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SOME RESULTS ON THE PHASE DIAGRAM OF THE ANTIFERROMAGNETIC POTTS MODEL IN AN EXTERNAL MAGNETIC FIELD

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ABSTRACT

Phase diagram of the q state antiferromagnetic Potts model on a square lattice is investigated in a neighborhood of the critical point $h = 4, T = 0$. Using a constructive criterion of uniqueness introduced by Dobrushin, Kolafa, and Shlosman for the antiferromagnetic Ising model, we show that the activity, $a_c(q) > 1/(q-1)$ and the slope increases from zero ($q = 2$) to infinity ($q = \infty$). The phase diagram bulges near the mentioned critical point for $2 < q < \infty$.

Introduction

The purpose of this paper is to extend the study of two dimensional Ising antiferromagnet done by Dobrushin, Kolafa, and Shlosman,[1] to the q state antiferromagnetic Potts model in neighborhood of ground state point h=4, T=0. In the following we shall use the same notations as in Ref.[1].

We consider the Potts antiferromagnet on a square lattice with spin variables $\sigma_t = 1, \ldots, q$ *attached to lattice site* $t \in \mathbb{Z}^2$ *and with the Hamiltonian*

$$H = \sum_{<st>} \delta_{\sigma_s \sigma_t} - h \sum_t \delta_{\sigma_t} \qquad (1.1)$$

where the first sum is over all nearest neighbours $s,t \in Z^2$, $|s-t| = 1$, and h is the magnetic field on every site $t \in Z^2$. for $0 < h < 2d = 4$, there are two different ground configurations $\sigma^{(1)}, \sigma^{(2)}$ in this model :

$$\sigma_t^{(1)} = \begin{cases} 1 & \text{whenever } t_1 + t_2 \text{ is even} \\ 2,\ldots,q & \text{whenever } t_1 + t_2 \text{ is odd} \end{cases}$$

and

$$\sigma_t^{(2)} = \begin{cases} 2,\ldots,q & \text{whenever } t_1 + t_2 \text{ is even} \\ 1 & \text{whenever } t_1 + t_2 \text{ is odd} \end{cases}$$

These configurations are stable i.e they generate extremal Gibbs states which differ in a region [2,3]

$$\beta\,(4 - h\,) > \mu \qquad\qquad \text{for } h > 0$$

with μ a positive constant .Using Finite-Size-Scaling method ,Bakchich, Benyoussef and Touzani [4] obtained the phase diagram (T , h) given in fig.1 , for q=4 and q=5 which show the phase transition between the antiferromagnetic phase and the paramagnetic phase. The slope of the transition line, between the antiferromagnetic and the paramagnetic phases, at h=4, T=0 increases with increasing q.

Our purpose consist of the proof with a mathematically rigorous method [1] , that the curve of phase transition bulges near the critical point h = 4, T=0 ,and show that the slope increases with increasing q at the mentioned critical point .

Nameley ,we have .

Theorem 1:

There exist $\Theta(q)$, $0 < \Theta(q) < \pi/2$, and $r > 0$ such that there is a non uniqueness antiferromagnet Gibbs state with parameters (h,T) in the Domain

$\{ (h,T) \,/\, h - 4 = \tilde{r} \sin\tilde{\Theta}$, $T = \tilde{r} \cos\tilde{\Theta}$, $-\pi/2 \le \tilde{\Theta} \le \Theta(q)$, $0 \le \tilde{r} \le r \}$.
(see fig.2)

Hence the curve of phase transition bulges above the half line $h > 4$ [at least near (h=4 , T=0)].

To prove this theorem we have used the method described in [1] which is based on a constuctive criterion of uniqueness presented by Dobrushin , and Shlosman [5] , and applied to Ising antiferromagnet by Dobrushin , Kolafa and Shlosman [1]. In Ref [5] a system of conditions Cv (for Gibbsian conditional distributions in a volume V) , $V \subset Z^2$, $V < \infty$ was constructed. These conditions have the property that if an interaction ϕ and β are such that condition Cv is true for a finite volume V then there is a unique Gibbs state for these interactions ϕ and β , as for interactions and β's in its neighbourhood . when a volume V enlarges , it becomes more complicated to verify the condition Cv , so it may be necessary to use computer . In this work we shall use the numerical result obtained by [1] .It turned out that the proof of bulging near (h=4 ,T = 0) for $2 < q < \infty$ is deduced from the condition Cv for a rectangular volume V=3 x 4 [1]. The slope increases with q for a fixed volume V , and goes to infinity for the limit $q \to \infty$. for fixed value of q ,the slope decreases when V enlarges and , in the limit $V \to \infty$, this slope goes to the value Ln(q-1).

