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PERIODIC ELECTROMAGNETIC VACUUM IN THE TWO-DIMENSIONAL YANG-MILLS THEORY WITH THE CHERN-SIMONS MASS

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ABSTRACT

The periodic vacuum structure formed from magnetic and electric fields is derived in the two-dimensional Yang-Mills theory with the Chern-Simons term. It is shown that both the magnetic flux quantization in the fundamental sell and conductivity quantization inherent to the vacuum. Hence, the quantum Hall effect gets its natural explanation.

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1. One of the interesting topics of nowadays physics is the investigation of the two-dimensional gauge field theories [1]. First of all this is stimulated by attempts to understand the quantum Hall effect and high temperature superconductivity [2]. Besides, a theory in two space dimensions possesses a number of interesting properties. One of them is the possibility to introduce a gauge field mass without destroying of gauge invariance. This mass, the well known Chern-Simons term [1], has the topological nature and effects various processes.

In the present letter we investigate the vacuum in a magnetic field H of the two-dimensional Yang-Mills field with the topological mass $m \neq 0$. As was shown in our paper [3], in this model the homogeneous magnetic field - the well known Savvidy state [4] - is spontaneously generated. In contrast to the massless case where this field is completely unstable because of tachyonic mode in the gluon spectrum [5],[6], for $m \neq 0$ there is a threshold H_0 of the instability appearance and for some big masses either the vacuum magnetization and stabilization of this state happen to be realized. Our aim here is to continue the vacuum investigations in some other line and determine a vacuum in magnetic fields below H_0. To do it we will derive the solution of the time-dependent classical field equations which describes a lattice of static magnetic and electric fields. This lattice can be realized uniquely due to the Chern-Simons term and actually exhibits the periodic structure of the new type. In it the magnetic flux quantization in the fundamental cell and conductivity quantization are realized. Hence, the quantum Hall effect gets its natural explanation. The most essential vacuum characteristics (the gluon condensate amplitude, current, magnetic and electric fields) will also be calculated.

2. Let us consider the Lagrangian of the two-dimensional $SU(2)$ gluodynamics with the Chern-Simons term:

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \frac{m}{4} \in^{\mu\nu\lambda} (G^a_{\mu\nu} A^a_\lambda - \frac{1}{3} g \in_{abc} A^a_\mu A^b_\nu A^c_\lambda) \ , \tag{1}$$

where A^a_μ is the gauge field potential, $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \in_{abc} A^b_\mu A^c_\nu$, $\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu + g \in^{abc} A^c_\mu$, g is a coupling constant, m is the Chern-Simons mass and $\mu = 0, 1, 2$; $a = 1, 2, 3$ are the space-time and internal indexes, respectively. For what follows it will be convenient to introduce "the charged basis" of fields,

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \pm i A^2_\mu), \quad A_\mu = A^3_\mu \ .$$

Then the Lagrangian takes the form:

$$\begin{aligned} \mathcal{L} = & -P^*_\mu W^+_\nu P^\mu W^{-\nu} + P^*_\mu W^+_\nu P^\nu W^{-\mu} - ig W^+_\mu W^-_\nu F^{\mu\nu} \\ & - \frac{g^2}{2} [(W^+_\mu W^{-\mu})^2 - W^+_\mu W^{+\mu} W^-_\rho W^{-\rho}] - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ & - im \in^{\mu\nu\lambda} W^+_\mu P_\nu W^-_\lambda + \frac{m}{2} \in^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \ , \end{aligned} \tag{2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the kinetic momentum $P_\mu = i\partial_\mu + g(\bar{A}_\mu + \hat{A}_\mu)$ includes the external field $\bar{A}_\mu$ and the field $\hat{A}$ generated in the system. Let us take the external field potential in the form: $\bar{A}_\mu = \delta_{\mu_2} H x_1$. The corresponding field equations are the following:

$$P^2_\lambda W^{-\mu} - P^\mu P_\lambda W^{-\lambda} + 2ig F^{\mu\lambda} W^-_\lambda + im \in^{\mu\lambda\nu} P_\lambda W^-_\nu$$

$$+ g^2(W^{-\mu}W^{+}_{\lambda}W^{-\lambda} - W^{+\mu}W^{-}_{\lambda}W^{-\lambda}) = 0 \ , \tag{3}$$

$$\partial_\lambda F^{\lambda\mu} = I^\mu \ , \tag{4}$$

$$I^\mu = g[(W^{-}_{\lambda} P^{*\mu} W^{+\lambda} - 2W^{-}_{\lambda} P^{*\lambda} W^{+\mu}_{+} \tag{5}$$

$$W^{-\mu} P^{*}_{\lambda} W^{+\lambda}) + h.c.] - m \in^{\mu\lambda\nu} (\partial_\lambda A_\nu - igW^{+}_{\nu}W^{-}_{\lambda})$$

