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RELATED TO A COLOURED BRAID GROUP  
REPRESENTATION**

**B. Basu-Mallick**

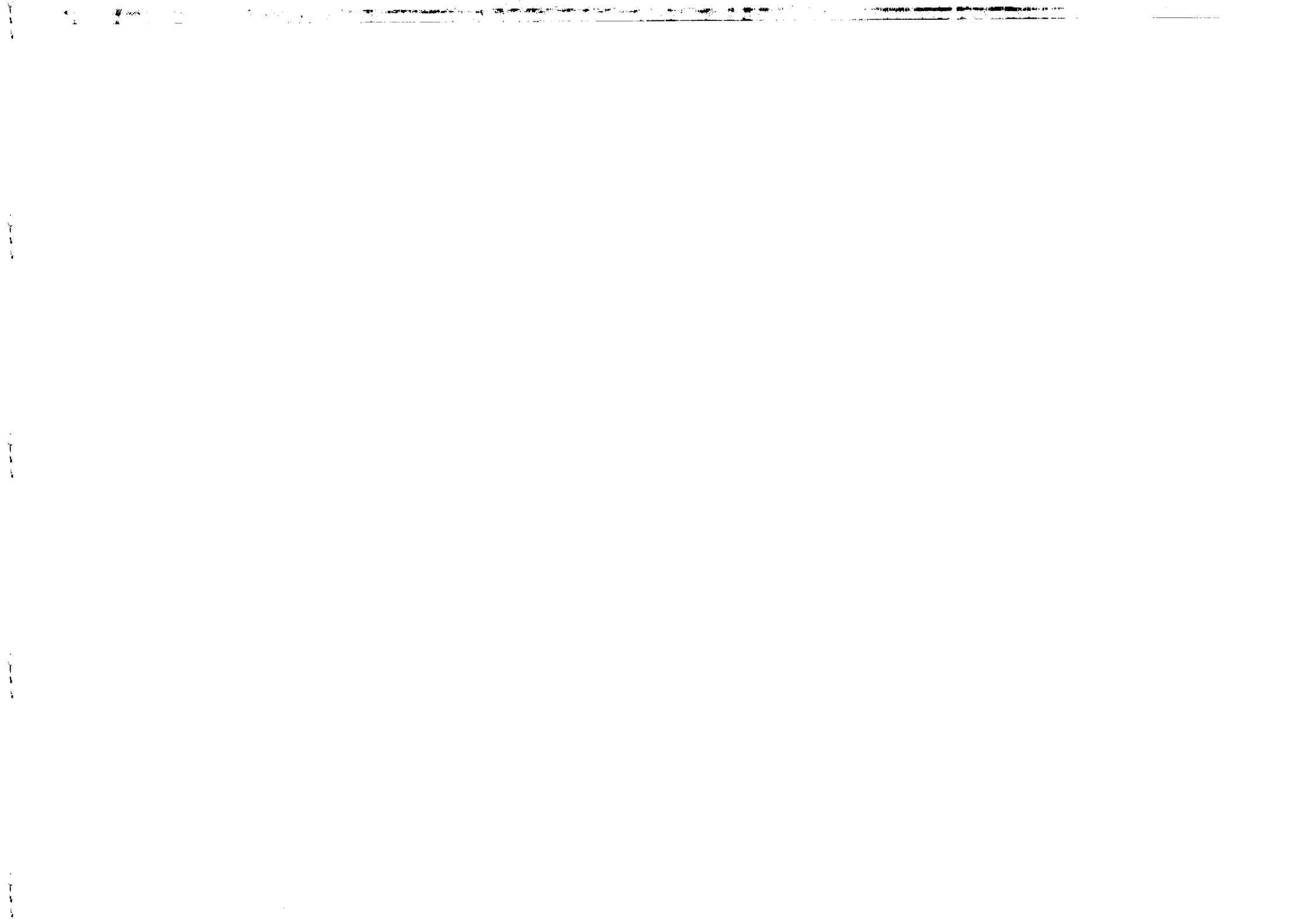


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QUANTUM GROUP AND MANIN PLANE  
RELATED TO A COLOURED BRAID GROUP REPRESENTATION

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ABSTRACT

By considering 'coloured' braid group representation we have obtained a quantum group, which reduces to the standard  $GL_q(2)$  and  $GL_{p,q}(2)$  cases at some particular limits of the 'colour' parameters. In spite of quite complicated nature, all of these new quantum group relations can be expressed neatly in the Heisenberg-Weyl form, for a nontrivial choice of the basis elements. Furthermore, it is possible to associate invariant Manin planes, parametrised by the 'colour' variables, with such quantum group structure.

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1. Introduction

In recent years quantum groups and related algebras are found to have a wide range of applications in different branches of physics and mathematics [1-12]. In particular, these algebraic structures manifested themselves in the study of quantum integrable models, as an abstraction of some basic relations like quantum Yang-Baxter equation [3,10-11]. From the mathematical point of view, quantum groups are also intimately connected to the braid group representations. To illustrate this, one may recall the case of well known  $GL_q(2)$  quantum group generated by the elements  $a, b, c, d$ , which satisfy the algebraic relations

$$\begin{aligned} ab &= q^{-1}ba, \quad ac = q^{-1}ca, \quad db = qbd, \quad dc = qcd, \\ bc &= cb, \quad [a, d] = -(q - q^{-1})cb, \end{aligned} \quad (1.1)$$

where the deformation parameter  $q$  is a nonzero complex number. Remarkably, the above bilinear relations can be expressed in a compact matrix form as [3]

$$R T_1 T_2 = T_2 T_1 R, \quad (1.2)$$

where  $T$  is a  $(2 \times 2)$ -matrix given by

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (1.3)$$

$T_1 = T \otimes 1$ ,  $T_2 = 1 \otimes T$  and  $R$  is a  $(4 \times 4)$ -matrix with usual  $c$ -number matrix elements:

$$R = \begin{pmatrix} q & & & \\ & 1 & (q - q^{-1}) & \\ & 0 & 1 & \\ & & & q \end{pmatrix}. \quad (1.4)$$

The expression (1.2) reveals that  $\Delta T = T \otimes T$  would be a coproduct of  $GL_q(2)$  quantum group, where the symbol  $\otimes$  signifies ordinary matrix multiplication with tensor multiplication of algebra.