The paper is organized as follows . In section 2 we discuss a connection between the Potts antiferromagnet and the hard square lattice-gas . In section 3 we rewrite the uniqueness criterion of [5] for the hard square lattice gas . Section 4 is reserved to some details of checking the condition Cv (V=1x1 , V=2x1 ,V=2x2 and V=3x4) for the hard square lattice gas .

2. The hard-square lattice-gas

In the following we shall consider a neighbourhood of the point (h=4, T=0). It can be directly shown that all configuration σ for which $|s-t| > 1$ whenever $\sigma_s = 2,\ldots,q$ and $\sigma_t = 2,\ldots,q$ are ground configurations at this point ($s = |s_1| + |s_2|$ for $s \in Z^2$) ; i.e spin values (2,...,q) cannot be neighbouring . These configurations are represented as follows :

unique				nonunique		
1	1	1		1	α'	1
1	1	1		α	1	δ
1	1	1		1	γ	1

where $\alpha=2,\dots,q$; $\alpha' = 2,\dots,q$; $\delta = 2,\dots,q$ and $\gamma = 2,\dots,q$.

We shall call compatible all configurations with this property . The antiferromagnetic Potts model have an infinite number of ground configurations at the point (h=4 , T=0). one would thus take for this ground states not every ground configuration itself but rather some special measures on these configurations . Such an approch is developed in detail in [3] where a ground state is defined as a state which is consistent with a ground specification . The ground specification , in its turn , is defined as a consistent set of conditionnal distributions which are concentrated for any boundary condition , on configurations which minimize a functional of relative energy . The model with such configurations is known as the model of hard square . Look now for measures arising naturally on these ground configurations of the Potts antiferromagnet at the point (0,4). to do this let us consider the following transformation :

$$\sigma_t \dashrightarrow s(\sigma_t) \text{ such that } s(\sigma_t)=1 \quad \text{for } \sigma_t = 1$$
$$s(\sigma_t)=-1 \quad \text{for } \sigma_t = 2,\dots,q \qquad (2.1)$$

Then $\delta_{\sigma_t 1}$ of the Hamiltonian (1.1) is given by

$$\delta_{\sigma_t 1} = \frac{1+s(\sigma_t)}{2}$$

Let us consider conditional distributions $q_V^{h,\beta}(.\,|\,.)$ of the model (1.1) in a volume V as

function of parameters h and β ,and let us consider the limit $\beta \dashrightarrow \infty$, $h \dashrightarrow 4$ along the direction

$$h = 4+\mu\beta^{-1} \qquad (2.2)$$

we obtain a specification $q_V^{\mu}(.\,|\,.)$ for the hard-squares with a chemical potentiel μ which corresponds to the interaction

$$\phi_A^{\mu}(s(\sigma_A)) = \begin{cases} +\infty & A=\{s,t\}\ |s-t|=1,\ s(\sigma_s) = s(\sigma_t) = -1 \\ \frac{\mu}{2} s(\sigma_t),\ A = \{t\} & \\ 0 & \text{otherwise} \end{cases}$$

The values $\sigma_t=2,\dots,q$ corresponds to a particle at the site t , while $\sigma_t=1$ to a vacancy . (In these variable the chemical potentiel μ has a value $(-\mu)$,and it is to expect that there exists a value $\mu_c(q)$, such that for $\mu > \mu_c(q)$ a Gibbs state of the potts antiferromagnet with h and β satisfying (2.2) ought to be unique , while for $\mu < \mu_c(q)$ and $\beta > \beta(\mu)$ the Gibbs state is nonunique. Then the value $\mu_c(q)$ determines an angle $\Theta(q)$ depending on the value of q , at which the line of the phase transition in the Potts antiferromagnet intersects the h axis . In section 4 we check the condition Cv and then we prove that every interaction which is close enought to the interaction of the hard-squares generates a unique Gibbs state whenever $\mu > Ln(q-1)$. In this way we prove the statement about the Potts antiferromagnet in Theorem1 .

3.Condition of uniqueness .