Now, let us apply Abrikosov's method [7] and determine a vacuum in magnetic fields H near the threshold H_0 (calculated below). The linearized equation (3) with $\hat{A} = 0$ reads,

$$\bar{P}^2_\lambda W^{-\mu} - \bar{P}^\mu \bar{P}_\lambda W^{-\lambda} + 2ig\bar{F}^{\mu\lambda} W^{-}_{\lambda} + im \in^{\mu\lambda\nu} \bar{P}_\lambda W^{-}_{\nu} = 0 \ , \tag{6}$$

where now $\bar{P}_\mu = i\partial_\mu + g\bar{A}_\mu$, $\bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu$.

Multiplying Eq. (6) on $\bar{P}_\mu$ and taking into account the commutation relation $[\bar{P}_\mu, \bar{P}_\nu] = ig\bar{F}_{\mu\nu}$ one obtains the constraint

$$W^{-}_{0} = 0 \ . \tag{7}$$

With this condition it is easy to check that there are no static solutions of Eq. (6). This is not a surprise because the Chern-Simons term mixes electric and magnetic components of A_μ and so both of them should be generated in a vacuum. The electric field will effect the charged components so only the stationary (not static) solutions for $W^{\pm}_{\mu}$ may be realized. To incorporate this idea let us extend the class of possible functions and introduce the following ansatz for W^{-}_{μ}:

$$W^{-}_{\mu}(x, y, t) = W^{-}_{\mu}(x, y) e^{-i\omega t} \ , \tag{8}$$

with ω to be determined below. Performing the Fourier transformation

$$W^{-}_{\mu}(x, y) = \int W^{-}_{\mu}(x, k) e^{iky} \frac{dk}{(2\pi)} \ , \tag{9}$$

let us rewrite Eq.(6) in the form:

$$(i\omega\partial_x - im(gHx - k))W^{-}_{1} + (-m\partial_x + \omega(gHx - k))W^{-}_{2} = 0 \ ,$$

$$(-\omega^2 + (gHx - k)^2)W^{-}_{1} + (-i(gH_x - k)\partial_x - im\omega + igH)W^{-}_{2} = 0 \ ,$$

$$(-i(gH_x - k)\partial_x + im\omega - 2igH)W^{-}_{1} + (-\partial^2_x - \omega^2)W^{-}_{2} = 0 \ . \tag{10}$$

This system has a solution

$$W^{-}_{\mu} = \frac{c}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}_{\mu} \exp\left[-\frac{gH}{2}(x - \frac{k}{gH})^2 + iky - i\omega t \right] \ , \tag{11}$$

where c is a constant, $\mu = 1, 2$ and ω satisfies the relation

$$\omega^2 - m\omega + gH = 0 \ . \tag{12}$$

The vector $b_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}_{\mu}$ is the spin operator eigenvector with eigenvalue $+1$. Notice that function (11) satisfies the condition $\bar{P}_\mu W^{-\mu} = 0$. From Eq. (12) one obtains:

$$\omega = \omega^{\pm} = \frac{m}{2} \pm \sqrt{\frac{m^2}{4} - gH} \ .$$

Hence it follows that the stationary solution exists in the fields $H \le H_0 = \frac{m^2}{4g}$. In the fields $H > H_0$ the frequency becomes imaginary and the stationary solution does not exist. So, in what follows we will consider $H < H_0$.

3. In order to determine a vacuum structure in fields $H \simeq H_0$ near the threshold let us, following to Abrikosov [7], take into account a degeneracy in k and introduce the linear combination of functions (11) with different momenta.

$$W_L(x,y,t) = e^{-i\omega t} \sum_n c_n \exp[inky - \frac{gH_0}{2}(x - \frac{nk}{gH_0})] , \tag{13}$$

where c_n are coefficients. To have a lattice solution it is necessary to subject c_n the periodicity condition

$$c_{n+\nu} = c_n ,$$

with ν to be a integer number. Then one obtains

$$W_L(x + \frac{\nu k}{gH_0}, y, t) = e^{-ik\nu y} W_L(x,y,t) . \tag{14}$$

As well known [7,8], the solution properties are weakly dependent on the particular choice of c_n and k. Only the form of the lattice depends on them. The degeneracy will be removed when the non-linearity of field system will be taken into account. It also leads to the normalization of function (13). Below the index L at W_L will be omitted.

