COURSE X

QUANTUM AND CLASSICAL PROPERTIES OF SOME BILLIARDS ON THE HYPERBOLIC PLANE

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M.-J. Giannoni, A. Voros and J. Zinn-Justin, eds.
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Chaos et Physique Quantique /Chaos and Quantum Physics
1. Introduction

As you already have heard the lectures of Colin de Verdière, I shall suppose in these two lectures that you know almost everything on the classical and quantum aspects of the dynamics on manifolds of negative curvature. Therefore I shall not recall any theoretical point about the ergodicity, and the mixing and chaotic properties of the classical motion on such compact manifolds. Neither shall I recall the proof of the Selberg trace formula which is an exact relation between the quantal spectrum and the periodic classical orbits. In fact, in these two lectures I shall present you some "experimental" results on the quantal spectrum of some billiards on two-dimensional manifolds of constant negative curvature. Besides we shall see how the use of the Selberg trace formula may bring some interesting new results on the properties of the classical motion.

But let us now turn to the problem that we shall treat. As you probably know, the main results about classical motion on manifolds of constant negative curvature have been obtained for compact manifolds. In two dimensions, such compact manifolds may be constructed by considering a polygon on the hyperbolic plane and by sewing or identifying pairs of sides. This operation is however possible only if the polygon tessellates the whole hyperbolic plane under translations. For instance, in the case of zero curvature there are only two compact regular polygons which tessellate the Euclidean plane, namely the square and the hexagon. For constant negative curvature, however, there are an infinite number of regular compact polygons which tessellate the hyperbolic plane, the simplest being the regular octagon which is presented in Figure 1. In this octagonal tessellation of the hyperbolic plane, the motion of a particle is completely chaotic as is perfectly illustrated in the paper of N.L. Balazs and A. Voros [1] to which I refer the reader for more details.

This classical system being completely chaotic, it constitutes a good example to check our hypothesis about the properties of its quantal spectrum [2]. Indeed, several "experimental" studies of quantum chaotic
systems have shown that their spectra differ markedly from the spectra of classically integrable systems. For integrable systems, one observes a purely random spectrum obeying Poisson's statistics, whereas for chaotic systems, the quantal spectra exhibit a quite strong level repulsion and rigidity and are fairly well described by the Random Matrix Theory [3]. Note that, although some theoretical works have been made to explain these facts [4], these conclusions are mainly experimental and other experimental evidences should be welcome. It will be the aim of the following lectures to present some new (and quite unexpected) results about the quantal spectrum of the octagon on the hyperbolic plane.

Fig. 1. Poincaré disk representation of the octagon which tessellates the hyperbolic plane under translation. The translation $g_0$ brings side $DE$ onto side $AH$; $g_1$: $EF$ onto $AB$; $g_2$: $FG$ onto $BC$; and $g_3$: $GH$ onto $CD$. 
1.1. Numerical methods

The general problem of finding the eigenvalues of the Laplace–Beltrami operator on a compact manifold is a rather difficult task for which we do not know any efficient numerical method. Besides, due to the many symmetries of the problem, the total spectrum will present numerous near-degeneracies which will make the numerical problem even more difficult. To avoid these difficulties, the easiest way is to treat separately each symmetry class, so that, first, the problem is reduced to a much simpler billiard problem and, second, the expected occurrence of near-degeneracies will be strongly reduced. For instance, it is easy to show that a wave function which is an eigenfunction of the Laplace–Beltrami operator in the triangle $\triangle OLM$ (see Fig. 1) with Dirichlet conditions on the boundary, is indeed a solution of the original problem (see Appendix 1). Thus the problem is reduced to a simple billiard problem which may be written as:

$$\frac{(1-r^2)^2}{4} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right] \Psi_E(\vec{r}) = -E \Psi_E(\vec{r}) \quad \vec{r} \in D$$

$$\Psi_E(\vec{r}) = 0 \quad \vec{r} \in OL, OM, LM$$

where we have explicitly used the expression of the Laplace–Beltrami operator in polar coordinates.

To solve this eigenvalue problem, we use an improved version of the "point-matching" or "collocation" method. In this method, the unknown function $\Psi_E(\vec{r})$ is expanded on a complete basis of the eigenfunctions of the Laplace–Beltrami operator:

$$\Psi_E(\vec{r}) = \sum_{m=M_{\text{min}}}^{M_{\text{max}}} a_m F_m(E,r) e^{im\varphi}$$

where the $F_m(E,r)$ are the regular solutions of the following differential equation:

$$\frac{(1-r^2)^2}{4} \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} \right] F_m(E,r) = -EF_m(E,r).$$

By introducing the new variable $Z = (1+\varphi)/(1-r^2)$ one easily gets the transformed differential equation:

$$\left[ (1-Z^2) \frac{d^2}{dZ^2} - 2Z \frac{d}{dZ} + E - \frac{m^2}{1-Z^2} \right] F_m(E,\sqrt{\frac{Z-1}{Z+1}}) = 0$$
in which one recognizes the differential equation of the Legendre function. Thus the eigenfunctions $F_m(E, r)$ are proportional to the regular Legendre functions

$$F_m(E, r) \propto P_{n-1/2}^m \left( \frac{1 + r^2}{1 - r^2} \right) \quad \text{with} \quad E = k^2 + \frac{1}{4}, \quad (5)$$

and therefore may be evaluated easily using their known properties.

Now, in the usual point-matching method, one imposes the condition that the wave function vanishes at $N$ given boundary points, where $N$ is equal to the number of unknown parameters $a_m$ in equation (2):

$$\Psi(r(\theta_i)) = \sum_m a_m F_m(E, r(\theta_i)) e^{im\delta_i} = 0, \quad i = 1, \ldots, N. \quad (6)$$

The condition for the existence of a non-trivial solution may then be written as:

$$\det\{d_{im}(E)\} = 0, \quad i = 1, \ldots, N, \quad m = M_{\text{min}}, \ldots, M_{\text{max}}; \quad (7)$$

with the matrix $d_{im}(E)$ given by:

$$d_{im}(E) = F_m(E, r(\theta_i)) e^{im\delta_i}.$$

Obvious drawbacks of the method are, first, that the determinant is not well behaved, and second and more important, that the number of boundary points used is equal to the number of partial waves. As it is well known, these drawbacks are rather severe and yield strong instability of the solutions. However, one may easily improve this method. In fact, on the boundary of the billiard, the wave function as given by equation (2) is a periodic function of the curvilinear abscissa $s$, its period being the billiard perimeter $L$ (we are using the negative curvature metric). It is clearly equivalent to impose the condition that the wave function vanishes on the boundary or that the Fourier coefficients of this periodic function vanish. Thus we impose the following equivalent conditions:

$$I_m(E) = \int_0^L ds \, e^{-2i\pi m's/L} \Psi(r(s)) = 0 \quad (8)$$

$$= \sum_{m=M_{\text{min}}}^{M_{\text{max}}} a_m J_{m'm}(E) \quad M_{\text{min}} \leq m' \leq M_{\text{max}},$$
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with

\[ J_{m',m}(E) = \int_0^\ell ds \ e^{-2i m' s / \ell} F_m(E, r(s)) e^{i m s}, \]

and the condition for the existence of a non-trivial solution may be written as:

\[ \det \{ J_{m',m}(E) \} = 0, \quad M_{\text{min}} \leq m', m \leq M_{\text{max}}. \] (9)