In this section we follow [1] to state the criterion of uniqueness in a form suitable for the hard-square model . Let us recall some notations . Let (X,ρ) be a metric space with a finite number of points and ξ,η two probability measures on X . A joint distribution of ξ,η is a probability measure P on X x X such that for every $Y \subset X$:

$$P(Y \times X) = \xi(Y)$$
$$P(X \times Y) = \eta(Y)$$

We denote by $P(\xi,\eta)$ the set of all joint distributions of measures ξ,η . The Kantorovich (Kantorovich -Rubinstein-Ornstein-Vasserstein) distance $R(\xi,\eta)$ is defined by the formula

$$R(\xi,\eta) = \min_{P\in P(\xi,\eta)} \sum_{x,y\in X} \bar{\rho}(x,y)\ P(x,y)$$

Let now $V\subset Z^2$ be an arbitrary finite volume and $t_0 \in \partial V$ with

$$\partial V = \{ t \in Z^2|_V \ / \ \mathrm{dist}(t,V) = 1 \}.$$

We introduce a metric on the set of all configurations

$$\Omega_V = \{ s(\sigma_v) \ / \ s(\sigma_v) : V \dashrightarrow \{-1,1\} \}$$

by

$$\rho(s(\sigma_v)^{(1)}, s(\sigma_v)^{(2)}) = \frac{1}{2} \sum_{t\in\partial V} \left| s(\sigma_v(t))^{(1)} - s(\sigma_v(t))^{(2)} \right|$$

Let $s(\tilde{\sigma}) \in \Omega_{\partial V/t0}$ be a configuration , we define the configurations $s(\tilde{\sigma})^{\pm} \in \Omega_{\partial V}$ by the formula

$$s(\tilde{\sigma}(t))^{\pm} = s(\sigma(t)) \quad \text{for } t \neq t_0$$

$$s(\tilde{\sigma}(t))^{\pm} = \pm 1 \quad \text{for } t = t_0$$

Let ϕ be a pair nearest neighbour interaction (i.e a translation invariant system of functions $\phi_{\{s\}}(\sigma), \phi_{\{s,t\}}(\sigma,\tau)$, $s,t \in Z^2$, $|s-t| = 1$, $\sigma,\tau = 2, \ldots, q$ with values in $R \cup \{\infty\}$) ,and $q_V^{\phi}(.\,|\, s(\tilde{\sigma})^{\pm})$ the corresponding conditonal Gibbs distribution in a volume V with the boundary condition $s(\tilde{\sigma})^{\pm}$. We set ,

$$K_{t_0}^{v} = \max_{s(\tilde{\sigma})} R(q_v^{\phi}(.|s(\tilde{\sigma})^{+}, q_v^{\phi}(.|s(\tilde{\sigma})^{-})$$

Theorem 2 :

Let $V\subset Z^2$ be a finite volume such that an interaction ϕ satisfies the following condition C_V :

$$\sum_{t\in\partial V} K_t^V < |V|$$

Then there is a unique Gibbs state for the interaction ϕ.

The proof is given in [5].

We say that pair nearest neighbour interactions ϕ' and ϕ'' are ε-close if

$$| \exp(-\phi'_s(\sigma)) - \exp(-\phi''_s(\sigma)) | < \varepsilon$$

and

$$| \exp(-\phi'_{s,t}(\sigma,\tau)) - \exp(-\phi''_{s,t}(\sigma,\tau)) | < \varepsilon$$

for all $\sigma,\tau = 1,\ldots,q$ and all points $s \in Z^2$ and all pairs $\{s,t\}$ of the nearest neighbours .

Corollary 1 :

Let an interaction ϕ satisfy the condition Cv with some $V\subset Z^2$. Then there exists $\varepsilon_V > 0$ such that the statement of uniqueness in theorem 2 is valid for every interaction ϕ' ,which is ε_V-close to ϕ .

Corollary 2 :

Suppose that the interaction ϕ^{μ} of the hard-square lattice system satisfy the condition Cv (where V may depend on μ) whenever $\mu \geq 0$. Then

(a) There exists $\mu_0(q) > 0$. such that the uniqueness takes place in the hard-square model whenever $\mu > -\mu_0(q)$.

(b) The phase diagram of the Potts antiferromagnet looks as described in theorem 1

Proof of corollary 2 .