As it is well known, the form of the algebraic relation (1.2) is quite general and can be applied to generate other quantum groups also, depending on the choice of the

corresponding  $R$ -matrix. Due to associativity of algebra (1.2), the  $R$ -matrix in general satisfy the spectral parameterless Yang-Baxter equation (YBE) given by

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \quad (1.5)$$

where we have used the standard direct product notation ( like  $R_{12} = R \otimes 1$  etc.). YBE (1.5), in turn, leads to a braid group representation (BGR) for the matrix  $\hat{R}^+ = \mathcal{P}R^+$  ( $\mathcal{P}$  being the permutation operator with the property  $\mathcal{P}A \otimes B = B \otimes \mathcal{P}A$ ):

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}.$$

We shall call however the  $R$ -matrix itself as a BGR in what follows, for the sake of convenience and to emphasise the close connection between them. Thus from the above discussion one finds that BGRs play a rather important role in constructing quantum groups through the defining relation (1.2). In this context we may mention about another  $(4 \times 4)$   $R$ -matrix, which contains two arbitrary parameters  $p, q$  and reduces to (1.4) at the limit  $p = q$  [13]. This BGR eventually leads to a  $p, q$ -deformed  $GL_{p,q}(2)$  quantum group, which has been studied extensively from different viewpoints [7,14-16].

Quite recently some 'coloured' generalisations of BGR have interestingly appeared in the literature [17-20], which satisfy

$$R_{12}^{(\lambda,\mu)} R_{13}^{(\lambda,\gamma)} R_{23}^{(\mu,\gamma)} = R_{23}^{(\mu,\gamma)} R_{13}^{(\lambda,\gamma)} R_{12}^{(\lambda,\mu)}, \quad (1.6)$$

where  $\lambda, \mu, \gamma$  are continuously variable 'colour' parameters. Though usually  $\hat{R}^{(\lambda,\mu)} = \mathcal{P}R^{(\lambda,\mu)}$  is defined as the coloured BGR (CBGR), we would call  $R^{+(\lambda,\mu)}$ -matrix itself as CBGR in analogy with the previous standard case. Now it is natural to enquire whether these CBGRs would also lead to a new class of quantum group relations. Furthermore, similar to the case of usual quantum groups, it should be much encouraging to investigate various mathematical and physical properties associated to such algebraic structures. In the present article we like to shed some light on these issues by concentrating on a  $(4 \times 4)$

CBGR given by

$$R^{(\lambda,\mu)} = \begin{pmatrix} q^{1-(\lambda-\mu)} & & & \\ & q^{\lambda+\mu} & (q-q^{-1})s^{-(\lambda-\mu)} & \\ & 0 & q^{-(\lambda+\mu)} & \\ & & & q^{1+(\lambda-\mu)} \end{pmatrix}, \quad (1.7)$$

which might be obtained from the fundamental representation of universal  $R$ -matrix related to  $U_{q,s}(gl(2))$  quantum algebra [19]. It is intriguing to notice that at the limit  $\lambda = \mu = 0$ , the above CBGR reduces to the BGR (1.4) associated to  $GL_q(2)$  quantum group. On the other hand, for  $\lambda = \mu \neq 0$  we will recover the two parameter dependent BGR [7,13] corresponding to  $GL_{p,q}(2)$  quantum group. So the quantum group which we are hoping to obtain at present through the CBGR (1.7), should be some 'coloured' generalisation of both  $GL_q(2)$  and  $GL_{p,q}(2)$  case. In sec.2 of this paper we shall first review the approach of ref.19 for generating the CBGR (1.7) and after that discuss how such CBGR, as well as its generalisations, might be obtained from the standard BGRs by 'colouring' them through some symmetry transformation of YBE.

Subsequently in sec.3 we attempt to construct the quantum group related to the CBGR (1.7). Since now colour parameters are present in the CBGR, the defining relation of the standard quantum group (1.2) should also be modified in a consistent way. Fortunately, such modified version already exist in the literature and was used to explore quantum groups related to infinite dimensional  $Z$ -graded vector spaces [21]. For the present purpose, we also fruitfully use this modified version and write down explicitly the quantum group relations corresponding to the CBGR (1.7).

Next we turn our attention to some interesting features possessed by the usual quantum groups and investigate whether these features remain meaningful even for the coloured case. For example, it is worth observing that the set of all algebraic relations corresponding to both  $GL_q(2)$  and  $GL_{p,q}(2)$  quantum groups can be recast in the Heisenberg-Weyl form, for unimodular values of the deforming parameters  $q$  and  $p$  [22,16]. Surprisingly we also find in sec.3 that, in spite of their much complicated nature, all independent bilinear relations appearing in the quantum group related to the CBGR (1.7) can be expressed finally

in the Heisenberg-Weyl form. This fact might be useful in building up representations for this coloured case.