This improved method presents several advantages. First, in the determinant of equation (9), the nearly diagonal elements are usually much larger than the far off-diagonal ones, so that this determinant is well-behaved and thus may be evaluated numerically with a good precision. The second advantage is that the precision in the evaluation of the matrix elements \( J_{m',m}(E) \) of the determinant may be increased by increasing the number of boundary points, and this without changing the number of partial waves. Thus, with this method, one can check the stability of the solutions with respect either to the number of partial waves or to the number of points used to evaluate the integral. Before going further, I would like to point out that this method may obviously be used for any billiard problem with any boundary conditions (Dirichlet, Neumann or mixed). For instance, we have used this method for the stadium billiard with a magnetic field where it proved to be very efficient [2]. We shall see now that this method is also very efficient and allows a rapid and precise evaluation of the eigenvalues of a triangular billiard on a negative curvature manifold.

In our specific case, the wave function expansion has to be modified to take into account the properties of our domain, so that it now reads as:

\[ \Psi_E(r) = \sum_{n=1}^N a_n F_{sn}(E, r) \sin(8n\theta). \] (10)

Thus, the Dirichlet conditions on sides \( OL \) and \( OM \) are automatically satisfied and the only boundary condition which has to be imposed is on the side \( LM \). The length of the latter being \( L_{LM} \), the condition for \( E \) to be an eigenvalue may be written as:

\[ \det \{ I_{n,m}(E) \} = 0, \quad 1 \leq n, m \leq N, \] (11a)

with

\[ I_{n,m}(E) = \int_{L_{LM}} ds \ \sin \left( \frac{n s S}{L_{LM}} \right) F_{sn}(E, r(s)) \sin(8m\theta(s)). \] (11b)
Finally, our numerical problem will be solved if we are able to compute the eigenfunctions $F_m(E, r)$ which, as we have seen, are proportional to the Legendre functions. To do so, we choose the functions $F_m(E, r)$ as:

$$F_m(E, r) = \lambda_{m,E} P^m_{ik-1/2} \left( \frac{1 + r^2}{1 - r^2} \right)$$  \hspace{1cm} (12)

with

$$\lambda_{m,E} = (-1)^m \prod_{n=1}^{m} \frac{1}{\sqrt{k^2 + (n - 1/2)^2}}, \quad E = k^2 + \frac{1}{4}.$$  \hspace{1cm} (13)

With this particular normalisation, the functions $F_m(E, r)$ are still real, and they bear some close resemblance to the Bessel functions that we would use for a plane billiard. For instance, the completeness relation of Legendre functions takes now the following form:

$$\sum_{m=-\infty}^{+\infty} F^2_m(E, r) = 1,$$  \hspace{1cm} (14)

whereas the recurrence relation becomes:

$$\sqrt{k^2 + (m + 1/2)^2} F_m(E, r) = (m + 1) \frac{1 + r^2}{r} F_{m+1}(E, r) - \sqrt{k^2 + (m + 3/2)^2} F_{m+2}(E, r).$$  \hspace{1cm} (15)

The behaviour of $F_m(E, r)$ as a function of $m$ is the same as that of the Bessel functions, i.e. for large $m$ ($> 2kr/(1 - r^2)$) the function $F_m(E, r)$ decreases as $(kr/m)^m$. This property allows the use of the recurrence relation (13), starting from $F_{M+2}(E, r) = 0$ and:

$$F_{M+1}(E, r) = \alpha \left( \frac{kr}{M + 1} \right)^{M+1}, \quad \text{with} \quad M = 8N + 30 > \frac{2kr}{1 - r^2}.$$  \hspace{1cm} (16)

The use of relation (13) then determines the value of the constant $\alpha$. This numerical method has been checked by evaluating the generating function of Legendre functions either directly or by summing the series. The two results were found to differ by less than $10^{-6}$, an accuracy which is fully sufficient for our purpose.
In our numerical calculations, it was found that a good stability of the solutions may be obtained, if the largest angular momentum in Eqs. (10) and (11), \( M_{\text{max}} = 8N \), is chosen as:

\[
M_{\text{max}} = \frac{2kr_{\text{max}}}{1 - r_{\text{max}}^2} + 24,
\]

where \( r_{\text{max}} \) is the largest Euclidean radial distance reached by the boundary (\( r_{\text{max}} = OL \) in the triangle \( OLM \)). Finally, the number of equally spaced points used in the evaluation of the integral \((\text{lib})\) is determined by the condition that the fastest varying phase varies by less than \( 2\pi \) between successive points.

2. Results

2.1. Tiling triangle

The algorithm described previously proved to be very efficient, since a one hour run on the UNIVAC 1191 yields the first 1500 eigenvalues of our triangular billiard \( OLM \). However, given this sequence of levels, one may wonder whether this sequence is pure and complete. Indeed, it could happen that the algorithm misses some levels; for instance, around almost degenerate levels, the determinant of Eq. (11a) could keep a constant sign, or could change sign twice in so small an interval of \( E \) that one could miss these two changes of sign and thus miss two levels. The easiest method to study the purity of the sequence is the method that we already used for the case of a billiard with a magnetic field [2] and is a graphical method. In fact, if one plots the deviation of the exact cumulative density from the averaged one as given by the Selberg trace formula, this quantity is expected to wander around zero if no level is missing, whereas it should wander around \( n \) if \( n \) levels are missed. On Figure 2, we present the plot of \( \delta_n \) as a function of \( n \), where \( \delta_n \) is given by:

\[
\delta_n = \tilde{N}(E_n) - \bar{N}(E_n) = \tilde{N}(E_n) - (n - 1/2)
\]

with

\[
\tilde{N}(E_n) = \frac{S}{4\pi}E - \frac{L}{4\pi}\sqrt{E} + \text{const.},
\]

where \( S \) is the billiard area (and \( L \) the perimeter) in the hyperbolic metric [1].
Fig. 2. Plot of the distance $\delta_n$ between the exact and averaged position of the $n$-th level as a function of $n$ for the tessellating triangle $OLM$ (the points have been joined by a curve to guide the eye).
One should note that this plot has already been published in a slightly different form in our study of the convergence of the generalised Zeta function evaluated at $s = 0$ [5].