Let us notice that the distance $R(q^1_V, q^2_V)$ is continuous function of values of measures $q^i_V(s(\sigma_v)^{(1)}), \ldots\ldots, q^i_V(s(\sigma_v)^{(2^{|v|})})$, i= 1, 2 , $s(\sigma_v)^{(j)} \in \Omega_V$. Then we obtain (a) from corollary 1 . Since $q^{h,\beta}_V$ tend to q^μ_V along the line $h = 4+\mu\beta^{-1}$ uniformly with respect to μ on any finite interval of inverse temperatures [β_μ, ∞] exists for which the uniqueness for the hard-squares with the chemical potentiel μ . And then it can be easily shown that for every $\varepsilon > 0$ there exists $\mu_q(\varepsilon) < \infty$ [1] such that for all $\mu > \mu_q(\varepsilon)$ the interactions $\beta\phi^h$ (ϕ^h is an interaction of the Potts antiferromagnet for $h=4+\mu\beta^{-1}$) and ϕ^μ are ε-close for all β .

4. A check of the uniqueness conditions

The *Kantorovich distance between distributions* $q^\mu_V(.\mid s(\sigma)^1)$, $q^\mu_V(.\mid s(\sigma)^2)$ has an important property that for any joint distribution $P \in P(q^\mu_V(.\mid s(\tilde\sigma)^+), q^\mu_V(.\mid s(\tilde\sigma)^-)$) we have an estimate ,

$$R(q^\mu_v(.|s(\tilde\sigma)^+), q^\mu_v(.|s(\tilde\sigma)^-) \le d(p) = \sum_{s(\sigma_v)^1, s(\sigma_v)^2 \in \Omega_v} P(s(\sigma_v)^1, s(\sigma_v)^2)\rho(s(\sigma_v)^1, s(\sigma_v)^2)$$

with :

$$q^\mu_v(s(\sigma_v)|s(\tilde\sigma)) = \frac{[(q-1)e^{-\mu}]^{|s(\sigma_v)|}}{\sum_{s(\sigma_v)\in\Omega_v(s(\tilde\sigma))} [(q-1)e^{-\mu}]^{|s(\sigma_v)|}}$$

where

$$|s(\sigma_v)| = \sum_{t\in v} s(\sigma_v(t))$$

$s(\sigma_v(t)) = 1$ for $\sigma_v(t) = 2, \ldots\ldots, q$

$s(\sigma_v(t)) = 0$ for $\sigma_v(t) = 1$

Now we shall look for a finite volume V for each $\mu \in$ [Ln(q-1), $+\infty$[, and a bound K^v_{t0} of K^v_t for every $t_0 \in \partial V$, so that the condition Cv

$$\sum_{t\in\partial V} K^V_t < |V|$$

takes place with $d(p) \le K^v_{t_0}$ for all boundary conditions $s(\tilde\sigma) \in \Omega_{\partial V | t_0}$

4.1 The estimate P" of the joint distribution .

A joint distribution P may be considered as a matrix $P(s(\sigma_v)^1, s(\sigma_v)^2)$ labelled in rows by *configurations* $s(\sigma_v)^1$, *compatible with the boundary condition* $s(\tilde\sigma)^+$, and *in columns* by configurations $s(\sigma_v)^2$ compatible with $s(\tilde\sigma)^-$. Let us denote the corresponding sets by $\Omega_V(s(\tilde\sigma)^\pm)$. Let us emphasize here that the configurations $s(\sigma_v)^1$ or $s(\sigma_v)^2$ itself need not be compatible . If we were interested in the hard-square model only we could confine ourselves to compatible boundary conditions $s(\tilde\sigma)^\pm$. However ,if we want to study a uniqueness of the antiferromagnet , we have to investigate the hard-square model in a way which is stable with respect to small perturbations , and we have thus to take into account all boundary conditions .