Another salient feature of the standard quantum groups is their close connection to noncommutative geometry [5-7, 23]. Quantum group structure emerges in fact in a natural way if one consider transformations on the noncommutative vector space or Manin plane, which preserve the form of algebra of the co-ordinates. So the stimulating question arises whether there exist some 'coloured' Manin planes on which the quantum group related to the CBGR (1.7) acts as endomorphism, i.e., generates transformations which preserve the related algebraic structures. In sec.4 we seek answer to this question and find the existence of such Manin planes. Sec. 5 is the concluding section.

## 2. Construction of CBGR

As it is well known for a quasitriangular Hopf algebra  $\mathcal{A}$ , there exists an invertible universal  $\mathcal{R}$ -matrix ( $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ ) such that it interrelates comultiplications  $\Delta$ ,  $\Delta'$  through  $\Delta(a)\mathcal{R} = \mathcal{R}\Delta'(a)$ , where  $a \in \mathcal{A}$  and satisfies the following conditions

$$(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \quad (\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (S \otimes id)\mathcal{R} = \mathcal{R}^{-1},$$

$S$  being the antipode. The above relations also imply that the  $\mathcal{R}$ -matrix would be a solution of YBE (1.5).

If one considers now the case of  $U_q(gl(2))$  quantised algebra, apart from the usual generators  $S_3$ ,  $S_{\pm}$  of  $U_q(sl(2))$ , a central element or Casimir like operator  $\Lambda$  is included in the picture with the commutation relations [7]

$$[S_3, S_{\pm}] = \pm S_{\pm}, \quad [S_+, S_-] = \frac{\sin(2\alpha S_3)}{\sin \alpha} \quad (2.1)$$

$$[\Lambda, S_{\pm}] = [\Lambda, S_3] = 0; \quad q = e^{i\alpha}.$$

As a result the standard comultiplication is also get modified to yield

$$\begin{aligned} \Delta(S_+) &= S_+ \otimes q^{-S_3} \cdot (qs)^{\Lambda} + \left(\frac{s}{q}\right)^{\Lambda} \cdot q^{S_3} \otimes S_+, \\ \Delta(S_-) &= S_- \otimes q^{-S_3} \cdot (qs)^{-\Lambda} + \left(\frac{s}{q}\right)^{-\Lambda} \cdot q^{S_3} \otimes S_-, \\ \Delta(S_3) &= S_3 \otimes \mathbf{1} + \mathbf{1} \otimes S_3, \quad \Delta(\Lambda) = \Lambda \otimes \mathbf{1} + \mathbf{1} \otimes \Lambda, \end{aligned} \quad (2.2)$$

where  $s$  is an arbitrary parameter appearing due to the symmetry of the algebra. The other Hopf algebraic structures like co-unit, antipode can be consistently defined and the universal  $\mathcal{R}$ -matrix may also be constructed as [19]

$$\mathcal{R} = q^{2(S_3 \otimes S_3 + S_3 \otimes \Lambda - \Lambda \otimes S_3)} \cdot \sum_{m=0}^{\infty} \frac{(1 - q^{-2})^m}{[m, q^{-2}]!} (q^{S_3} (qs)^{-\Lambda} S_+)^m \otimes \left( q^{-S_3} \left(\frac{s}{q}\right)^{\Lambda} S_- \right)^m, \quad (2.3)$$

where  $[m, q] = (1 - q^m)/(1 - q)$  and  $[m, q]! = [m, q] \cdot [m - 1, q] \cdots 1$ .

Denoting now the eigenvalue of the Casimir like operator  $\Lambda$  by  $\lambda$  and the corresponding  $n$ -dimensional irreducible representation of algebra (2.1) as  $\Pi_{\lambda}^n$ , we may obtain the 'colour' representation  $(\Pi_{\lambda}^n \otimes \Pi_{\mu}^n)\mathcal{R}$ , giving a finite dimensional CBGR  $R^{(\lambda, \mu)}$  satisfying (1.6). In particular, for the simple two dimensional representation  $\Pi_{\lambda}^2$  through identity operator and Pauli matrices :  $\Lambda = \lambda \mathbf{1}$ ,  $S_3 = \frac{1}{2}\sigma_3$ ,  $S_{\pm} = \sigma_{\pm}$ , one gets the CBGR (1.7).

In another recent developement [24,25] similar CBGR, as well as its generalisations in arbitrary dimensions, were obtained directly from the standard BGRs by using a symmetry transformation of YBE. It has been shown that if  $\mathcal{R}$ -matrix is a solution of eqn. (1.5) with the 'particle conserving' constraint, i.e. its elements  $R_{ij}^{kl}$  are non-zero only when the 'incoming particles'  $(i, j)$  are some permutations of the outgoing ones  $(k, l)$ , then one can construct the CBGR  $R^{(\lambda, \mu)}$  satisfying (1.6) with elements given by

$$\left[ R^{(\lambda, \mu)} \right]_{ij}^{kl} = R_{ij}^{kl} \frac{u_i^{(1)}(\lambda) u_j^{(2)}(\lambda)}{u_i^{(1)}(\mu) u_k^{(2)}(\mu)}. \quad (2.4)$$