On this figure, we immediately see that our sequence of levels is pure and complete, since the curve does indeed wander around zero. Moreover, this figure exhibits another property of the spectrum, namely its strong rigidity. Indeed, we may note that the amplitude of the deviation $|\delta_n|$ is always smaller than 2, which indicates that in all this sequence, the distance between a given eigenvalue and its theoretical averaged position is always smaller than two level spacings. For a pure random sequence the variance of this distance would increase as $n$, so that on such a sample one would observe levels displaced by 30 or 40 level spacings from their averaged theoretical positions. Thus this spectrum appears to be extremely rigid, as expected for a quantum chaotic system. Before studying more precisely the rigidity, let us first study the nearest neighbour spacing distribution, whose histogram is presented in Fig. 3.*

* Note: In all the following, we shall work on the "unfolded" spectrum, which has a constant averaged level spacing equal to 1, and which is obtained from the true spectrum by the following transformation:

$$\epsilon_n = \bar{N}(E_n) - 1/2,$$

with $\bar{N}(E)$ given in equation (16).
We find the surprising fact that this histogram is clearly very different from the Wigner distribution (full curve) and from the Poisson one (dashed curve), being more or less between the two. This histogram presents two essential features: first, the spectrum exhibits almost no level repulsion, and second, large spacings occur much more frequently than predicted by the random matrix theory. Thus, for the spacing distribution, this spectrum appears to be rather different from a G.O.E. one. Let us see now whether its rigidity exhibits the same peculiar fea-
tures. As is well known [2], the spectral rigidity may be evaluated by studying the two quantities \( \Sigma^2(L) \) and \( \Delta^*_n \). The quantity \( \Sigma^2(L) \) is the variance of the number of levels in an interval of length \( L \). For a pure random sequence of levels (Poisson spectrum) this quantity increases as \( L \), whereas for a G.O.E. spectrum it increases only as \( \log L \). The values obtained for \( \Sigma^2(L) \) with our spectrum are presented in figure 4, together with the Poisson (dashed curve) and G.O.E. (full curve) predictions. There again, we observe the same features as in the spacing distribution. Indeed, for small \( L \) \((L < 1)\), our results are in rather good agreement with the Poisson curve, which indicates that at small scale the spectrum is not rigid. However, as \( L \) increases, we observe a very rapid transition to an almost constant value of \( \Sigma^2(L) \) which is reached for an \( L \) value of the order of three. Thus, on a scale of the order of 6 level spacings the spectrum appears to be as rigid as a G.O.E. spectrum, whereas on a larger scale it appears to be more rigid. On the next figure (Fig. 5), we present the results obtained for \( \Delta^*_n \), which measures the deviation of a sample of \( n \) successive levels from an equally spaced spectrum. Here again, we observe the same features: at small scale \((n < 5 \text{ or } 6)\), the spectrum is of Poisson type, whereas at larger scale, after a very rapid transition to an almost constant value of \( \Delta^*_n \), the spectrum is as rigid as a G.O.E. spectrum \((n = 30)\), and becomes even more rigid \((n > 40)\).

Thus, for all three quantities which we have studied, we find that the spectrum is quite different from a G.O.E. spectrum, and that at small scale its properties are close to those of a Poisson spectrum. These facts are extremely surprising since for all quantum chaotic systems already studied [2] the calculated spectra were in extremely good agreement with those of the random matrix theory. However, such results could be obtained if the studied system possessed some symmetry [2]. In such a case, the spectrum would be the superposition of two or more uncorrelated spectra, so that it would no longer exhibit any level repulsion nor any small scale rigidity. This argument does not seem however to apply to our triangular billiard which does not exhibit any symmetry. Another possible explanation of this disagreement could be that we only consider the lower part of the spectrum for which the asymptotic regime would not be reached. In our study of a billiard with magnetic field [2], we observed such an effect: for a small enough value of the magnetic field, the asymptotic G.U.E. regime was not reached in the lower part of the spectrum, whose properties were indeed intermediate between G.O.E. and G.U.E. . In order to investigate this possibility, we divided our se-
Billiards on the hyperbolic plane

Fig. 6. Plot of the $\Sigma^2(l)$ values as a function of $l$ for the triangle $OLM$. The full curves correspond to Poisson (straight line) and G.O.E. The circles $\circ$ correspond to sample 1, the triangles $\triangle$ to sample 2, the crosses $+$ to sample 3, the $\times$ to sample 4 and the squares $\square$ to sample 5.

Fig. 7. Plot of the $\Delta^*_n$ values for the tiling triangle. The dashed curve indicates the Poisson prediction, whereas the five series of points correspond to the five samples, the higher the points the higher the sample number.
Table 1. Tessellating triangle. The values of the level spacing variance are given in the first column for the total spectrum (first line), for the five samples of 500 levels (lines 2 to 6), for a Poisson spectrum (line 7), and for a G.O.E. spectrum (line 8). In the second column we give the values of the 2-level correlation function at $r = 0$, and in the third one the values of its derivative.

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_2^2$</th>
<th>$a_0$</th>
<th>$a_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total spectrum</td>
<td>.479 ± .041</td>
<td>.386 ± .036</td>
<td>-.107 ± .131</td>
</tr>
<tr>
<td>Sample I</td>
<td>.312 ± .035</td>
<td>.613 ± .044</td>
<td>-.264 ± .162</td>
</tr>
<tr>
<td>0 → 500</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sample II</td>
<td>.405 ± .058</td>
<td>.410 ± .055</td>
<td>+.041 ± .197</td>
</tr>
<tr>
<td>level 250 → 750</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sample III</td>
<td>.543 ± .080</td>
<td>.316 ± .061</td>
<td>-.061 ± .227</td>
</tr>
<tr>
<td>level 500 → 1000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sample IV</td>
<td>.605 ± .088</td>
<td>.296 ± .067</td>
<td>-.243 ± .253</td>
</tr>
<tr>
<td>level 750 → 1250</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sample V</td>
<td>.583 ± .085</td>
<td>.126 ± .067</td>
<td>+.109 ± .262</td>
</tr>
<tr>
<td>level 1000 → 1500</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Poisson</td>
<td>1.000</td>
<td>0.</td>
<td>0.</td>
</tr>
<tr>
<td>G.O.E.</td>
<td>.273</td>
<td>1.</td>
<td>$-\pi^2/6 = -1.645$</td>
</tr>
</tbody>
</table>

The first sample goes from level 1 to level 500, the second one from level 251 to level 750, and so on. For each sample, we evaluated the spacing variance, the value of the two-point correlation function and of its derivative at the origin, and the quantities $\Sigma^2(L)$ and $\Delta^*_r$. The results of this analysis are presented in Figs. 6 and 7 and Table 1.