The most simple way to define a joint distribution is to set [1]

$$P(s(\sigma_v)^1, s(\sigma_v)^2) = q^\mu_v(s(\sigma_v)^1 \mid s(\tilde\sigma)^+) q^\mu_v(s(\sigma_v)^2 \mid s(\tilde\sigma)^-)$$

$(s(\sigma_v)^1, s(\sigma_v)^2) \in A \equiv \Omega_V(s(\tilde\sigma)^-) \times \Omega_V(s(\tilde\sigma)^+)$ i.e by pairs $(s(\sigma_v)^1, s(\sigma_v)^2)$,which are compatible with both $s(\tilde\sigma)^-$ and $s(\tilde\sigma)^+$ (see fig.3). It can be easily shown that the matrix [1]

$$P'(s(\sigma_v)^1,s(\sigma_v)^2)=\begin{cases} q_v^{\mu}(s(\sigma_v)^1 \mid s(\tilde{\sigma})^+)\ , & s(\sigma_v)^1 = s(\sigma_v)^2 \quad (s(\sigma_v)^1\ ,\ s(\sigma_v)^2\ ) \in A \\ 0\ , & s(\sigma_v)^1 \neq s(\sigma_v)^2 \quad (s(\sigma_v)^1\ ,\ s(\sigma_v)^2\ ) \in A \\ P(s(\sigma_v)^1,s(\sigma_v)^2) & \text{otherwise} \end{cases}$$

is again a joint distribution of $q_v^{\mu}(\ .\mid s(\tilde{\sigma})^+)$, $q_v^{\mu}(\ .\mid s(\tilde{\sigma})^-)$ and $d(P') \leq d(P)$.

Where $s(\sigma_v)^1 = s(\sigma_v)^2$ if $s(\sigma_v(t))^1 = s(\sigma_v(t))^2$ for all $t \in V$.

Now we shall introduce a block B of a joint distribution matrix . Let $t_1 \in V$ be a nearest neighbour of $t_0 \in \partial V$ (there is only one such t_1 if the volume V is rectangular).We say that $(s(\sigma_v)^1\ ,\ s(\sigma_v)^2) \in B$ if $s(\sigma_v(t_1))^1 = -1$

$s(\sigma_v(t_1))^2 = +1$ and $s(\sigma_v(t_j))^2 = +1$ whenever $t \in V$, $|t - t_1| = 1$

Let us denote the former set of configurations by $\Omega_V(\text{-})$ and the latter by $\Omega_V(++)$ (see fig.3). In the same way as it was done with the block A we may change the symmetric square block B of the matrix P' into " diagonal " submatrix with a diagonal given by $s(\sigma_v(t))^1 = s(\sigma_v(t))^2$ for $t \in V|_{t_1}$. Thus we obtain the matrix

$$P''(s(\sigma_v)^1,s(\sigma_v)^2)=\begin{cases} q_v^{\mu}(s(\sigma_v)^1 \mid s(\tilde{\sigma})^+) \sum\limits_{s(\sigma_v)\in\Omega_v(++)} q_v^{\mu}(s(\sigma_v) \mid s(\tilde{\sigma})^-) \\ \quad = q_v^{\mu}(s(\sigma_v)^2 \mid s(\tilde{\sigma})^-) \sum\limits_{s(\sigma_v)\in\Omega_v(-)} q_v^{\mu}(s(\sigma_v) \mid s(\tilde{\sigma})^+) \ \text{for} \\ \quad s(\sigma_v(t))^1 = s(\sigma_v(t))^2\ , t\in V|_{t_1} \quad (s(\sigma_v)^1\ ,\ s(\sigma_v)^2\ ) \in B \\ 0\ , \quad s(\sigma_v(t))^1 \neq s(\sigma_v(t))^2 \\ \quad \text{for at least one } (s(\sigma_v)^1\ ,\ s(\sigma_v)^2\ ) \in B \\ P'(s(\sigma_v)^1,s(\sigma_v)^2) \quad \text{otherwise} \end{cases}$$

We have again $d(P'') \leq d(P')$. Finally , we denote by C the block of pairs $(s(\sigma_v)^1\ ,\ s(\sigma_v)^2)$ which are not in A or B .

4.2 Eliminating of some boundary conditions

To find a " good " joint distributions for all sites $t_0 \in \partial V$ and all boundary conditions $s(\tilde{\sigma}) \in \Omega_{\partial V|_{t_0}}$. Taking into account that their number is considerable ($|\partial V|\ 2^{|\partial V|_{t_0}|}$) , let us first eliminate some of them for a case of rectangular volume using the following rules (see fig.4) .

(a) We use a space symmetry

(b) if t_1 , $t_2 \in \partial V|_{t_0}$ are neighbours of a corner-site $s \in V$, then all three boundary conditions with ($s(\tilde{\sigma}(t_i)) = -1$) , for i =1 or i = 2 or both are equivalent (they give the same sets $\Omega_V(s(\tilde{\sigma})^{\pm})$.