Here the indices  $i, j, k, l$  run from 1 to  $N$  and  $u_i^{(1)}(\lambda)$ ,  $u_i^{(2)}(\lambda)$  are  $2N$  number of arbitrary colour parameter dependent functions. Starting now from standard BGR  $R^{\pm}$  related to the fundamental representation of  $U_q(sl(N))$  [26] :

$$R^{\pm} = \sum_i q^{\pm 1} e_{ii} \otimes e_{ii} + \sum_{i \neq j} \phi_{ij} \cdot e_{ii} \otimes e_{jj} \pm (q - q^{-1}) \sum_{\substack{i < j \\ (i > j)}} e_{ij} \otimes e_{ji}, \quad (2.5)$$

which evidently satisfies the 'particle conserving' restriction and using further (2.4) one

derives the corresponding CBGR as

$$R^{\pm(\lambda, \mu)} = \sum_i q^{\pm 1} \frac{u_i^{(1)}(\lambda) u_i^{(2)}(\lambda)}{u_i^{(1)}(\mu) u_i^{(2)}(\mu)} e_{ii} \otimes e_{ii} + \sum_{i \neq j} \phi_{ij} \frac{u_j^{(1)}(\lambda) u_j^{(2)}(\lambda)}{u_i^{(1)}(\mu) u_i^{(2)}(\mu)} e_{ii} \otimes e_{jj} \\ \pm (q - q^{-1}) \sum_{\substack{i < j \\ (i > j)}} \frac{u_i^{(1)}(\lambda) u_j^{(2)}(\lambda)}{u_i^{(1)}(\mu) u_j^{(2)}(\mu)} e_{ij} \otimes e_{ji}, \quad (2.6)$$

where elements of the matrix  $e_{ij}$  are given by  $(e_{ij})_{kl} = \delta_{ik} \delta_{jl}$  and  $\phi_{ij}$  are arbitrary constants with the condition  $\phi_{ij} \cdot \phi_{ji} = 1$ . Now it is interesting to observe that in the particular case  $N = 2$  along with the choice

$$\phi_{12} = 1, \quad u_1^{(1)}(\lambda) = 1, \quad u_2^{(1)}(\lambda) = (qs)^\lambda, \quad u_1^{(2)}(\lambda) = q^{-\lambda}, \quad u_2^{(2)}(\lambda) = s^{-\lambda}, \quad (2.7)$$

the form of CBGR  $R^{+(\lambda, \mu)}$  in (2.6) reduces exactly to the CBGR (1.7), which was obtained earlier from the universal  $\mathcal{R}$ -matrix related to  $U_q(gl(2))$  in its fundamental representation. Thus one finds here a rather convenient method to generate CBGRs from the standard BGRs in the 'particle conserving' case, by simply 'colouring' the BGRs through a symmetry transformation of YBE. We may hope that the CBGR (2.6), with arbitrary  $N$ , would be similarly related to the fundamental representation of the  $U_q(gl(N))$  quantised algebra. Notice that other type of CBGRs can also be constructed by restricting the deforming parameter  $q$  at root of unity [17-18,20], in contrast to the present case where it is arbitrary.

Interestingly, one can Yang-Baxterise the CBGR (1.7) and generate a solution of YBE depending on two-component spectral parameters [25]. Moreover, realisation of Faddeev-Reshetikhin-Takhtajan (FRT) algebra [3,21] corresponding to this CBGR is also possible. Yang-Baxterisation of such FRT algebra leads to 'coloured' generalisations of various well known quantum integrable models like lattice sine-Gordon model, Ablowitz-Ladik model etc. [25]. However, at present our aim is to focus on the quantum group relations corresponding to the CBGR (1.7) and to explore some related mathematical properties.

### 3. Quantum group related to a CBGR

As already mentioned earlier, we need to modify the defining relation of usual quantum group (1.2) for the case dealing with a CBGR. Such modified relation was previously employed to construct quantum groups related to infinite dimensional  $Z$ -graded vector spaces and may be expressed as [21]

$$R^{(\lambda, \mu)} T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R^{(\lambda, \mu)}. \quad (3.1)$$

Notice that the operator valued elements of the matrix  $T(\lambda)$  appearing above are explicitly dependent on the colour parameter and the coproduct in this case might be given by  $\Delta T(\lambda) = T(\lambda) \otimes T(\lambda)$ . Taking now  $R^{(\lambda, \mu)}$  as (1.7), the  $(2 \times 2)$   $T(\lambda)$ -matrix in the form

$$T(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}, \quad (3.2)$$

and inserting them in (3.1), we get the following relations among the elements of our coloured version of the quantum group:

$$a(\lambda)b(\mu) = q^{-1+2\lambda} b(\mu)a(\lambda), \quad a(\lambda)c(\mu) = q^{-1-2\lambda} c(\mu)a(\lambda), \quad (3.3a, b)$$

$$d(\lambda)b(\mu) = q^{1+2\lambda} b(\mu)d(\lambda), \quad d(\lambda)c(\mu) = q^{1-2\lambda} c(\mu)d(\lambda), \quad (3.3c, d)$$

$$b(\lambda)c(\mu) = q^{-2(\lambda+\mu)} c(\mu)b(\lambda), \quad [a(\lambda), d(\mu)] = -(q - q^{-1}) q^{-(\lambda+\mu)} s^{-\lambda+\mu} c(\lambda)b(\mu), \quad (3.3e, f)$$

which might be considered as a  $\lambda, \mu$ -dependent generalisation of (1.1). Moreover, in addition to (3.3), we also get some extra independent relations which after a little manipulation may be expressed as

$$a(\lambda)b(\mu) = (qs)^{\lambda-\mu} a(\mu)b(\lambda), \quad a(\lambda)c(\mu) = (qs)^{-\lambda+\mu} a(\mu)c(\lambda), \quad (3.4a, b)$$

$$d(\lambda)b(\mu) = (qs)^{\lambda-\mu} d(\mu)b(\lambda), \quad d(\lambda)c(\mu) = (qs)^{-\lambda+\mu} d(\mu)c(\lambda), \quad (3.4c, d)$$

$$b(\lambda)c(\mu) = s^{-2(\lambda-\mu)} b(\mu)c(\lambda), \quad a(\lambda)d(\mu) = a(\mu)d(\lambda), \quad (3.4e, f)$$

$$a(\lambda)a(\mu) = a(\mu)a(\lambda), \quad b(\lambda)b(\mu) = q^{2(\lambda-\mu)} b(\mu)b(\lambda), \quad (3.4g, h)$$

$$c(\lambda)c(\mu) = q^{-2(\lambda-\mu)} c(\mu)c(\lambda), \quad d(\lambda)d(\mu) = d(\mu)d(\lambda). \quad (3.4i, j)$$