Now, let us calculate the current (5) in the vacuum. To do it let us write down Eq. (13) in the form: $W = |W(x,y)| \exp[i\theta(x,y) - i\omega t]$ and substitute it and the vector b_μ to the Eq.(5). We have,

$$\begin{aligned} I_0 &= -2g\omega|W|^2 - mF_{12} + mg|W|^2 , \\ I_1 &= \partial_2(g|W|^2 + m\hat{A}_0) , \\ I_2 &= -\partial_1(g|W|^2 + m\hat{A}_0) . \end{aligned} \tag{15}$$

Here, the terms $\sim \hat{A}|W|^2$ were omitted because they are small as compare $|W|^2$. Substituting (15) into (4) one obtains the relation between the field $\hat{F}_{12}$ generated in the vacuum and the field $|W|$:

$$\hat{F}_{12} = g|W|^2 + m\hat{A}_0 . \tag{16}$$

As it is seen, the potential A_0 contributes to the magnetic field strength due to the topological mass presence. With Eq.(16) the Eq.(4) for the component $\hat{A}_0$ reads

$$(\partial_1^2 + \partial_2^2 - m^2)\hat{A}_0 = 2g\omega|W|^2 + mH . \tag{17}$$

The second term in the right-hand side is just the induced charge in a magnetic field. It does not produce an electric field in the (x,y)-plane and does not effect a vacuum structure. So, we shall omit it in what follows. Then the solution to Eq.(17) gets the form,

$$\hat{A}_0(\vec{r}) = -\frac{g\omega}{\pi} \int d\vec{r}' K_0(m|\vec{r} - \vec{r}'|)|W(\vec{r}')|^2 , \tag{18}$$

where $K_0(x)$ is the MacDonald function, $\vec{r}, \vec{r}'$ are radia in the (x,y)-plane. It is very essential that $\hat{A}_0(\vec{r})$ is time independent one. Hence, the strength (16) is gauge invariant because the static $\hat{A}_0$ potential is gauge invariant itself.

A more simple local relation between $\hat{A}_0(\vec{r})$ and $W(x,y)$ can be derived in the approximation $\frac{\Delta \hat{A}_0}{A_0} \ll m^2$. In this case from Eq.(17) one obtains:

$$\hat{A}_0(x,y) = -\frac{2g\omega}{m^2}|W(x,y)|^2 \, . \tag{19}$$

This approximation was used in Ref.[9] for the lattice description in the electroweak theory and is a good one. It represents all the peculiarities of the vacuum structure and moreover differs from results obtained with the exact non-local relation (18) only in 10% [10]. So, in what follows we restrict ourselves by the discussion of relation (19). Hence, for magnetic and electric fields we have:

$$\hat{F}_{12} = g(1 - \frac{2\omega}{m})|W|^2 \, ,$$

$$\hat{F}_{ko} = -\frac{2g\omega}{m^2}\partial_k |W|^2 \, . \tag{20}$$

To derive the normalization condition for W let us take into account the non-linearity of the system. The energy of fields is given by the expression

$$U = \int d^2x T^{00} =$$

$$= \int d^2x [i\partial_0 W_\mu^+ P_0 W^{-\mu} - i\partial_0 W_\mu^- P_0^* W^{+\mu} + m \in^{\nu 0 \mu} W_\nu^+ \partial_0 W_\mu^- - \mathcal{L}] \tag{21}$$

where $\mathcal{L}$ is the Lagrangian (2). Then assuming that W_μ and A_μ give a minimum of energy and substituting $W \to (1+\varepsilon)W$; $\varepsilon \ll 1$ in equation (21) one obtains for energy variation