From the inspection of the Table, one notes that the spacing variance increases steadily from the first sample to the fifth. For the first sample...
the estimated value is in rather good agreement with the G.O.E. value, whereas for the last sample we are off by a factor of two. The same conclusions hold for the two-point correlation function. In fact, the level repulsion decreases steadily from sample 1 to sample 5 and is much smaller for the last sample than for the first one. Besides, the slope of the two-point correlation function at the origin is, in all five samples, consistent with the Poisson value \( a_1 = 0 \). As for the spectral rigidity, we observe that the values of \( \Delta_n^* \) and \( \Sigma^2(L) \) increase from the first sample to the fifth, and that the higher the sample, the closer the curve to the Poisson curve. From this study, it appears clearly that the spectrum is not homogeneous, i.e. that its properties depend on the sample studied and that the asymptotic regime is certainly not reached in our sample of 1500 levels. However, as all quantities evolve towards Poisson values, one may conclude that the asymptotic regime would be the Poissonian rather than the G.O.E. regime. Another conclusion of this study is that our spectrum is not simply the superposition of two or three uncorrelated G.O.E. spectra, since in such a case the spectrum properties would still be independent of the sample studied. In fact, the observed evolution of the spectrum properties toward Poissonian ones is exactly what is observed in an integrable system such as a plane circular billiard. In this case, the spectrum is a superposition of several rigid spectra, the number of spectra increasing with the energy, which in turn induces an evolution of the properties towards those of a Poisson spectrum.

Thus, our study has shown that the spectrum of the triangular billiard OLM is asymptotically a Poisson spectrum, i.e. a spectrum which exhibits neither level repulsion nor spectral rigidity, although the associated classical system is fully chaotic. This finding is extremely surprising since in all previous studies of chaotic systems the quantal spectrum has always been found to be of the G.O.E. type (level repulsion and strong rigidity). Now one may wonder why this particular system does not seem to present the usual properties. A possible explanation of this fact could be that the triangular billiard OLM is indeed pathological, in the same way as the plane equilateral triangle is pathological. Indeed, the equilateral triangle does tessellate the plane while the triangle OLM tessellates the hyperbolic plane. In the plane case it is well known that tiling triangles are rather peculiar since they are integrable, which is not a generic property of a plane triangle. In the same way, it could be that, although not an integrable system, a triangle which tessellates the hyperbolic plane is somehow singular and thus exhibits peculiar properties. To check this hypothesis, we have made the same analysis on a
Fig. 8. Same as Fig. 2 for the non-tessellating triangle.
slightly different triangular billiard with Dirichlet boundary conditions. This second triangle has angles equal to $\pi/8$, $\pi/2$ and $67\pi/200$, an area equal to $\pi/25$, and is thus extremely close to the original triangle $OLM$. However, due to the value of the third angle, $\theta_3 = 67\pi/200$, this triangle does not tile the hyperbolic plane and is thus more alike a generic (although rational) triangle. Note however that the properties of the classical system are not known and that for such a billiard it is only conjectured that it is a fully chaotic system.

2.2. Non-tiling triangle

Using the same algorithm, we have evaluated the first 1500 levels of this triangular billiard. The results of the analysis of the spectrum are presented in figures 8, 9, 10 and 11. On figure 8 we plot the deviation of the exact spectrum from the averaged one, as in Fig. 2 for the triangle $OLM$. As in the previous case, this curve indicates that the sequence of levels is pure and that no level is missing. Furthermore it indicates the strong rigidity of the spectrum. The comparison with figure 2 shows that the amplitude of the fluctuations is larger for the tiling triangle than for the non-tiling one. Thus the rigidity for the second spectrum appears to be stronger than for the first one. In figure 9 we present the histogram of the next-neighbour spacings, compared with Poisson and G.O.E. curves. The agreement of our data with the G.O.E. curve is now extremely good,
contrary to what was observed for the tiling triangle (see Fig. 3 for comparison). In figures 10 and 11, we present the values of $\Sigma^2(L)$ and $\Delta_n^*$ obtained for the second spectrum (for comparison see Figs. 4 and 5). There again, we find a good agreement of the experimental values with the G.O.E. predictions at least for small enough values of $L$. However, as in the previous case we find a very rapid transition to a regime where $\Sigma^2(L)$ and $\Delta_n^*$ are almost constant, this transition occurring very early, i.e. for quite small values of $L$ or $n$.

Fig. 10. Same as Fig. 4 for the non-tessellating triangle.

Fig. 11. Same as Fig. 5 for the non-tessellating triangle.
Fig. 12. Same as Fig. 6 for the non-tessellating triangle.

Fig. 13. Same as Fig. 7 for the non-tessellating triangle.

Thus, this spectrum exhibits all the properties that one expects for a chaotic quantum system, namely a strong level repulsion and a strong rigidity, in agreement with all our previous studies. Furthermore, we have studied the homogeneity of the spectrum by repeating the analysis done for the tiling triangle. The results are presented in figures 12 and 13 and in Table 2. In Figs. 12 and 13 we may see that, except for the first sample, the rigidity does not change appreciably with the sample considered, and that it becomes closer and closer to the G.O.E. predictions when the energy increases. As for the spacing variance, we see in Table 2 that for all five samples it agrees fairly well with the
G.O.E. value and that it does not exhibit any significant drift. The same conclusion holds for the value of the two-point correlation function at $r = 0$. On the contrary, the slope of the two-point correlation function at $r = 0$ appears to be too small, which indicates that the spectrum is more correlated than expected. Nevertheless, it appears that this spectrum is fairly homogeneous and that it possesses all properties that one is used to find for the spectra of quantum chaotic systems.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$\sigma_k^2$</th>
<th>$a_0$</th>
<th>$a_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total spectrum</td>
<td>.262 ± .011</td>
<td>.913 ± .021</td>
<td>-1.035 ± .078</td>
</tr>
<tr>
<td>Sample I</td>
<td>.248 ± .019</td>
<td>.875 ± .035</td>
<td>- .849 ± .128</td>
</tr>
<tr>
<td>Sample II</td>
<td>.254 ± .017</td>
<td>.920 ± .035</td>
<td>-1.032 ± .133</td>
</tr>
<tr>
<td>Sample III</td>
<td>.258 ± .020</td>
<td>.971 ± .033</td>
<td>-1.196 ± .128</td>
</tr>
<tr>
<td>Sample IV</td>
<td>.291 ± .021</td>
<td>.934 ± .037</td>
<td>-1.230 ± .139</td>
</tr>
<tr>
<td>Sample V</td>
<td>.280 ± .018</td>
<td>.924 ± .036</td>
<td>-1.171 ± .136</td>
</tr>
<tr>
<td>Poisson</td>
<td>1.000</td>
<td>0.</td>
<td>0.</td>
</tr>
<tr>
<td>G.O.E.</td>
<td>.273</td>
<td>1.</td>
<td>-1.645</td>
</tr>
</tbody>
</table>

Table 2. Same as Table 1 for the non-tessellating triangle.