(c) Let us denote by $s(\tilde{\sigma})_{-1}$ some boundary conditions with t_1 , t_2 and $s(\sigma(t_i))$ as above and with t'_1 , $t'_2 \in \partial V|_{t_0}$ such that $|t_j - t'_j| = 1$, j = 1 , 2 (thus $|t'_1 - t'_2| = 4$) , $s(\tilde{\sigma}(t'_j))_{-1} = -1$, moreover , let $s(\tilde{\sigma})_{+1}$ be such that $s(\tilde{\sigma}(t'_j))_{+1} = -1$,$s(\tilde{\sigma}(t_j))_{+1} = +1$,

then

$\Omega_V(s(\tilde{\sigma})^{\pm}_{+1}) = \Omega_V(s(\tilde{\sigma})^{\pm}_{-1})$ X { $s(\sigma(s))=+1$, $s(\sigma(s)) = -1$ } and knowing a joint distribution for $s(\tilde{\sigma})_{-1}$ we can easily obtain a joint distribution for $s(\tilde{\sigma})_{+1}$ with the same distance .

(d) If $t_0 \in \partial V$ and $t \in \partial V|_{t_0}$ are neighbours of a corner-site $s \in V$, then a configuration $s(\tilde{\sigma}) \in \Omega_{\partial V|_{t_0}}$ for $s(\sigma(t)) = -1$ gives the zero distance since $\Omega_V(s(\tilde{\sigma})^+) = \Omega_V(s(\tilde{\sigma})^-)$ and thus P can taken as a diagonal square matrix .

(e) Let t'_j , t_j be as in (c) but to stays insteed of t_2 and $s(\tilde{\sigma}(t_1))_{+1} = +1$, then

$$R(q^{\mu}_v(\,.\,|\,s(\tilde{\sigma})^+),\, q^{\mu}_v(\,.\,|\,s(\tilde{\sigma})^-)) = \frac{q-1}{q-1+e^{\mu}}$$

which is the same as for V : 1 X 1 = { s }

4.3 Uniqueness conditions for V= 1 X 1 , V= 2 X 1 , V= 2 X 2

As an illustration we shall consider V= 1 X 1 , V= 2 X 1 and V= 2 X 2 . we have respectively one ,one and four non equivalent (in the sens described in precedent section) boundary conditions $s(\tilde{\sigma}) \in \Omega_{\partial V|_{t_0}}$ (see fig.5 ,a,b,c) . The computation of d(P") for each of one case (V= 1 X 1 , V= 2 X 1) and four cases (V= 2 X 2) can be easily made "by hand " . Fig.6,a,b,c show an example of the matrix P" (for boundary condition respectively for V= 1 X 1 , V= 2 X 1 and V= 2 X 2. The results are :

$$\underset{\sigma}{\mathrm{Max}}\, d(P")_{1\times1} = \frac{a}{1+a}$$

$$\underset{\sigma}{\mathrm{Max}}\, d(P")_{2\times1} = \frac{a}{1+a}$$

$$\underset{\sigma}{\mathrm{Max}}\, d(P")_{2\times2} = \frac{a(1+2a)}{1+3a+a^2}$$

with $a = (q-1)\, e^{-\mu}$

According to the condition Cv we need :

$$K_t^{v=1\times1} < \frac{|V|}{|\partial v|} = \frac{1}{4}$$

$$K_t^{v=2\times1} < \frac{|V|}{|\partial v|} = \frac{1}{3}$$

$$K_t^{v=2\times2} < \frac{|v|}{|\partial v|} = \frac{1}{2}$$

for all $t \in \partial V$. Then $d(P) \le K^V_{t_0}$ holds for :

$$\mu > \mu_{1\times1}(q) = \mathrm{Ln}(3) + \mathrm{Ln}(q-1)$$

$$\mu > \mu_{2\times1}(q) = \mathrm{Ln}(2) + \mathrm{Ln}(q-1)$$

$$\mu > \mu_{2\times2}(q) = \mathrm{Ln}\left(\frac{6}{1+\sqrt{13}}\right) + \mathrm{Ln}(q-1)$$

So it is easy to show that the critical value $\mu_c(q)$ of the model (1.1) is related to that of Ref [1] by the following relation (for fixed V) :

$$\mu_c^V(q) = \mu_c^V(2) + \mathrm{Ln}(q-1)$$

and for V= 3 x 4 ($\mu_c(2) = 0$) [1]

$$\mu_c(q) = \mathrm{Ln}(q-1)$$

So the slope increases when q increases such that :

$$\mu_c(2) = 0 \;\text{ and }\; \mu_c(\infty) = \infty$$

These results give the proof of theorem1.

