \to i+n$ for $n \geq 2$ are not to be considered.

The most important remark to make about Eq.11) is that this relation is an "equivalent" version of the Yang-Baxter equation. It is easy to check that the matrix $R=P\Lambda$ satisfies the Y-B equation

$$R_{12}\,R_{13}\,R_{23} = R_{23}\,R_{13}\,R_{12} \tag{13}$$

In what follows, we will deal with matrix deformations Λ such that the matrices $R=P\Lambda$ are Y-B operators. Thus, the obtained deformed algebra is then an associative one. Consequentely, an arbitrary three deformed form can be written as follows :

$$\omega^{(3);q} = \omega_{ijk}\, \sigma^i \wedge;^{\sim} \sigma^j \wedge;^{\sim} \sigma^k \tag{14}$$

with ω_{ijk} arbitrary complex numbers. The space $\Omega^{(3);q}(V)$ of all q-three-forms is then the vector space generated by the set $\{ \sigma^i \wedge;^{\sim} \sigma^j \wedge;^{\sim} \sigma^k\, ;\, i,j,k =1,\ldots,r\}$.

Finally and by induction on p, we set $\Omega^{(p);q}(V)$ to be the space of deformed p-forms $\omega^{(p);q}$; these latters are given by :

$$\omega^{(p);q} = \omega_{i_1 \ldots\, i_p}\, \sigma^{i_1} \wedge;^{\sim} \sigma^{i_2} \wedge;^{\sim} \ldots \wedge;^{\sim} \sigma^{i_p} \tag{15}$$

with $\omega_{i_1 \ldots\, i_p} \in \mathbb{C}$. $\Omega^{(p);q}(V)$ is generated by the set $\{\sigma^{i_1} \wedge;^{\sim} \sigma^{i_2} \wedge;^{\sim} \ldots \wedge;^{\sim} \sigma^{i_p},\ i_1,\ldots, i_p=1,\ldots,r\}$.
Notice that the dimension of Ω\s((p);q)(V) is greater than C\s(p;r) = \f(r!;(r-p)!p!) = dim $\Omega^{(p);q=1}(V)$. The equality is obtained when the skewsymmetry property is restored i. e. if and only if we have $\Lambda^{ij}{}_{;kl} = \delta^{i;l}\,\delta^{j;k}$ i.e. $\Lambda=P$.

Let then $\omega^{(n);q}$ and $\omega^{(m);q}$ be two elements of $\Omega^{(n);q}(V)$ and $\Omega^{(m);q}(V)$, respectively. Their exterior product $\omega^{(n);q} \wedge;\tilde{} \ \omega^{(m);q}$ is an element of $\Omega^{(n+m);q}(V)$ and we have:

$$\omega^{(n);q} \wedge;\tilde{} \ \omega^{(m);q} = \varphi_{i_1 \dots i_n} \psi_{j_1 \dots j_m} \sigma^{i1} \wedge;\tilde{} \ \sigma^{i2} \wedge;\tilde{} \dots \wedge;\tilde{} \ \sigma^{in} \wedge;\tilde{} \ \sigma^{j1} \wedge;\tilde{} \ \sigma^{j2} \wedge;\tilde{} \dots \wedge;\tilde{} \ \sigma^{jm} \tag{16}$$

with:

$$\omega^{(n);q} = \varphi_{i_1 \dots i_n} \sigma^{i1} \wedge;\tilde{} \ \sigma^{i2} \wedge;\tilde{} \dots \wedge;\tilde{} \ \sigma^{in} \quad ; \quad \omega^{(m);q} = \psi_{j_1 \dots j_m} \sigma^{j1} \wedge;\tilde{} \ \sigma^{j2} \wedge;\tilde{} \dots \wedge;\tilde{} \ \sigma^{j}.$$

The deformed exterior associative algebra $\Omega_q(V)$ we are interested in is defined, by analogy with the classical limit, to be the direct sum of all $\Omega^{(p);q}(V)$; $p \geq 0$, i.e. :

$$\Omega_q(V) = \oplus;^{+\infty}_{;p=0} \Omega^{(p);q}(V) \ ; \ \Omega^{(0);q}(V) = \mathbb{C} \tag{17}$$

At this step, an important feature of $\Omega_q(V)$ appears. It is an infinite dimensional vector space contrary to the classical case where we have

$$\dim \ \Omega_{q=1}(V) = 2^{\dim V} \tag{18}$$

An equivalent situation was encountered in ref [8] where it has been shown by completely different construction that the space $\Omega_q(V)$ is infinite dimensional.

Up to now, the deformation of $\Omega(V) \equiv \Omega_{q=1}(V)$ was defined through the matrix Λ or equivalently the matrix $R=P\Lambda$. However, the matrix R is subject to satisfy just the Yang-Baxter equation and thus R is not completely determined. Thus, we hope to find another constraint fixing then R. The latter will be an "RTT" type equation [9] :

$$R_{12} \, t_1 \, t_2 = t_2 \, t_1 \, R_{12} \tag{19}$$

The above relation will be then the main tool to define the quantum group associated to the deformed exterior associative algebra $\Omega_q(V)$.

Notice that the following equality is obvious

$$R^{ij}{}_{;kl} \ \delta^{k;p} \delta^{l;q} = \delta^{j;l} \delta^{i;k} \ R^{kl}{}_{;pq} \tag{20}$$

Eq.(20) can be written as

$$R^{ij}{}_{;kl} \ (\sigma^k \otimes \sigma^l)(e_p, e_q) = (\sigma^j \otimes \sigma^i)(e_l, e_k) \ R^{kl}{}_{;pq} \tag{21}$$

where $\{e_p, p=1, \dots, r\}$ is the dual basis of $\{\sigma^p ; p=1, \dots r\}$.
By requiring that the Eq.(21) is unchanged under any transformation of the basis

$$e_p \rightarrow e'_{;p} = e_j \, t^{j;p} \quad , \quad p = 1,2 \dots, r = \dim V \tag{22}$$

where ($t^{i;j}$) are entries of a certain matrix (for instance) of GL(r,$\mathbb{C}$). One can check that the t's satisfy the constraint

$$R^{ij}{}_{;kl}\ t^{k;p}\ t^{l;q} = t^{j;l}\ t^{i;k}\ R^{kl}{}_{;pq} \tag{23}$$

The meaning of the above equation is that the elements $t^{i;j}$ do not in fact commute and so can be used to construct an associative algebra generated by r^2 elements satisfying Eq.(23). Notice that the associativity of this algebra follows from the Y-B Eq.(13). The latter algebra is denoted by $GL_q(r,\mathbb{C})$.

This result agrees with one obtained in Ref.[7]. There it was shown with a different approach that the quantum group associated to $\Omega_q(p)$ is of type $GL_q(r,\mathbb{C})$.

Finally, we note that an extension of this work can be developed on a general manifold V rather than a vector space. This may lead to the construction of a quantum differential calculus on V. This question will be examined elsewhere [10].

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