Trace Formulae for Arithmetical Systems

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Abstract

For quantum problems on the pseudo-sphere generated by arithmetic groups there exist special trace formulae, called trace formulae for Hecke operators, which permit the reconstruction of wave functions from the knowledge of periodic orbits. After a short discussion of this subject we present the Hecke operators trace formulae for the Dirichlet problem on the modular billiard, which is a prototype of arithmetical systems. The results of numerical computations for these semiclassical type relations are in good agreement with the directly computed eigenfunctions.

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1. The problem of semiclassical quantization of ergodic systems has attracted wide attention last years (see e.g. [1], [2] and references therein). The main tool here is the trace formula [3] which gives a connection between the density quantum energy levels and a sum over all classical periodic trajectories of the system.

In general, this relation is valid only in the limit of the Planck constant tending to zero. The only exception where such formulae are exact is the problem of finding the spectrum of the Laplace-Beltrami operator on surfaces of constant negative curvature generated by discrete groups (see e.g. [15], [4]). In these cases there exist the famous Selberg trace formula [5], [6], giving an exact relationship between the quantum spectrum and classical periodic orbits. The numerical computations for different models [7] – [11], have confirmed that such formulae are not only of a pure theoretical interest.

The purpose of this note is to emphasize that for specific subclasses of models on constant negative curvature surfaces, namely for ones generated by arithmetic groups, there exists another type of trace formula; it permits (in principle) to obtain not only the energy eigenvalues but also the corresponding eigenfunctions directly from periodic orbits. Though in general one can build a semiclassical expression for wave functions through periodic orbits [12], the formula discussed below is of quite different origin and seems to be not easily generalized for other systems. Nevertheless, it is important for semiclassical computations to investigate how classical periodic orbits conspire to reproduce quantum eigenfunctions.

2. Arithmetic groups are a specific subclass of discrete groups. We shall not give here the precise mathematical definitions, they can be found e.g. in [13], [6], [14]. One can say as a crude analogy that arithmetic groups are among discrete groups as integers among rationals. The simplest and the most investigated example of arithmetic groups is the modular group (and its subgroups) (see e.g. [15]) which is defined as the group of all 2x2 matrices where all entries are integers and the determinant is one.

The peculiarities of quantum problems for arithmetic groups were stressed in [16] where it was shown that the arithmetic nature of these groups leads to exponentially big degeneracies of periodic orbits and to non-universal energy levels statistics contrary to what was expected for ergodic systems [17].

Here we shall explore another property of such systems, namely the existence of infinite many operators commuting with the quantum hamiltonian (and with themselves) (see also [18]). These operators are called Hecke operators and are of pure arithmetical origin [14], [15].

For simplicity we consider (as usual in this subject) the case of modular group, but most formulae could be generalized for other arithmetic groups [14].

Let us consider the set of the 2x2 matrices with integer entries as for the modular domain, but with the determinant being a certain integer p ($p \neq 1$, $p \neq 0$):

$$M_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \text{ are integers, } ad - bc = p \right\}. \tag{1}$$

Their importance come from the (easily checked) fact that different matrices of the modular group could be conjugated by these matrices though they do not form a group, being not stable by multiplication.

An arbitrary matrix G of this form can be uniquely represented by the product [14], [15]:

$$G = g.\alpha_p \tag{2}$$

where g belongs to the modular group and α_p is one of the following fixed matrices:

$$\alpha_p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \; a, b, d \text{ integers, } ad = p,\; d > 0,\; 0 \le b \le d-1. \tag{3}$$

Let $\psi(x, y)$ be an automorphic function of the modular group [15], i.e. it will obey: $\psi(g(z)) = \psi(z)$ for all matrices g from the modular group.

It is easy to see [15] that the function :

$$\phi(z) = (T_p\psi)(z) = \frac{1}{p^{1/2}} \sum_{a,b,d} \psi(\frac{az+b}{d}), \tag{4}$$

where the summation is done over all a, b, d being as in Eq.(3), will be also an automorphic function for the modular group. This is a kind of symmetrization of $\psi(z)$ over the images of z by the elements of M_p.