5.Conclusion :

We have calculated the slope of the curve of phase transition near the critical point h = 4 , T = 0, however this slope increases with increasing q such that $\mu_c(q) = \mathrm{Ln}\,(q-1)$. Then the phase diagram of the antiferromagnet Potts model (1.1) bulges near the critical point (h = 4 ,T =0) for $2 < q < \infty$. and in particular case (q = 2) the slope is equal zero [1].

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Figures captions

Fig.1 : Phase diagram in (h,T) plane within finite-size-scaling method.

Fig.2 : Phase diagram near the critical point h = 4 , T=0.

Fig.3 : Blocks A, B, and C of the joint distribution matrix.

Fig.4 : Equivalence of boundary conditions

$t_0 = +$ when the variable $s(\bar{\sigma}(t_0)) = +1$

$t_0 = -$ when the variable $s(\bar{\sigma}(t_0)) = -1$

Fig.5 Distances for non equivalent boundary conditions in the (a),V = 1x1 ; (b), V= 2x1 ; and (c) ,V = 2x2 , with

$$M = \frac{1}{(1+a)(1+3a+a^2)}$$

Fig.6 : Example of the joint distribution matrix ($a = (q-1)e^{-\mu}$) ; (a),V=1x1 ; (b), V=2x1 ; (c) V = 2x2 .

In (b)

$$N = \frac{1}{(1+a)(1+2a)}$$

In (c)