$$\begin{aligned} \delta U \; = \; & 2\varepsilon \int d^2x \{ -i(P_0^* + gA_0^{(1)}) W^{+\mu}\partial_0 W_\mu^- + i(P_0 + gA_0^{(1)}) \cdot \\ & \cdot W^{-\mu}\partial_0 W_\mu^+ + m\varepsilon^{\mu 0 \nu} W_\nu^+ \partial_0 W_\mu^- - W^{+\nu}(P_\nu P_\mu + gA_\nu^{(1)} P_\mu \\ & + gP_\nu A_\mu^{(1)}) W^{-\mu} + W_\nu^+ (P_\mu + 2gA_\mu^{(1)}) P^\mu W^{-\nu} + im \in^{\mu\lambda\nu} W_\mu^+ (P_\lambda + gA_\lambda^{(1)}) W_\nu^- \\ & + 2ig(F_{\mu\nu} + F_{\mu\nu}^{(1)}) W^{+\mu} W^{-\nu} + im \in^{\mu\lambda\nu} W_\mu^+ (P_\lambda + gA_\lambda^{*(1)}) W_\nu^- \\ & + g^2 (W_\mu^+ W^{-\mu})^2 + \frac{1}{2} F^{\mu\nu} F_{\mu\nu}^{(1)} - m \in^{\mu\lambda\nu} A_\mu \partial_\lambda A_\nu^{(1)} \} = 0 \end{aligned} \tag{22}$$

where the terms of the order $\sim \varepsilon^2$ have been omitted. This expression was derived with the field A_μ written in the form, $A_\mu = \bar{A}_\mu^{(0)} + A_\mu^{(1)}$, where $A_\mu^{(1)}$ appears due to the W presence and deviation H from H_0. besides, Eq.(6) and the condition, $\bar{P}_\mu W^{-\mu} = 0$ was also used. Taking into account that $F_{12}^{(1)} = H - H_0 + g(1 - \frac{2\omega}{m})|W|^2$, the following normalization conditions can be derived:

$$\left[\frac{\omega^3}{m} + g(H_0 - H)\right] \overline{|W|^2} = \frac{4g^2\omega^2}{m^2} \overline{|W^4|} \, , \tag{23}$$

where $\overline{|W|^n}$ denotes functions integrated over the volume V of the system. Written in this form the normalization condition plays an essential role because it does not depend on the specific structure of the lattice. Notice that for $m = 0$ from Eq.(23) and relation (12) follows $\overline{|W|^4} = 0$, that is the structure does not realized.

The obtained normalization condition (23) differs in an essential way from that of superconductivity [7] and electroweak theory [9]. In particular, the gluon condensate W, does exist at $H = 0$ for the frequency $\omega = \omega^+$ in Eq.(12). The energy calculation described below shows that solution with $\omega = \omega^+$ is favorable. So, it should to be realized. That is what below we will discuss results for this one. As it was mentioned before, the choice of the lattice structure can be done by fixing the specific values of k and ν. This gives a possibility to calculate explicitly the ratio

$$\beta = \frac{\overline{|W|^4}}{(\overline{|W|^2})^2} \; . \tag{24}$$

As is well known [8], for the triangular lattice one has $\nu = 2$, $\beta = 1,16$ and for square lattice - $\nu = 1$, $\beta = 1,18$. This information is sufficient for calculation of the macroscopic vacuum characteristics.

4. First, let us calculate the magnetic induction B - the average value of the microscopic magnetic field. With Eq.(20) it can be written in the form:

$$B = H + g(1 - \frac{2\omega}{m})\overline{|W|^2} \; . \tag{25}$$

Using the normalization condition (23) and relation (24) one obtains,

$$B = H + \frac{m^2}{4g\omega^2\beta}(1 - \frac{2\omega}{m})(\frac{\omega^3}{m} + g(H_0 - H)) \; . \tag{26}$$

As follows from Eq.(26) for the choice $\omega = \omega^+$ the gluon condensate partially screens the external field, and for $\omega = \omega^-$ - antiscreens it.

The mean energy density ($\varepsilon = U/V$) of fields can be calculated with Eqs. (21), (3)-(5):

$$\varepsilon = \frac{H^2}{2} + \frac{\omega^3}{m}\overline{|W|^2} - \frac{6g^2\omega^2}{m^2}\overline{|W|^4} \; , \tag{27}$$

Using once again Eqs. (23), (24) let us rewrite this expression as follows,

$$\varepsilon = \frac{H^2}{2} - \frac{1}{8g^2\omega^2\beta}[\omega^3 + 3mg(Ho - H)][\omega^3 + mg(H_0 - H)] \tag{28}$$

As numerical calculations show ε is decreasing with the decreasing of β at fixed B, therefore, the triangular lattice should be realized in the vacuum.