The operators T_p defined in (4) are called Hecke operators. They form a commutative algebra and commute also with the Laplace-Beltrami operator [14], [15]. Therefore, if there is no degeneracy of the eigenvalues of the Laplace-Beltrami operator, one has:

$$T_p\psi_n(x,y) = c_p(n)\psi_n(x,y), \tag{5}$$

i.e. an eigenfunction of the Laplace-Beltrami operator will be simultaneously an eigenfunction for all the Hecke operators.

Any eigenfunction of the Laplace-Beltrami operator for the modular group corresponding to the discrete spectrum can be written as the following Fourier decomposition (see e.g. [15]):

$$\psi_n(x,y) = y^{1/2}\sum_{p=1}^{\infty} c_p(n)K_{s-1/2}(2\pi py)e^{2\pi ipx}, \tag{6}$$

where s is connected with a Laplace-Beltrami eigenvalue λ by the relation $\lambda = s(s-1)$, and $K_\nu(x)$ is the Hankel function.

Using properties of the Hecke operators one can show [15] (assuming non-degeneracies of the eigenvalues of the Laplace-Beltrami operator) that $c_p(n)$ in Eq.(5) coincide with $c_p(n)$ in Eq.(6), i.e. the eigenvalues of the p^{th} Hecke operator are connected with the p^{th} Fourier coefficients of the expansions of the eigenfunctions of the Laplace-Beltrami operator. This property is of particular importance because then the knowledge of the eigenvalues of all Hecke operators permits to reconstruct wave functions.

3. To compute the trace formula for Hecke operators one has to consider the following sum:

$$\int_F \sum_{G\in M_p} k(z,Gz)d\mu(z), \tag{7}$$

where the kernel $k(z,z')$ depends only on the hyperbolic distance between z and z', and F is the fundamental domain of the modular group. Then one regroups terms into conjugacy classes with respect to the group $SL(2,Z)$. In [19] this trace formula was presented for p prime, and $p>0$. We found that for numerical computations it is more convenient to use a different trace formula corresponding to the Hecke operator applied to the Dirichlet kernel on a billiard defined in half the modular domain. More precisely, instead of applying T_p to $k(z,z')$ as in (7) one applies $\frac{1}{2}(T_p - T_{-p})$; as shown in [20] this gives a trace formula for a billiard with Dirichlet boundary conditions on half the modular domain (which is called the Artin billiard [11]). The group Γ with respect to which we compute the conjugacy classes is the full billiard modular group (B.M.G.), i.e. the set of all 2x2 matrices with integer entries and determinant ± 1.

This formula to our knowledge has not been published before. We present here the final result of this computation; details will be published elsewhere [21]:

$$\begin{aligned}\sum_{n=0}^{\infty} c_p(n)h(r_n) = &\frac{1}{\sqrt{p}}\sum_{hyperbolic}\frac{\ell_p}{2\sinh(L_p/2)}g(L_p)\\ &-\frac{1}{\sqrt{p}}\sum_{hyperbolic}\frac{\ell_{-p}}{2\cosh(L_{-p}/2)}g(L_{-p})\\ &+\frac{1}{2\sqrt{p}}\sum_{elliptic}\frac{1}{2m\sin\theta}\int_{-\infty}^{+\infty}\frac{e^{-2\theta r}}{1+e^{-2\pi r}}h(r)dr\\ &+g(\ln p)[\ln\left(\frac{p-1}{p+1}\right)-\frac{\ln X(p-1)}{p-1}+\frac{\ln X(p+1)}{p+1}]\\ &+\int_{\ln p}^{\infty} g(u)\frac{du}{e^{u/2}p^{1/2}-e^{-u/2}p^{-1/2}}\end{aligned} \qquad (8)$$

Here $h(r)$ is any analytic function in the strip $|Imr| \le \frac{1}{2}+\delta$ such that $h(-r)=h(r)$ and $|h(r)| < A|1+|r||^{-2-\delta}$ $A>0, \delta>0$.