Thus from this study one may draw two important conclusions. First, the spectrum of a generic triangular billiard on the hyperbolic plane exhibits the same properties as those found in other chaotic quantum systems. This point gives a further confirmation of our hypothesis according
to which the Random Matrix Theory describes the spectral properties of quantum chaotic systems. The second conclusion is that whenever the billiard is non-generic, for instance if it tessellates the hyperbolic plane, its spectral properties are non-generic too and more alike those of an integrable system. Now these rather peculiar quantum spectral properties must reflect some peculiar properties of the classical system. It will be the aim of the next part to investigate this point.

2.3. From quantum to classical properties

In generic systems there exists a relation, known as the Gutzwiller relation, between the periodic orbit sum and the quantal spectrum [6]. This relation is usually asymptotic, whereas for a tessellation of the hyperbolic plane this relation is exact and is known as the Selberg trace formula. Note that in most applications the Selberg trace formula or the Gutzwiller formula are used to obtain information on the quantum system from the knowledge of some property of the classical system: starting from the geodesic spectrum one attempts to calculate some eigenvalue of the quantal spectrum [7]. In the following we shall adopt the opposite attitude: having at our disposal a sequence of 1500 levels of the tessellating triangle we shall make use of the Selberg trace formula to obtain some information on the classical system. Let us first recall the general expression of the Selberg trace formula for a billiard:

\[
\sum_{k} h(\rho_{k}) = \frac{a}{4\pi} \delta - b \mathcal{L} + c + \sum_{n=1}^{\infty} \sum_{(p)} W_{n}(l(p)) \tilde{h}(n l(p)),
\]

where \( h(\rho) \) is a real, even, analytic function in \( |\text{Im} \rho| < 1/2 + \epsilon \), such that \( h(\rho) \sim o(|\rho|^{-2-\epsilon}) \) for \( |\rho| \to \infty \),

\[
\tilde{h}(l) = \frac{1}{2\pi} \int d\rho \, e^{il\rho} h(\rho),
\]

\[
a = -\int_{-\infty}^{+\infty} \frac{dl}{\sinh(l/2)} \frac{d}{dl} \tilde{h}(l),
\]

\[
b = \frac{1}{4} \tilde{h}(0),
\]

\[
c = \sum_{r} \sum_{n=1}^{m_r-1} \frac{\varphi(r)}{8 \sin \frac{\pi n r}{2}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{h}(l) dl}{\sin \left( \frac{\pi \varphi(r)}{2} + \frac{r l}{2} \right)},
\]
where this sum runs over the three corners $\varphi(r) = \pi/m_r$ of the triangle.

In Eq. (17), $S$ is the area and $\mathcal{L}$ the perimeter of the triangle. In the left hand side of the equality, the sum runs over all eigenvalues of the Laplace–Beltrami operator, $p_k = \sqrt{E_k} - 1/4$, whereas in the right hand side the sum runs over all primitive inconjugate closed orbits $p$ of length $l(p)$. In the r.h.s., the weight $W_n(l(p))$ of a given geodesic measures the stability of the orbit and is given by:

\[ W_n(l) = \frac{l}{2 \sinh nl/2} \sim l e^{-nl/2} \quad \text{for even geodesics (even number of reflections)} \]

\[ W_n(l) = \frac{(-1)^n l}{\left(e^{nl/2} - (-1)^n e^{-nl/2}\right)} \sim (-1)^n l e^{-nl/2} \quad \text{for odd geodesics} \]

\[ W_n(l) = \frac{l}{4 \left(\frac{1}{\sinh nl/2} - \frac{1}{\cosh nl/2}\right)} \sim l e^{-3nl/2} \quad \text{for edge terms}. \]

Now, if we choose the function $h(p)$ as:

\[ h(p) = 4 \sqrt{\pi \beta} e^{-\beta l^2} \cos(l_0 p) \]

one obtains

\[ \tilde{h}(l) = e^{-l(l+l_0)^2/4 \beta} + e^{-l(l-l_0)^2/4 \beta}, \]

and the Selberg trace formula takes the following form:

\[ F_\beta(l_0) = \sum_{k=1}^{\infty} 4 \sqrt{\pi \beta} e^{-\beta \rho_n^2} \cos(l_0 \rho_n) \approx \sum_{(p)} \epsilon_p l_p \cos(l_0 l_p) e^{-l(l+l_0)^2/4 \beta} + \ldots \]

(18)

where, in the r.h.s. of the equation, we have neglected all the repetition terms and also the terms due to the zero length geodesics (area, perimeter and corner terms which contribute only for $l_0 \sim 0$).

With this choice of $h(p)$ (see [7]), we see that the function $F_\beta(l_0)$ will exhibit a peak of width $\sqrt{\beta}$ when $l_0$ is near a geodesic length $l_p$, the height of the peak being equal to $l_p \exp(-l_p/2)$, and its sign being equal to $\pm 1$ depending on the number of reflections of the geodesic. Thus, in order to obtain the best resolution on the geodesic length spectrum, one should choose $\beta$ as small as possible. However, the sum in the l.h.s. runs over all eigenvalues whereas we only know the first 1500 levels. Thus the parameter $\beta$ has to be chosen so that the cut-off error due to the use of a finite sum in the l.h.s. is small enough. In our practical evaluation of the function $F_\beta(l_0)$, we chose $\beta$ such that $e^{-\beta \rho_n^2} = .001$, where $\rho_n^2$ is the
last eigenvalue of our sample. With this choice, the cut-off error should be small and the width of each peak will be of the order of .01, so that in the geodesic length spectrum all details smaller than this value will be washed out. The plot of the function $F(I)$ is presented in Fig. 14 for values of $I$ ranging from 0 to 20. This maximum value of $I$ is imposed, first by our rather limited precision on our eigenvalues, and second by the fact that for such a large $I$ value the expected density of geodesics is of the order of $10^7$. On figure 14 one may note that the function $F(I)$ gives indeed a very precise and accurate description of the geodesic length spectrum, at least for small enough values of $I$. On this curve one could easily measure the length, parity and degeneracy of the first 50 geodesics or so. One may also observe the repetitions of the first geodesics. This very accurate result is probably due to the fact that
the Selberg trace formula is exact, whereas for a generic billiard, the relation being only asymptotic, one would obtain a curve blurred by quantum corrections. Besides, there are two important features which may be noted on figure 14. First, the geodesic spectrum appears to be divided into two parts: in the lower part of the spectrum the density of peaks appears to increase almost linearly with \( l \), while for larger values of \( l \) it seems to remain approximately constant. Second, the height of the peaks remains almost constant throughout the spectrum. These two points are extremely surprising since one knows that, first, the density of peaks should increase exponentially, and second, that their height should decrease exponentially. In order to confirm these two qualitative conclusions, we have made a quantitative analysis of the geodesic length spectrum.

