LIMIT ANALYSIS OF A SPHERICAL SHELL UNDER RING LOAD AND THE EXPERIMENTAL VERIFICATION

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ABSTRACT
The upper bound limit theorem has been used here to study the limit load of spherical shell under ring load. A closed form solution for the limit load has been found. A series of experiments was performed with different sizes and different thicknesses of the spherical shell under different diameters of ring loads. Good agreement was found between the theory and the experimental results.

INTRODUCTION
The space grid joint connects many bars of the space grid, which is usually made of a spherical shell with the ratio of diameter to thickness of about 30 - 40. Loads from bars are ring loads, either tension or compression. Tan and Yang [1] used experimental data to get an empirical solution statistically, and they also proved experimentally that damage of the spherical shell joint depended on a single bar load, mainly controlled by compression load. Zhang [2] simplified the classical elastic shell equations and analyzed the shell under concentrated loads as an arch on an elastic foundation. By making the maximum stress in the shell less than the allowable stress, Zhang [2] obtained a solution for the bearing capacity of the joint. This study uses the method of shell limit analysis by Qian [3] to study the problem. From the limit analysis of a spherical shell under ring load, a very simple, closed form equation is proposed here. The equation is found to be consistent with experimental results.

THEORETICAL MODEL
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The displacement can be expressed as

\[
\begin{align*}
    w &= A \frac{a+l}{l} (1 - \frac{r}{a+l}) \\
    u &= A \frac{a(a+l)}{l} \left(1 - \frac{r}{a+l}\right)
\end{align*}
\]

(1)

With the coordinate system shown in figure 1, the strain displacement relations are [from 4]:

\[
\begin{align*}
    e_1 &= \frac{1}{R} \frac{\partial u}{\partial \theta} \\
    e_2 &= \frac{u}{R \theta} \\
    \gamma_{12} &= 0
\end{align*}
\]

(2)

and

\[
\begin{align*}
    \chi_1 &= -\frac{1}{R} \frac{\partial (u + \frac{1}{R} \partial w)}{\partial \theta} \\
    \chi_2 &= -\frac{1}{R \theta} \frac{\partial (u + \frac{1}{R} \partial w)}{\partial \theta} \\
    \chi_{12} &= 0
\end{align*}
\]

(3)

STRAIN OF THE SHELL

Substituting Eq. (1) into (2) and (3), one obtains

\[
\begin{align*}
    e_1 &= -\frac{A}{IR} \left(\frac{\sqrt{R^2 - r^2}}{\sqrt{R^2 - a^2}} a + a + l - r\right) \\
    e_2 &= -\frac{a+l}{IR} \left(\frac{\sqrt{R^2 - r^2}}{\sqrt{R^2 - a^2}} (a - \frac{a}{r} + (1 - \frac{r}{a+l}))\right) \\
    \gamma_{12} &= 0
\end{align*}
\]

(4)

and

\[
\begin{align*}
    \chi_1 &= -\frac{A}{IR^2} (r - \frac{\sqrt{R^2 - r^2}}{\sqrt{R^2 - a^2}} a) \\
    \chi_2 &= -\frac{A}{IR^2 \theta} \left(\frac{a}{\sqrt{R^2 - a^2}} (a+l-r) - R \cos \theta\right) \\
    \chi_{12} &= 0
\end{align*}
\]

(5)

Due to the fact that the distance between two plastic hinges has the same order as the shell thickness, that is one to three times the thickness of shell, in \(a < r < a+l\), one can take \(\sqrt{R^2 - r^2} \approx \sqrt{R^2 - a^2} \equiv 1\). At the same time, omitting the term of \(1/R^2\), one has

\[
\begin{align*}
    e_1 &= -\frac{A}{IR} (2a + l - r) \\
    e_2 &= -\frac{A}{IR} (2a + l - r - a(a+l)) \\
    \gamma_{12} &= 0
\end{align*}
\]