$g(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(r)e^{-iru}dr$ is the Fourier transform of $h(r)$. The summation in r.h.s. is taken over all eigenvalues of the Laplace - Beltrami operator $\lambda_n = \frac{1}{4}+r_n^2$, and the $c_p(n)$ are the eigenvalues of the T_p Hecke operator. (We stress that there exist a trace formula for each p). $X(n) = \prod_{k \bmod n}(k,n)$, (k,n) being the greatest common divisor of k and n.

The first term corresponds to the matrices in M_p whose trace is greater than $2\sqrt{p}$ and which do not belong to a conjugacy class of some α_p; equivalently those are matrices whose traces are greater than $2\sqrt{p}$ but not equal to $(p+1)$.

The corresponding term in the usual trace formula involves hyperbolic conjugacy classes, *which are in one-to-one correspondence with the periodic orbits of the system.* Here we do the same, taking a representative G for each conjugacy class with respect of the B.M.G. in M_p. We can associate a length L_p to G as to every hyperbolic fractional transformation by :

$$2\cosh\frac{L_p}{2} = |\mathrm{Tr}\frac{G}{\sqrt{p}}| \tag{9}$$

Then one computes the *commutant* of it i.e. matrices g of the full B.M.G. which commute with G: $Gg = gG$. This set is, as previously, generated by a single element of the B.M.G. whose associated periodic orbit has length l_p. The formula for hyperbolic classes looks the same as in the usual case, but in the usual case $L_p = nl_p$, n being integer, whereas here there is no simple relation between them. In fact, our results show that in some cases l_p can be enormously greater than L_p. In other words G as a transformation maps one point on the periodic orbit of length l_p into another point on the same periodic orbit which is not connected in a simple way to the starting point. We can say that those L_p of hyperbolic classes of M_p are the hyperbolic distances between a point and its image.

The second term gives the same as above for $-p$; the matrices of M_{-p} have negative determinant and corresponds to odd boosts. Here as above L_{-p} is the length associated to an hyperbolic class in M_{-p} with respect to the B.M.G.:

$$2\sinh\frac{L_{-p}}{2} = |\mathrm{Tr}\frac{G}{\sqrt{p}}|; \tag{10}$$

the trace of the matrices in M_{-p} should be different from $(p-1)$. l_{-p} is the length of the generator of the commutant in the B.M.G., and corresponds to the length of a periodic orbit of the billiard.

The matrices of M_{-p} with trace equal to zero have an additional factor 1/2 due to the existence of a commuting element which square is identity.

The third term corresponds to the conjugacy classes of elliptic matrices G in M_p, i.e. whose traces is less than $2\sqrt{p}$ (there is no such classes inM_{-p}); then we can write Tr(G)=$2\sqrt{p}\cos\theta$ with $0 < \theta < \pi$. The only possible matrices of the billiard modular group commuting with those matrices are elliptic matrices corresponding to a primitive rotation of angle $2\pi/m$. It is this integer m which enter Eq.(8) ($m = 1$ corresponds to the identity matrix).

The last two terms are connected with the existence of the infinite cusp which usually leads to difficulties in deriving trace formulae for the modular group [6], [19], [20]. They correspond to the conjugacy classes of the matrices α_p and α_{-p} of Eq.(3). The commutant of those matrices is trivial, being only the identity matrix. We note that two different α_p can belong to the same B.M.G.-conjugacy class.

4. We can interpret this formula as the usual trace formula applied to a system with symmetry, by looking which periodic orbits of the B.M.G. will appear in Eq.(8). The only ones which will be selected will be those whose corresponding matrix T commute with one matrix G of M_p (or M_{-p}). From Eq.(2) one knows that G can be written $G = g.\alpha_p$, g being in $SL(2,Z)$ and α_p having the form (3). It is easy to show that if T commute with $g.\alpha_p$ then $g^{-1}Tg = \alpha_p T \alpha_p^{-1}$.