Observe that for the case of usual groups relations like (3.4) do not occur at all, since they become trivial in the monochromatic limit  $\lambda = \mu$ . Thus the relations (3.3) and (3.4) define together our 'coloured' version of quantum group and reduce to the well known  $GL_q(2)$  case (1.1) when  $\lambda = \mu = 0$ . On the other hand, by taking the limit  $\lambda = \mu \neq 0$  one can similarly reproduce the two parameter deformed  $GL_{p,q}(2)$  quantum group [7]. Notice that many other interesting relations like

$$a(\lambda)b(\mu) = q^{-1+\lambda+\mu} s^{\lambda-\mu} b(\lambda)a(\mu), \quad d(\lambda)a(\mu) = d(\mu)a(\lambda),$$

$$[a(\lambda), d(\mu)] = -(q - q^{-1})q^{\lambda+\mu} s^{\lambda-\mu} b(\lambda)c(\mu),$$

etc. are also derivable from the basic ones (3.3) and (3.4).

After obtaining the quantum group related to the CBGR (1.7), we intend to study some of its mathematical properties. For this purpose, one may first observe that the set of all algebraic relations (1.1) corresponding to  $GL_q(2)$  quantum group can be recast in the Heisenberg-Weyl form, for unimodular values of  $q$  [22]. To achieve this a quantum determinant is usually defined as

$$D = ad - q^{-1}bc,$$

which can be shown to commute with all elements  $a, b, c, d$  through (1.1). Now one can choose a new basis of  $GL_q(2)$  with elements  $D, b, c, d$ , since by using the above relation of quantum determinant the element  $a$  can be expressed as ( assuming the invertibility of  $d$  )

$$a = (D + q^{-1}bc) d^{-1}.$$

Evidently, at this new basis all algebraic relations corresponding to  $GL_q(2)$  quantum group take the Heisenberg-Weyl form for unimodular values of  $q$ . Similar conclusion can be drawn even for the two parameter deformed  $GL_{p,q}(2)$  quantum group [16]. However for this case the quantum determinant has to be defined in a slightly modified way, and it no longer commutes with all elements of  $GL_{p,q}(2)$ . The fact that these quantum groups can be recast in the Heisenberg-Weyl form, plays a crucial role in finding their representations in terms of commuting pairs of canonically conjugate operators and matrices [22,16].

At this point the natural question arises whether relations (3.3) and (3.4), corresponding to our coloured version of quantum group, can also be expressed similarly in the Heisenberg-Weyl form. To accomplish this, we need to choose first generators  $O_i(\lambda)$  ( $i \in [1, 4]$ ) such that the following relations are satisfied :

$$O_i(\lambda)O_j(\mu) = P_{ij}(\lambda, \mu) O_j(\mu)O_i(\lambda), \quad O_i(\lambda)O_j(\mu) = Q_{ij}(\lambda, \mu) O_i(\mu)O_j(\lambda), \quad (3.5a, b)$$

where we have not used any summation convention for repeated indices and  $P_{ij}(\lambda, \mu), Q_{ij}(\lambda, \mu)$  are  $c$ -number functions of the colour parameters. Notice that similar to (3.4), the relations (3.5b) would become trivial in the monochromatic limit  $\lambda = \mu$ . Interestingly, the elements of the matrices  $P$  and  $Q$  occurring in (3.5a,b) can be related through the symmetry conditions given by

$$P_{ii}(\lambda, \mu) = Q_{ii}(\lambda, \mu), \quad P_{ji}(\lambda, \mu) = \frac{1}{P_{ij}(\mu, \lambda)}, \quad Q_{ij}(\lambda, \mu) = \frac{1}{Q_{ij}(\mu, \lambda)}, \quad (3.6)$$

$$Q_{ji}(\lambda, \mu) = Q_{ij}(\mu, \lambda)P_{ij}(\lambda, \mu)P_{ji}(\lambda, \mu).$$

It may also be observed that if one writes down expressions like

$$O_i(\lambda)O_j(\mu) = S_{ij}(\lambda, \mu) O_j(\lambda)O_i(\mu),$$

then the elements  $S_{ij}$  would be completely determined through  $P$  and  $Q$  matrices:  $S_{ij}(\lambda, \mu) = Q_{ij}(\lambda, \mu) P_{ij}(\mu, \lambda)$ . Now, in analogy with the case of  $GL_q(2)$  and  $GL_{p,q}(2)$  quantum groups, we define the quantum determinant for our coloured case as

$$D(\lambda) = a(\lambda)d(\lambda) - q^{-1+2\lambda} b(\lambda)c(\lambda). \quad (3.7)$$

Subsequently by using (3.3) and (3.4), it is not difficult to arrive at the following algebraic relations

$$a(\lambda)D(\mu) = D(\mu)a(\lambda), \quad b(\lambda)D(\mu) = q^{-4\mu} D(\mu)b(\lambda), \quad c(\lambda)D(\mu) = q^{4\mu} D(\mu)c(\lambda), \quad (3.8a, b, c)$$

$$d(\lambda)D(\mu) = D(\mu)d(\lambda), \quad D(\lambda)D(\mu) = D(\mu)D(\lambda). \quad (3.8d, e)$$

We present the derivation of (3.8a) in Appendix A as an illustration. Thus one finds that, relations of the type (3.5a) can be obtained if the elements of the basis are chosen as  $D(\lambda)$ ,  $b(\lambda)$ ,  $c(\lambda)$  and  $d(\lambda)$ . However for the coloured case we need to get also relations like (3.5b), one of which would be of the form

$$D(\lambda)b(\mu) = f(\lambda, \mu) D(\mu)b(\lambda), \quad (3.9)$$

where  $f(\lambda, \mu)$  is a c-number function. Notice that if one substitutes the expression of quantum determinant (3.7) to the above equation, elements carrying the 'colour'  $\lambda$  would occur twice in each term of the l.h.s.. On the other hand, only one such element of 'colour'  $\lambda$  would be present in each term of the r.h.s. of (3.9). So by using quantum group relations (3.3) and (3.4), which preserve the number of elements of any particular colour in both sides of the equation, it seems to be unlikely to get expressions like (3.9) for the general case  $\lambda \neq \mu$ .