Now, let us consider ε as a function of H. The numerical calculation shown that ε is an increased function of H and its value is alway smaller as compared with the energy of homogeneous field applied. At some critical value of $H = H_c$ the total energy gets to be zero. For the choice $\omega = \omega^+$ this value is $H_c \simeq 0,912 H_0$, that is close enough to the threshold H_0. For smaller external fields energy becomes negative. This means that actually the lattice structure can be spontaneously generated. As an example let us note the particular values of $\varepsilon(H)$; For $H = H_0$, $\varepsilon = \frac{H_0^2}{2}(1 - \frac{1}{4\beta})$, for $H = 0$, $\varepsilon = -\frac{35}{8\beta}H_0^2$. It is interesting to note that latter value is smaller than then energy of Savvidy's state with the same m.

5. Let us discuss other properties of the lattice. Applying standard arguments as in superconductivity [7]-[10] it is easy to determine the magnetic field quantization in the fundamental cell with the flux quantum $\phi_0 = \frac{2\pi}{g}$ as in the electroweak theory [9],[10].

The second essential property is the conductivity quantization. The vacuum current I_i and electric field are connected in a standard way: $I_i = \sigma_{ij} E_j$, where σ_{ij} is the conductivity tensor. If one compares this expression with Eqs. (15), (20), the following explicit form for σ_{ij} can be derived,

$$\sigma_{ij} = \varepsilon_{ij} m(1 - \frac{m}{2\omega}) \equiv \varepsilon_{ij}\sigma, (i,j = 1,2) \ , \qquad (29)$$

ε_{ij} is the antisymmetric unit tensor. As it is seen, the current and field appears to be orthogonal to each other and the conductivity σ is a function of the external field as in the Hall effect. The most simple expression for σ will be in the limit $H \to 0$; $\sigma = \frac{m}{2}$. As is well known [1], the Chern-Simons mass must be quantized,

$$m = \frac{g^2}{4\pi} n \ , \qquad (30)$$

n is an integer number. Hence, the conductivity quantization immediately follows:

$$\sigma = \frac{g^2}{8\pi} n \ . \qquad (31)$$

If one compares this expression with the Hall conductivity $\sigma_H = \frac{g^2}{2\pi} n_H$ [2], the following relation can be derived:

$$n_H = \frac{n}{4} \ .$$

Thus, in the model under consideration the quantum Hall effect gets natural explanation.

6. The two-dimensional Yang-Mills theory with the Chern-Simons mass term exhibits an attractive feature and has a non-trivial vacuum state - the lattice of magnetic and electric fields with the cells $\sim \frac{1}{m}$. This periodic vacuum has an energy lower than the Savvidy state and the latter one does not realized. The lattice divides space in cells and in this way the tachyonic mode is excluded.

It is very essential that in two dimensions the magnetic field confines the charged fields. So, the static and stationary time-dependent configurations are to be physically equivalent ones. As it also has been determined, all the observables - electric and magnetic fields, current, energy, etc.- are time-independent too. The magnetic flux quantization and the conductivity quantization derived above are a strict consequence of $m \neq 0$. So this mechanism would work for other models with $m \neq 0$.

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References

1. S. Deser, R. Jackiw and S. Templeton, Ann. Phys. (NY) **140** 91982) 372.

2. Quantum Hall Effect, Eds. R.E. Prange and S.M. Girvin, Springer-Verlag, New York, 1987.

3. V.V. Skalozub, S.A. Vilensky and A.Yu. Zaslavsky, Yad. Fiz. **56** (1993) 230.

4. G.K. Savvidy, Phys. Lett. **B71** (1977) 133.

5. V.V. Skalozub, Sov. J. Nucl. Phys. **28** (1978) 113.

6. N.K. Nielsen and P. Olesen, Nucl. Phys. **B144** (1978) 376.

7. A.A. Abrikosov, JETP **32** (1952) 1442.

8. W.H. Kleiner, L.M. Roth and S.H. Autler, Phys. Rev. **133** (1964) 1226.

9. V.V. Skalozub, Sov. J. Nucl. Phys. **43** (1986) 665; **45** (1987) 1058.

10. S.W. MacDowell and O. Törnkvist, Phys. Rev. **D45** (1992) 3833.