(6)

and

\[
\begin{align*}
    \chi_1 &= 0 \\
    \chi_2 &= \frac{A}{IR} \cos^2 \theta \equiv \frac{A}{IR} \cos^2 \alpha \\
    \chi_{12} &= 0
\end{align*}
\]

(7)

ENERGY RATE CALCULATION

From Qian [3], the energy rate per unit area can be expressed as

\[
T = \int_{h/2}^{h/2} \sigma_1 (|e_1| + |e_2| + |e_1 + e_2|) dz
\]

(8)

where \(t\) is the energy rate per unit area; \(\sigma_1\) is the yielding stress of the material; and \(h\) is the thickness of the spherical shell. Calculating each term separately:

Energy rate of \(e_1\)

From Eq. (6), we know that \(e_1 < 0\) when \(r < 2a + l\); and \(e_1 > 0\) when \(r > 2a + l\). Henceforth, \(e_1\) is negative when \(r\) is in \((a, a+l)\). Thus

\[
T_1 = \int_{a}^{a+l} \sigma_1 (|e_1| + |e_2| + |e_1 + e_2|) dz dr
\]

(9)

\[
= \frac{\pi A \sigma_1 h}{6} (6a^2 + 6al + l^2)
\]

(10)
Energy rate of $e_2$

Let $e_2 = 0$, we have $z$ coordinate of neutral axis,

$$z_0 = \frac{1}{R \cos^2 \alpha} (a + l - r)(r - a).$$

From this equation, we know that $e_2 < 0$ when $z < z_0$; $e_2 > 0$ when $z > z_0$. If we set $z_0 = \pm h/2$, then

$$r_0 = \frac{2a + l}{2} \left( 1 \pm \sqrt{1 - \frac{4a(a + l) \pm 2Rh \cos^2 \alpha}{(2a + l)^2}} \right). \quad (9)$$

As mentioned above, $l$ is of same order as $h$. Therefore, considering $2Rh \cos^2 \alpha > l^2$, we have

$$\frac{4a(a + l) \pm 2Rh \cos^2 \alpha}{(2a + l)^2} > 1.$$  

Henceforth, Eq. (9) has no root for $z_0 = h/2$, while for $z_0 = -h/2$ the roots are

$$r_1 = \frac{2a + l}{2} \left( 1 - \sqrt{1 - \frac{4a(a + l) - 2Rh \cos^2 \alpha}{(2a + l)^2}} \right), \quad (10)$$

$$r_2 = \frac{2a + l}{2} \left( 1 + \sqrt{1 - \frac{4a(a + l) - 2Rh \cos^2 \alpha}{(2a + l)^2}} \right).$$

It is easy to show that $r_1 \leq a$ and $r_2 \geq a + l$. The energy rate per unit area due to $e_2$ is

$$T_2 = \int_{z_0}^{z_2} \sigma e_2 dz + \int_{z_2}^{h/2} \sigma e_2 dz$$

$$= \frac{\sigma A R}{2Ir \cos^2 \alpha} \{ (2a + l - r)^2 - 2a(2a + l - r)(a + l) \}$$

$$+ \frac{a^2 (a + l)^2}{r} \left( \frac{\sigma h}{4} + \frac{r^2}{4} \right) \cos^4 \alpha. \quad (11)$$

Integrating over $r$

$$T_2 = \int_{z_0}^{z_2} 2\pi r dr$$

$$= \frac{\pi A \sigma R}{IR \cos^2 \alpha} \left[ (2a + l)^2 + (2a + l) \frac{1}{3} [(a + l)^3 - a^3] \right]$$

$$- (2a + l)^2 (a + l) a - \frac{1}{2} (2a + l) [(a + l)^4 - a^4]$$

$$+ \frac{1}{5} [(a + l)^3 - a^3] + a^2 (a + l)^2 l + \frac{R^2 h^2}{4} \cos^4 \alpha. \quad (12)$$

Energy rate of $e_1 + e_2$

Similar to $e_2$, the energy rate of $e_1 + e_2$ can be written as

$$T_3 = \int_{a}^{a+l} 2\pi r dr \sigma \left[ \frac{\sigma h}{4} \frac{P^2}{16\pi^2 R^2 \sigma_t} \right] (2a + l). \quad (13)$$

Energy Rate of Plastic Hinges

The energy rate of plastic hinges can be expressed as

$$T_4 = \frac{2\pi A \sqrt{R^2 - (a + l/2)^2}}{IR} \left( \frac{\sigma h}{4} - \frac{P^2}{16\pi^2 R^2 \sigma_t} \right) (2a + l). \quad (14)$$

where $P$ is the external ring load.