To have a way out from this difficulty, let us closely examine the expression of  $a(\lambda)$  which may be obtained from (3.7) as

$$a(\lambda) = \tilde{O}(\lambda) + q^{-1+2\lambda} O(\lambda), \quad (3.10)$$

where

$$\tilde{O}(\lambda) = D(\lambda)d^{-1}(\lambda), \quad O(\lambda) = b(\lambda)c(\lambda)d^{-1}(\lambda),$$

and the existence of inverse of the operator  $d(\lambda)$  has been assumed for all values of  $\lambda$ . Now the key observation is that, in the expression of  $a(\lambda)$  (3.10) the operator  $D(\lambda)$  is not appearing individually, but as a part of the composite operator  $\tilde{O}(\lambda)$ . Consequently, we have the freedom to choose the basis of present coloured version of quantum group as

$$O_1(\lambda) = \tilde{O}(\lambda) = D(\lambda)d^{-1}(\lambda), \quad (3.11)$$

$$O_2(\lambda) = b(\lambda), \quad O_3(\lambda) = c(\lambda), \quad O_4(\lambda) = d(\lambda).$$

Surprisingly, as we would find below, with the new choice of basis (3.11) all quantum group relations can be expressed nicely in the desired form (3.5). As a first step towards this, by

using eqns. (3.3) and (3.8) one can easily obtain the expressions

$$O_1(\lambda)b(\mu) = q^{-1+2\lambda}b(\mu)O_1(\lambda), \quad O_1(\lambda)c(\mu) = q^{-(1+2\lambda)}c(\mu)O_1(\lambda), \quad O_1(\lambda)d(\mu) = d(\mu)O_1(\lambda). \quad (3.12)$$

The only independent relations which are now needed to derive are of the form  $O_1(\lambda)O_j(\mu) = Q_{1j}(\lambda, \mu) O_1(\mu)O_j(\lambda)$ . At this point we notice that by using eqns. (3.3d) and (3.4d) it is possible to arrive at the expression

$$c(\lambda)d^{-1}(\lambda) = \left(\frac{s}{q}\right)^{\lambda-\mu} c(\mu)d^{-1}(\mu). \quad (3.13)$$

It is curious to observe that in contrast to eqns. (3.3) and (3.4), elements of a particular 'colour' are present in unequal numbers in the l.h.s. and r.h.s. of the above relation. By using now this important relation as well as the original ones (3.3) and (3.4), we can obtain the expressions like

$$O(\lambda)b(\mu) = \left(\frac{s}{q}\right)^{\lambda-\mu} O(\mu)b(\lambda), \quad O(\lambda)c(\mu) = q^{-3(\lambda-\mu)}s^{-\lambda+\mu} O(\mu)c(\lambda) \quad (3.14a, b)$$

$$O(\lambda)d(\mu) = q^{-2(\lambda-\mu)} O(\mu)d(\lambda), \quad O(\lambda)a(\mu) = q^{-2(\lambda-\mu)} O(\mu)a(\lambda), \quad (3.14c, d)$$

$$a(\lambda)O(\mu) = q^{2(\lambda-\mu)} a(\mu)O(\lambda), \quad O(\lambda)O(\mu) = O(\mu)O(\lambda), \quad (3.14e, f)$$

where the operator  $O(\lambda)$  is defined as in (3.10). We present the derivation of (3.14a) in Appendix B and the other relations appearing above can also be derived in a similar fashion. Now it is rather easy to verify the validity of the relations

$$O_1(\lambda)b(\mu) = (qs)^{\lambda-\mu} O_1(\mu)b(\lambda), \quad O_1(\lambda)c(\mu) = (qs)^{-\lambda+\mu} O_1(\mu)c(\lambda) \quad (3.15a, b)$$

$$O_1(\lambda)d(\mu) = O_1(\mu)d(\lambda), \quad O_1(\lambda)O_1(\mu) = O_1(\mu)O_1(\lambda), \quad (3.15c, d)$$

by first substituting to them

$$O_1(\lambda) = a(\lambda) - q^{-1+2\lambda} O(\lambda),$$

from (3.10) and then using the eqns. (3.4a,b,f,g) as well as (3.14).



Notice that the expressions (3.12), (3.15) and those of (3.3), (3.4) which do not contain the operator  $a(\lambda)$  play a crucial role for our purpose. Because by starting from them and exploiting the symmetry conditions (3.6), one can finally generate all of the desired relations (3.5). Therefore we see that for the nontrivial choice of the basis (3.11), unimodular values of the deforming parameters  $q$ ,  $s$  and real values of the colour parameters  $\lambda$ ,  $\mu$ , the new 'coloured' quantum group relations can also be cast in the Heisenberg-Weyl form (3.5). Moreover, such relations form a complete set, since by inverting them one can recover all of the original ones (3.3) and (3.4). Thus from the above discussions one may conclude that, some properties of our 'coloured' quantum group are very similar to that of its standard counterparts. In the next section we try to extend this area of similarity further, by investigating whether there exist some invariant Manin planes associated to such coloured quantum group.