Thus, let us introduce the probability density \( p_\epsilon(l) \) of finding a length \( l \) with parity \( \epsilon \), \( d_\epsilon(l) \) the multiplicity of this length, i.e. the number of different geodesics having the same length \( l \) and parity \( \epsilon \). With these notations, the function \( F(l) \) may be written as:

\[
F(l) = \sum_{\epsilon = \pm 1} \epsilon \int d l' \ p_\epsilon(l') \ d_\epsilon(l') \ W(l') \ e^{-(l-l')^2/4\beta},
\]

where

\[
W(l) = l e^{-l/2}.
\]

Then, the local average of \( F(l) \) takes the following form:

\[
\langle F(l) \rangle = \frac{1}{2a\sqrt{4\pi\beta}} \int_{l-a}^{l+a} F(l') \ dl' = \sum_{\epsilon = \pm 1} \epsilon \ p_\epsilon(l) \ d_\epsilon(l) W(l),
\]

where \( p_\epsilon(l) \) is the mean probability density of finding a length \( l \) with parity \( \epsilon \). This quantity measures the difference between the weighted densities of even and odd geodesics. In our numerical calculations (using \( a = 1 \)) this quantity is always small and compatible with zero, which indicates that these two probability densities are equal (note that this result may be obtained analytically as indicated in the lecture notes of Colin de Verdière). Thus one has:

\[
\bar{p}_+(l) = \bar{p}_-(l) = \bar{p}(l)/2,
\]

and

\[
\bar{d}_+(l) = \bar{d}_-(l) = \bar{d}(l).
\]
In order to obtain more information on $\bar{p}(l)$ and $\bar{d}(l)$, one may also evaluate the local averages of $|F(l)|$, $F^2(l)$ and $F^4(l)$. If one neglects correlations, these different averages may be written as:

\[
S_1(l) = \frac{1}{2a\sqrt{2\pi}} \int_{l-a}^{l+a} |F(l')| \, dl' \approx W(l)\bar{d}(l)\bar{p}(l),
\]

\[
S_2(l) = \frac{1}{2a\sqrt{2\pi}} \int_{l-a}^{l+a} F^2(l') \, dl' \approx (W(l)\bar{d}(l))^2\bar{p}(l),
\]

\[
S_4(l) = \frac{1}{2a\sqrt{2\pi}} \int_{l-a}^{l+a} F^4(l') \, dl' \approx (W(l)\bar{d}(l))^4\bar{p}(l).
\]

Note that the above expression of $S_1(l)$ is exact as long as there are no correlations, whereas the expressions of $S_2(l)$ and $S_4(l)$ are valid only when different peaks do not overlap, i.e. as long as the density of peaks is smaller than 100. As for the averaged value of $F^2(l)$ given by the above expression, one may note that it is exactly the weighted density of geodesics evaluated by J.H. Hannay and A.M. Ozorio de Almeida [8]. These authors have shown that this quantity behaves differently depending on the nature of the classical system: for an integrable system, the geodesics' density increases polynomially and the stability decreases polynomially, with the result that asymptotically the weighted sum tends to be constant and independent of $l$. For a chaotic system, on the other hand, the exponential proliferation of geodesics together with an exponentially increasing instability results in a linear increase of the weighted density. The results of our numerical evaluation of $S_2(l)$ are presented in Fig. 15.

On this figure, one observes an almost linear increase of $S_2(l)$ for values of $l$ smaller than 8, while for higher values of $l$ the weighted density remains approximately constant and equal to 8. This transition could be interpreted as a sign that our classical system exhibits for long enough geodesics the same behaviour as an integrable system. However, we have checked that this transition in $S_2(l)$ is due to our finite resolution in $l$. If for instance we use half our sample of levels, which is equivalent to multiplying the peaks' widths by $\sqrt{2}$, we observe the same behaviour as previously but now the transition occurs for an $l$ value of the order of 6. Therefore, this transition is an artefact of our finite resolution, and the only meaningful part of the curve is the lower part ($l < 8$). Thus it appears that in this range of lengths the geodesic spectrum behaves as that of a chaotic system. Furthermore, it appears that for these small
Fig. 15. Plot of the averaged weighted density of geodesics $S_2(l)$ as a function of $l$.

lengths the correlation effects are unimportant. Then, by using the calculated values of $S_1(l)$ and $S_4(l)$, one may extract the total density of peaks $\rho(l)$ and the intensity of each peak $V(l)d(l)$:

$$W(l)d(l) \approx S_2(l)/S_1(l), \quad \rho(l) \approx S_2^2(l)/S_2(l), \quad (22a)$$

or

$$W(l)d(l) \approx \sqrt{S_4(l)/S_2(l)}, \quad \rho(l) \approx S_4^2(l)/S_4(l). \quad (22b)$$

The numerical values obtained for $W(l)d(l)$ and $\rho(l)$ using expressions (22a) are presented in Figures 16 and 17. On these figures one may note that the intensity of each peak remains constant and equal to .6, while the density of lengths $\rho(l)$ increases almost linearly until $l$ equals 8 and thereafter remains approximately constant and equal to 20. If, instead of using expression (22a), we use expression (22b), we observe the same behaviour, the curves being now more irregular. The value of the weight $W(l)d(l)$ is then slightly larger and equal to .7, and the density of peaks is comparatively smaller, increasing linearly until $l$ equals 9 or 10 and remaining almost constant and equal to 16 for larger values of $l$. These two results are thus in good agreement, and the slight discrepancies observed may easily be explained by the fluctuations of the degeneracy $d(l)$. 
Fig. 16. Plot of the averaged density $\tilde{p}(l)$ of geodesics' lengths as a function of $l$.