$$N = \frac{1}{(1+2a)(1+3a+a^2)}$$

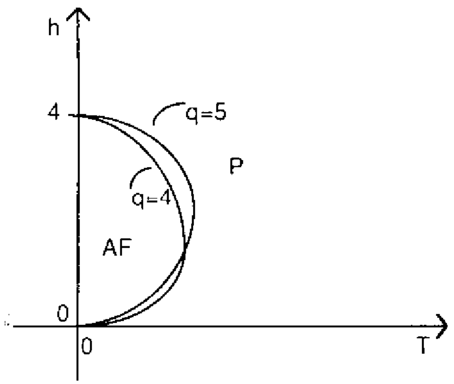

Fig.1

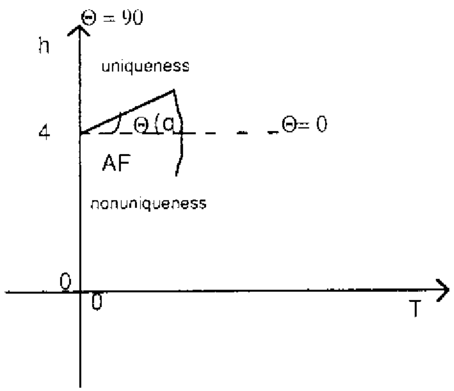

Fig.2

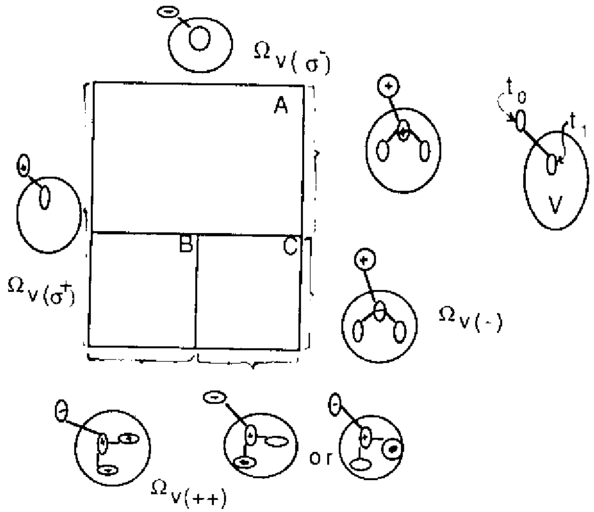

Fig.3

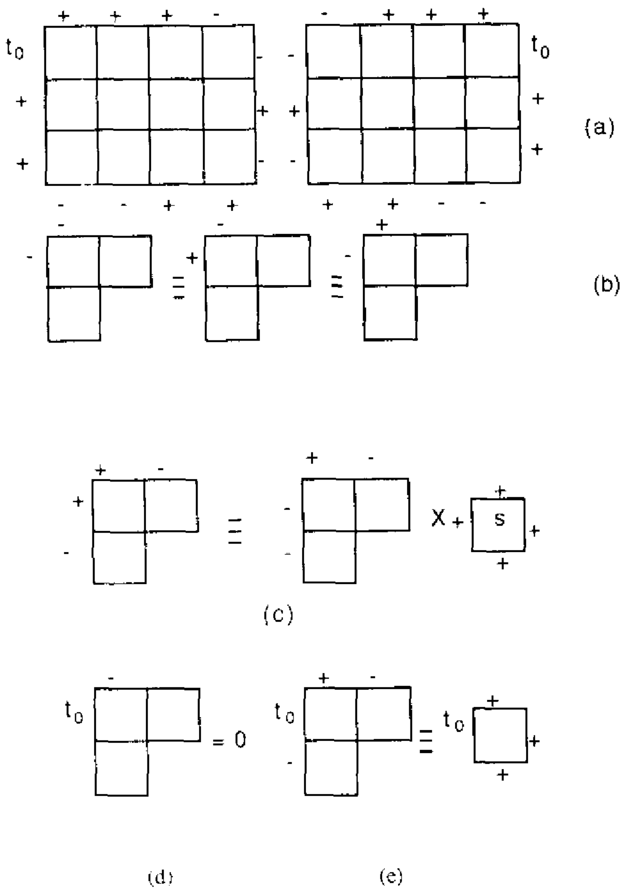

Fig.4

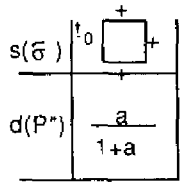

Fig.5(a)

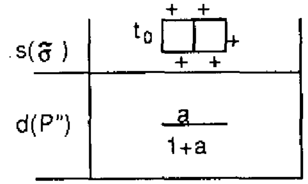

Fig.5(b)

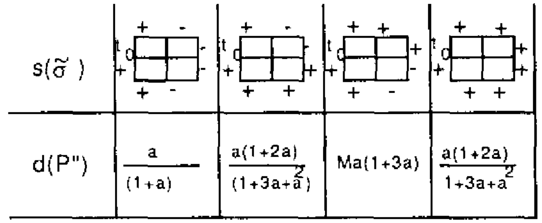

Fig.5(c)

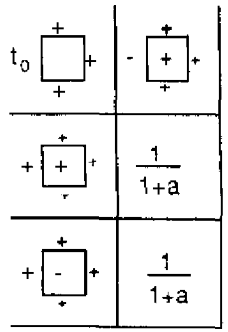

Fig.6(a)

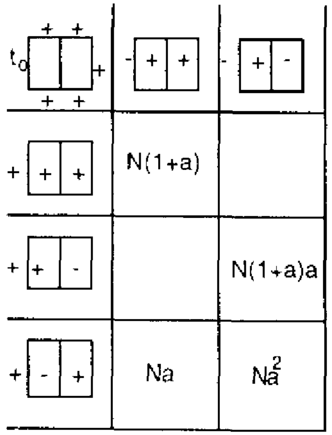

Fig.6(b)

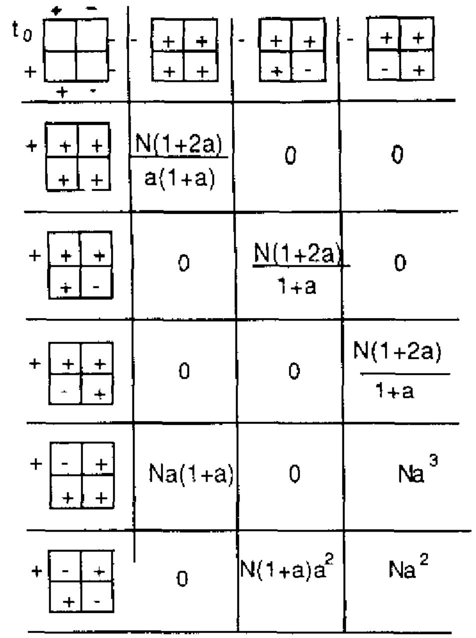

Fig.6(c)