#### 4. Manin plane related to CBGR

A rather interesting approach towards quantum groups is found by deforming the coordinates of a vector space to be noncommuting objects, which obey a set of bilinear product relations [6-7,23]. Then the quantum group might be identified as the operator which acts on such noncommutative spaces or Manin planes and preserves the form of the algebra of coordinates even after transformation. For example, in the case of  $GL_q(2)$  group (1.1) we may take the coordinates of a two dimensional vector space as  $x_1, x_2$  and with the help of  $T$ -matrix (1.3) define a transformation like

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad (4.1)$$

Now by assuming the quantum group elements to be commuting with the coordinates and using the relations (1.1), one can show that ( for  $q^2 \neq -1$  ) there exist only two types of bilinear product relations between the coordinates, which remain invariant under the action (4.1). These two types of bilinear relations give us the well known  $q$ -plane and its

exterior plane, respectively [23]:

$$x_1 x_2 = q^{-1} x_2 x_1 ; \quad \xi_1^2 = \xi_2^2 = 0, \quad \xi_1 \xi_2 = -q \xi_2 \xi_1. \quad (4.2a, b)$$

Consequently, the transformed coordinates  $x'$  and  $\xi'$  would also satisfy the above form of commutation relations like  $x$  and  $\xi$ . One can also argue on the other way round and try to find the relations among the elements of  $T$ -matrix (1.3), which would keep the form of commutation relations (4.2a,b) invariant under transformation. The  $GL_q(2)$  quantum group structure (1.1) emerges in a natural way from such requirement. The two parameter deformed  $GL_{p,q}(2)$  quantum group can also be obtained in a similar fashion [7,13,16], by slightly modifying the commutation relations of the coordinates and its differentials (4.2a,b).

Now we turn our attention to the 'coloured' quantum group relations (3.3), (3.4) and attempt to find out whether there exist any invariant Manin plane related to this case. Since at present the transformation matrix  $T(\lambda)$  (3.2) is a function of  $\lambda$ , the coordinates should also be naturally dependent on such colour parameter. So in analogy with (4.1), we may define the transformation as

$$\begin{pmatrix} x_1'(\lambda) \\ x_2'(\lambda) \end{pmatrix} = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix} \begin{pmatrix} x_1(\lambda) \\ x_2(\lambda) \end{pmatrix}. \quad (4.3)$$

Next we make an ansatz of the bilinear product relations between the coordinates of different colours:

$$x_1(\lambda)x_1(\mu) = \alpha(\lambda, \mu) x_1(\mu)x_1(\lambda), \quad x_1(\lambda)x_2(\mu) = \beta(\lambda, \mu) x_2(\mu)x_1(\lambda), \quad (4.4a, b)$$

$$x_1(\lambda)x_2(\mu) = \gamma(\lambda, \mu) x_1(\mu)x_2(\lambda), \quad x_2(\lambda)x_2(\mu) = \delta(\lambda, \mu) x_2(\mu)x_2(\lambda), \quad (4.4c, d)$$

where  $\alpha(\lambda, \mu)$ ,  $\beta(\lambda, \mu)$ ,  $\gamma(\lambda, \mu)$  and  $\delta(\lambda, \mu)$  are  $c$ -number functions. Notice that at the limit  $\lambda = \mu$  associated to the standard cases, the relations (4.4a,c,d) become trivial and from their consistency requirement we should have

$$\alpha(\lambda, \lambda) = \gamma(\lambda, \lambda) = \delta(\lambda, \lambda) = 1.$$

Subsequently we demand that the transformation (4.3) for the coordinates with colour  $\lambda$  and a similar transformation with matrix  $T(\mu)$  for colour  $\mu$ , will keep the form of the relations (4.4) invariant. In other words, the commutation relation between transformed coordinates would be obtained by just replacing  $x_i(\lambda)$  and  $x_j(\mu)$  in (4.4) by  $x'_i(\lambda)$  and  $x'_j(\mu)$  respectively. Exploiting such invariance condition, assuming the matrix elements of  $T(\lambda)$  to be commuting with the coordinates  $x_i(\mu)$  for all values of  $\lambda, \mu$  and using the coloured quantum group relations (3.4), (3.5), it is possible to get after a long but straightforward calculation two sets of solutions for the coefficients  $\alpha(\lambda, \mu), \beta(\lambda, \mu), \gamma(\lambda, \mu), \delta(\lambda, \mu)$  in (4.4). Denoting the coordinates of the invariant 'coloured' planes corresponding to these two sets of solutions by  $x_i(\lambda)$  and  $\xi_i(\lambda)$ , we write down the desired commutations relations as

$$\begin{aligned} x_1(\lambda)x_1(\mu) &= q^{\lambda-\mu} x_1(\mu)x_1(\lambda), \quad x_1(\lambda)x_2(\mu) = q^{-(1+\lambda+\mu)} x_2(\mu)x_1(\lambda), \\ x_1(\lambda)x_2(\mu) &= s^{-\lambda+\mu} x_1(\mu)x_2(\lambda), \quad x_2(\lambda)x_2(\mu) = q^{-\lambda+\mu} x_2(\mu)x_2(\lambda), \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} \xi_1(\lambda)\xi_1(\mu) &= \xi_2(\lambda)\xi_2(\mu) = 0, \\ \xi_1(\lambda)\xi_2(\mu) &= -q^{1-\lambda-\mu} \xi_2(\mu)\xi_1(\lambda), \quad \xi_1(\lambda)\xi_2(\mu) = s^{-\lambda+\mu} \xi_1(\mu)\xi_2(\lambda). \end{aligned} \quad (4.6)$$

It is interesting to notice that at the limit  $\lambda = \mu = 0$ , (4.5) and (4.6) reduce to the commutation relations (4.2a) and (4.2b) respectively, related to the  $GL_q(2)$  quantum group. On the other hand, for  $\lambda = \mu \neq 0$  one would similarly recover the commutation relations of the  $q$ -plane and its exterior plane [7], corresponding to the two parameter deformed  $GL_{p,q}(2)$  case. Thus we surprisingly find that invariant Manin planes can also be attached to the present coloured quantum group relations, in analogy with its standard counterparts.