Fig. 17. Plot of the averaged weight $W(l)\tilde{d}(l)$ of a geodesic length in the Selberg trace formula ($\tilde{d}(l)$ is the degeneracy of the length and $W(l) = l \exp(-l/2)$ is its instability).
This quantitative analysis of the geodesic length spectrum thus confirms our qualitative conclusions. However these results are quite surprising. Indeed, the fact that the intensity of a peak remains constant implies that the degeneracy of each geodesic length increases exponentially:

$$\tilde{d}(l) \sim .6 \frac{e^{l/2}}{l}.$$ 

This experimental result is essentially the same as the one obtained by R. Aurich and F. Steiner [7] for the total octagon, where the observed degeneracy is of the order of $e^{l/2}/l$ (see Appendix 2). As for the density of lengths $\rho(l)$, the fact that it increases linearly with $l$ is extremely unsatisfactory, since then the total density of geodesics of length $l$ would be of the order of $\rho(l) = \tilde{\rho}(l) \tilde{d}(l) = e^{l/2}$ whereas it should increase as $e^l/l$. A possible explanation of this fact could be that there are numerous degeneracies between even and odd geodesics so that they would never appear in our geodesic spectrum. Another possibility is that the distinction between a linear and an exponential behaviour requires higher values of $l$ which we cannot reach with our sample of levels.

2.4. Summary and conclusions

In this lecture we have presented an algorithm to calculate the spectrum of the Laplace-Beltrami operator for any compact billiard with Dirichlet conditions on the pseudosphere. This algorithm has been applied to two triangular billiards, one which tessellates the pseudosphere and the other which does not. The spectrum of the tessellating triangle exhibits neither level repulsion nor spectral rigidity and there are strong evidences that asymptotically the spectrum is of Poisson type, although this billiard is known to be a strongly chaotic system. The spectrum of the non-tessellating triangle, whose classical properties are not known but which is probably a chaotic system too, exhibits the essential features of a generic chaotic system, namely the level repulsion and the spectral rigidity of G.O.E., as already observed in other chaotic systems. For the tessellating triangle the use of the Selberg trace formula has allowed us to evaluate the weighted geodesic length spectrum. The qualitative and quantitative study of this geodesic spectrum shows that in this triangle the geodesic lengths are exponentially degenerated. Furthermore, there are indications that additional degeneracies between even and odd geodesics might be the rule. These conclusions show that, contrary to what is usually done, the use of the Selberg trace formula or
the Gutzwiller formula may bring some information on the properties of the classical system.

Appendix 1.

The octagon presented in figure 1 exhibits numerous symmetries that one has to remove in order to obtain a pure spectrum. For the complete desymmetrisation of the octagon we refer the reader to the paper of Balazs and Voros [1]. Here we shall only present the method for a particular case.

Due to the invariance of the octagon and of the translation group under a rotation $R$ of $\pi/4$, there exist wave functions which are eigenfunctions of the Laplace-Beltrami operator and which are also invariant under this rotation:

$$R\psi = \psi.$$

Now, this rotation may be expressed as the product of a symmetry with respect to the axis $\Pi_0$ and a symmetry with respect to the axis $\Pi_1$ (see Fig. 1),

$$R = \Pi_1\Pi_0 \quad \iff \quad \Pi_1 = R\Pi_0.$$

If we choose a wave function antisymmetric under $\Pi_0$, $\Pi_0\psi = -\psi$,

then this wave function must also be antisymmetric under $\Pi_1$: $\Pi_1\psi = -\psi$. It may easily be seen that this wave function is also antisymmetric under any of the symmetries $\Pi_2, \Pi_3, \ldots, \Pi_7$ (Fig. 1). Besides, the translation ($g_0$) which brings side $DE$ on side $AH$ may be expressed as the product of a symmetry with respect to $\Pi_4$ and a symmetry (inversion) with respect to the side $AH$:

$$g_0 = \Pi_{AH}\Pi_4 \quad \iff \quad \Pi_{AH} = g_0\Pi_4.$$

Then, as $g_0\psi = \psi$ (translational invariance) and $\Pi_4\psi = -\psi$, one has $\Pi_{AH}\psi = -\psi$, i.e. the wave function has to be antisymmetric with respect to the side $AH$ too. Thus, we have shown that there exist solutions of the initial problem which are also solutions of a billiard problem in the triangle $OIA$ with Dirichlet conditions on all three sides of the triangle. However, the triangle $OIA$ still possesses some symmetry ($OI = IA$), so
that one is led to consider a billiard problem with Dirichlet conditions in the triangle $OIK$. Unfortunately, this triangle is half the equilateral triangle $OIJ$, so that we may still expect some degeneracy in the spectrum of the triangle $OIK$. In order to eliminate these degeneracies one has to further desymmetrise the equilateral triangle $OIJ$. Thus we are left with a billiard problem in the triangle $OLM$ which is equal to $1/6$ of the triangle $OIJ$ and to $1/96$ of the total octagon. As may be seen on figure 1, the angles of this triangle are equal to $\pi/8$ (O), $\pi/2$ (M) and $\pi/3$ (L), so that its area is equal to $\pi/24$. With this triangle and Dirichlet conditions on all three sides we shall thus obtain $1/96$ of the total spectrum of the octagon.

Appendix 2.

As shown in Ref. [1], the octagon presented in Fig. 1 tessellates the hyperbolic plane under a discrete group of translations. This group has four generators $g_i$, $i = 0$ to 3, and their inverses $g_i$, $i = 4$ to 7, whose explicit expression is given by:

$$g_n = \begin{pmatrix} \alpha & \beta e^{in\pi/4} \\ \beta e^{-in\pi/4} & \alpha \end{pmatrix} \quad \text{with} \quad \alpha = 1 + \sqrt{2}, \quad \beta = \sqrt{1 + \sqrt{2}/2}. \quad (A.1)$$

Due to the non-commutativity of these generators, to each regular word of the group corresponds a unique image of the original octagon and conversely, the union of all the images covers the hyperbolic plane. Note that in the plane case the translations do commute, so that there are many regular words of the group which bring the original domain (for instance a square or a hexagon) onto another one. By a regular word we mean a word, consisting of a product of an arbitrary number of generators which cannot be simplified by using the identities:

$$g_n g_{n+4} = 1, \quad g_n g_{n+5} g_{n+2} g_{n+7} g_{n+4} g_{n+1} g_{n+6} g_{n+3} = 1. \quad (A.2)$$