To use the upper bound theorem of limit theory, we need the total internal energy rate. Summing Eqs. (8), (12), (13), and (14), the total internal energy rate is

$$T = \frac{A \sigma R}{6} \left[ \frac{1}{6} (6a^2 + 6al + l^2) + \frac{1}{6hR \cos \alpha} (6a^4 + 12a^3 l + 10a^2 l^2 + 4al^3 + 14^3 + 3R^2 h^2 \cos^4 \alpha) \right] +$$

$$2\pi A (2a + l) \sqrt{R^2 - (a + l/2)^2} \left( \frac{\sigma h}{4} - \frac{P^2}{16\pi^2 R^2 \sigma_t} \right). \quad (15)$$

The total external energy rate is not difficult to determine and is

$$V = PA \frac{R}{\sqrt{R^2 - a^2}}. \quad (16)$$

By applying upper bound theorem, $T = V$, we have

$$P \bar{g}_1 + 2\overline{P - g_2} = 0. \quad (17)$$

where
Solving Eq. (17) yields
\[
\bar{P} = \frac{P}{4\pi R^2 \sigma_i},
\]
\[
g_1 = (2a + l)\sqrt{R^2 - a^2} \frac{\sqrt{R^2 - (a + l/2)^2}}{R^2}
\]
\[
g_2 = \frac{h^2 g_1}{4 R^2} + \frac{h\sqrt{R^2 - a^2}}{2 R^4} \left[ \frac{1}{6} (6a^2 + 6al + l^2) + (6a^4 + 12a^2l^2 + 10a^2l^2 + 4al^3 + l^4 + 3R^2h^2 \cos^2 \alpha) \frac{1}{6hR \cos^2 \alpha} \right].
\]
(18)

The length of the plastic region should minimize the value of \( P \). By solving
\[
\frac{\partial P}{\partial l} = 0,
\]
we can get the corresponding \( l_{\text{min}} \). Then substituting back to Eq. (19) or (20), we obtain the solution for the limit load of a spherical shell under ring load.

COMPARISON WITH EXPERIMENTAL RESULTS

Table 1 is a comparison of experimental results with the results calculated using Eq. (19).

It is apparent that the calculated results from equation (19) are very consistent with the experimental results. If the ratio of the diameter of the shell to the diameter of the loading pipe is less than 1/3, the difference between the theoretical value here and the experimental value is within 10%.

Note that, although equation (19) is derived by the upper bound theorem of limit analysis theory, the experimental results in table 1 are not always less than the values from equation (19). The reasons may be many. One possible reason is that the thicknesses of the spherical shells are not uniform due to manufacture process, while the thicknesses used in equation (19) are always taken to be 0.86 times the plate thickness, from which the spherical shells were made. When the load is applied to thick part, the test limit load could be higher than that from equation (19) and vice versa. Many other factors, such as experimental uncertainties, material deviation and others, can also contribute to the difference.

CONCLUSION

A very simple closed form solution has been derived from the upper bound theorem of limit analysis of shell structure. The solution has been proved to be consistent with experimental results. However, if using the equation for a structure other than a space grid joint (for which the equation is intended to be used for), one should be careful not to apply the solution to a case with a larger ratio of load radius to shell radius. In such a case, the assumed damage mechanism might not reflect the real damage form of structure. For a space grid joint, this ratio is always less than 1/3. In this case, the limit load from equation (19) agrees well with test data.

REFERENCES

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