## 5. Conclusion

In this paper, we have investigated the quantum group related to a 'coloured' braid group representation (CBGR). Interestingly, the well known  $GL_q(2)$  and  $GL_{p,q}(2)$  quantum

groups can be recovered as some special cases, from this 'coloured' quantum group (CQG). In spite of their quite complicated nature, all of these new quantum group relations can be expressed neatly in the Heisenberg-Weyl form, in analogy with its standard counterparts. However to achieve this, a nontrivial choice of the corresponding basis elements seems to play a crucial role. Furthermore, it is possible to associate invariant Manin planes, parametrised by the colour variables, with our CQG structure.

These results might have implications in several directions. Since the new CQG relations can be recast in the Heisenberg-Weyl form, we hope that in analogy with the standard cases its elements may also be realised through mutually commuting pairs of canonically conjugate operators and matrices. Such realisations could be important in the context of quantum integrable models, if one interprets the 'colour parameters' as the 'spectral parameters'. Moreover, it should be interesting to study the CQG relations corresponding to other kind of CBGRs and examine whether they can also be expressed in the Heisenberg-Weyl form. The possibility of attaching invariant Manin planes to the CQG relations might also be much promising. Such approach may lead us to a whole class of noncommutative quantum planes parameterised by the colour variables. However the geometrical interpretation of such colour parameters seems to be yet lacking and whether one can build up differential geometry on these 'coloured' quantum planes might be an interesting problem for future study.

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### Appendix A. Derivation of the relation (3.8a)

By using first the expression of quantum determinant (3.7) we get

$$a(\lambda)D(\mu) = a(\lambda)a(\mu)d(\mu) - q^{-1+2\mu}a(\lambda)b(\mu)c(\mu). \quad (A.1)$$

Now application of the relations (3.4g) and (3.3f) in order yields

$$\begin{aligned} a(\lambda)a(\mu)d(\mu) &= a(\mu)a(\lambda)d(\mu) \\ &= a(\mu)d(\mu)a(\lambda) - (q - q^{-1})q^{-(\lambda+\mu)}s^{-\lambda+\mu}a(\mu)c(\lambda)b(\mu). \end{aligned} \quad (A.2)$$

Next, with the help of the relations (3.3b) and (3.4b) one obtains

$$a(\mu)c(\lambda) = q^{-(1+\lambda+\mu)}s^{\lambda-\mu}c(\mu)a(\lambda). \quad (A.3)$$

By using now relations (A.3), (3.3a) and (3.3e) in order we get

$$\begin{aligned} a(\mu)c(\lambda)b(\mu) &= q^{-(1+\lambda+\mu)}s^{\lambda-\mu}c(\mu)a(\lambda)b(\mu) \\ &= q^{-2+\lambda-\mu}s^{\lambda-\mu}c(\mu)b(\mu)a(\lambda) \\ &= q^{-2+\lambda+3\mu}s^{\lambda-\mu}b(\mu)c(\mu)a(\lambda). \end{aligned} \quad (A.4)$$

Substitution of (A.4) into (A.2) then gives

$$a(\lambda)a(\mu)d(\mu) = a(\mu)d(\mu)a(\lambda) - (q - q^{-1})q^{-2+2\mu}b(\mu)c(\mu)a(\lambda). \quad (A.5)$$

On the other hand, by using relations (3.3a) and (3.3b) in order one obtains

$$\begin{aligned} a(\lambda)b(\mu)c(\mu) &= q^{-1+2\lambda}b(\mu)a(\lambda)c(\mu) \\ &= q^{-2}b(\mu)c(\mu)a(\lambda). \end{aligned} \quad (A.6)$$

Substituting now (A.5) and (A.6) in the r.h.s. of (A.1) we finally arrive at the desired relation (3.8a).

### Appendix B. Derivation of the relation (3.14a)

The application of the relations (3.3e) and (3.3c) in order yields at first

$$b(\lambda)c(\mu)d^{-1}(\mu) = q^{-2(\lambda+\mu)}c(\mu)b(\lambda)d^{-1}(\mu) = q^{1-2\lambda}c(\mu)d^{-1}(\mu)b(\lambda). \quad (B.1)$$

Now by using the definition of operator  $O(\lambda)$  from (3.10) and applying the relations (B.1), (3.4h), (B.1) and (3.13) one by one in order, we would obtain :

$$\begin{aligned} O(\lambda)b(\mu) &= b(\lambda)c(\lambda)d^{-1}(\lambda)b(\mu) = q^{1-2\lambda}c(\lambda)d^{-1}(\lambda)b(\lambda)b(\mu) \\ &= q^{1-2\mu}c(\lambda)d^{-1}(\lambda)b(\mu)b(\lambda) = b(\mu)c(\lambda)d^{-1}(\lambda)b(\lambda) \\ &= \left(\frac{s}{q}\right)^{\lambda-\mu}b(\mu)c(\mu)d^{-1}(\mu)b(\lambda) \\ &= \left(\frac{s}{q}\right)^{\lambda-\mu}O(\mu)b(\lambda). \end{aligned}$$

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