The problem of finding the periodic orbits of this tessellation amounts then to find the shortest path which connects the fundamental domain and its image by a regular word $W$, the length of the corresponding periodic orbit being given by:

$$l_W = \inf d(z, W(z))$$
with the hyperbolic distance function $d(z, z')$ given by:

$$
cosh d(z, z') = 1 + \frac{2(z - z')^2}{(1 - |z|^2)(1 - |z'|^2)}, \quad (A.3)
$$

where $W(z)$ is the image of $z$ by the word $W$:

$$
W = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}; \quad \text{then} \quad W(z) = \frac{\alpha z + \beta}{\beta^* z + \alpha^*}.
$$

As shown in Ref. [1], the length of the geodesic corresponding to the word $W$ is simply given by:

$$
cosh \frac{l_W}{2} = \frac{1}{2} |\text{Tr}(W)|. \quad (A.4)
$$

In the enumeration of periodic orbits, one should however notice that the hyperbolic plane is invariant under translations. Thus, the distance between two points $z$ and $z'$ is equal to the distance between their images $M(z)$ and $M(z')$ where $M$ is any translation or any word in the discrete group. Then one has:

$$
d(z, W(z)) = d(M(z), M(W(z))) = d(z', MWM^{-1}(z')) \quad \text{with} \quad z' = M(z).
$$

This equality means simply that in the tessellation, there are an infinite number of pairs of octagons at the same distance, a fact which is obvious for a tessellation of the plane. Therefore, in order to enumerate correctly the inconjugate periodic orbits, one has to eliminate all words which under circular permutation reduce, after simplification, to a word which is already counted.

With these rules, the enumeration of periodic orbits may be done quite easily on a computer. However, if one undertakes such an enumeration, one finds numerous degeneracies (for instance, in an enumeration of all geodesics' lengths corresponding to words of at most 8 letters, we found a length which was 1024 times degenerate). Obviously, due to the symmetry of the octagon, one expects some geometrical degeneracy. However, this degeneracy should not be greater than 32. Indeed, given a domain connected to the fundamental one by a geodesic, the invariance of the tessellation under rotations of $\pi/4$ will yield 8 images, the invariance under a symmetry with respect to the $z$-axis will yield another factor two and finally the inverse word (the time reversed geodesic) will
yield another factor two. In order to explain the observed degeneracy, one has to consider the structure of the words of the group. Due to the particular form of the generators of the group (Eq. (A.1)), it is easy to show that a general word has the following form:

\[ W = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \]  

(A.5)

with \( \alpha = n_1 + m_1\sqrt{2} + i(n_2 + m_2\sqrt{2}) \), \( \beta = ((n_3 + m_3\sqrt{2} + i(n_4 + m_4\sqrt{2}))\sqrt{1 + \sqrt{2}}, \) where \( n_i \) and \( m_i \) are integers. Furthermore, one can prove easily that \( n_1 \) is an odd integer, while \( n_2 \) and \( n_3 + n_4 \) are even. Finally, using the fact that the determinant of the matrix of a word is unity, one obtains:

\[ \alpha\alpha^* - \beta\beta^* = (n_1 + m_1\sqrt{2})^2 + (n_2 + m_2\sqrt{2})^2 - (\sqrt{2} + 1)((n_3 + m_3\sqrt{2})^2 + (n_4 + m_4\sqrt{2})^2) = 1. \]  

(A.6)

This equation in integer numbers is in fact equivalent to two equations, one corresponding to the integer terms, the other to the terms containing \( \sqrt{2} \). Any combination of these two equations has to be satisfied and in particular the combination which amounts to changing the sign of \( \sqrt{2} \). Thus one should have:

\[ (n_1 - m_1\sqrt{2})^2 + (n_2 - m_2\sqrt{2})^2 + (\sqrt{2} - 1)((n_3 - m_3\sqrt{2})^2 + (n_4 - m_4\sqrt{2})^2) = 1. \]

The condition on the determinant is now expressed as a sum of positive terms. The condition for the existence of a solution may then be expressed as:

\[ |n_1 - m_1\sqrt{2}| < 1, \quad |n_2 - m_2\sqrt{2}| < 1, \]

\[ |n_3 - m_3\sqrt{2}| < \sqrt{2} + 1, \quad |n_4 - m_4\sqrt{2}| < \sqrt{2} + 1. \]

If one returns to the lengths of the geodesics as given by Eq. (A.4), one finds that they are given by

\[ \cosh\frac{t}{2} = |n_1 + m_1\sqrt{2}| \]

with \( n_1 = 2k + 1 \) and \( |n_1 - m_1\sqrt{2}| < 1, \)
which is the result found empirically in Ref. [7]. Finally, the constraints on \(n_1\) and \(m_1\), together with the total density of geodesics, imply a strong degeneracy of each length. Indeed, the total number of geodesics of length smaller than \(l\) is given by \(N(l) = e^{l/2}\), while the total number of different lengths smaller than \(l\), \(n(l)\), is given by:

\[
n(l) = \sum_{n,m} 1 \quad \text{with} \quad |2n + 1 + m\sqrt{2}| < \frac{e^{l/2}}{2} \quad \text{and} \quad |2n + 1 - m\sqrt{2}| < 1,
\]

which may be approximated by

\[
n(l) \approx \bar{n}(l) = \frac{1}{4\sqrt{2}} \int_{-\sqrt{2}e^{l/2}}^{+\sqrt{2}e^{l/2}} dx \int_{-1}^{+1} dy \sim \frac{1}{2\sqrt{2}} e^{l/2}.
\]

The density of lengths \(\bar{p}(l) = \frac{dn(l)}{dl}\) is then given by:

\[
\bar{p}(l) \approx \frac{1}{4\sqrt{2}} e^{l/2}.
\]

Thus, in the octagon, the average degeneracy of each length, \(\bar{d}(l)\), is of the order of

\[
\bar{d}(l) = \frac{e^{l/2}}{\bar{p}(l)} \approx 4\sqrt{2} \frac{e^{l/2}}{l}.
\]

Note that this estimation of the average degeneracy is a lower bound since we have greatly overestimated the cumulative density of lengths \(n(l)\). Besides, if one assumes that, when one considers a triangle obtained by desymmetrisation of the total octagon, the geodesic spectrum is unchanged except for the geometrical degeneracies which are suppressed, one obtains the averaged degeneracy of geodesics in the triangle as

\[
\bar{d}_{OLM}(l) = \frac{\bar{d}(l)}{16} \approx \frac{3\sqrt{2}e^{l/2}}{l}.
\]

This estimated value of the degeneracy is actually in good agreement with the experimentally observed one.

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