This means that $\alpha_p T \alpha_p^{-1}$ is a matrix of the B.M.G. belonging to the same conjugacy class as T; then $\alpha_p T \alpha_p^{-1}$ corresponds to a periodic orbit of the Artin billiard and this periodic orbit is the one given by T. So the periodic orbits of the Artin billiard which are selected are those which are invariant by the action of α_p. For example, T_{-1} acts as the symmetry with respect to the axis $x = 0$, and the corresponding trace formula for $\frac{1}{2}(T_1 - T_{-1})$ gives the usual trace formula for the Dirichlet problem on the Artin billiard [20].

5. We have computed the r.h.s. of Eq.(8) for a few value of p. We chose as function $h(r)$ a gaussian function: $h(r) = \exp(-A(r-r_0)^2) + \exp(-A(r+r_0)^2)$ for which the r.h.s. of the trace formula (8) considering as the function of k_0 should have peaks at true eigenvalues, the amplitudes of which being equal to the Fourier coefficients of the expansion (6). For each

conjugacy class of hyperbolic elements of M_p we computed the matrix of the B.M.G. of minimal length which commute with it; this matrix is the generator of the commutant. We have found for some of the M_p matrices unexpectedly big length for the commuting matrices. For example, the matrix with entries 295, 274; 267, 248, has determinant equal to 2, and the matrix of the B.M.G. of minimal trace which commute with it has integer entries of order of 10^{171}. The details of the method used will be given elsewhere [21].

The hyperbolic terms give the main contribution to the formula (8); the elliptic ones are exponentially small at the energies of computation, and the 'cusp' terms give more or less a smooth slowly oscillating function of period $2\pi/\ln p$ with small amplitude.

We have computed the sum in the r.h.s. of Eq.(8) using approximately 15000 periodic orbits and have compared it with the results of direct computations of the Fourier coefficients and eigenvalues for this problem. As an example we presented in Figs. 1 and 2 the results of the computations for $p = 3$, $A = 30$ and for $p = 5$, $A = 28$. The classical and quantum computations seem almost undistinguishable at the scale used. The results for other values of p are of the same quality. For the first eigenvalues our computations are in perfect agreement with the results of Hejhal [22]

So this method seems to be efficient to explore the eigenfunctions of arithmetical systems; it does not use a lot of computer time to get the results we show.

We have also studied the number of the conjugacy classes in M_p with respect to their length L_p for different p. It was found that this number grows slowly than in usual hyperbolic systems; for these ones, Huber's law [23] states that the number of periodic orbits of length less than L is $N(l < L) \sim \exp L/L$. Our computation gives a dependence like:

$$\log \mathrm{N}(L_p < L) \sim \frac{3}{4} L. \tag{11}$$

The coefficient 3/4 is a numerical one and we can not exclude a slow dependence of this factor with p.

6. In conclusion we emphasize the following points:

- The existence of infinite number of commuting Hecke operators is the characteristic property of models generated by arithmetic groups.
- For each value of p there exist a subset of periodic orbits of the modular domain, each of which remains invariant under one of transformations (3). The number of these orbits grows approximately as in (11).
- It is these invariant periodic orbits which give the contributions to the trace formulae for Hecke operators. The latter are a new kind of trace formulae which permit to reconstruct quantum wave functions of arithmetical systems directly from classical periodic orbits.
- The Hecke-type trace formula for Artin's billiard was derived and checked numerically; a good agreement between quantum and classical calculations was found.

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FIGURE CAPTIONS

Figure 1: The Hecke operators trace formula for p=3; the dashed curve is our result with 15000 periodic orbits, the continued one is the quantum calculation.

Figure 2: The same as Figure 1 but for p=5.

Figure 1

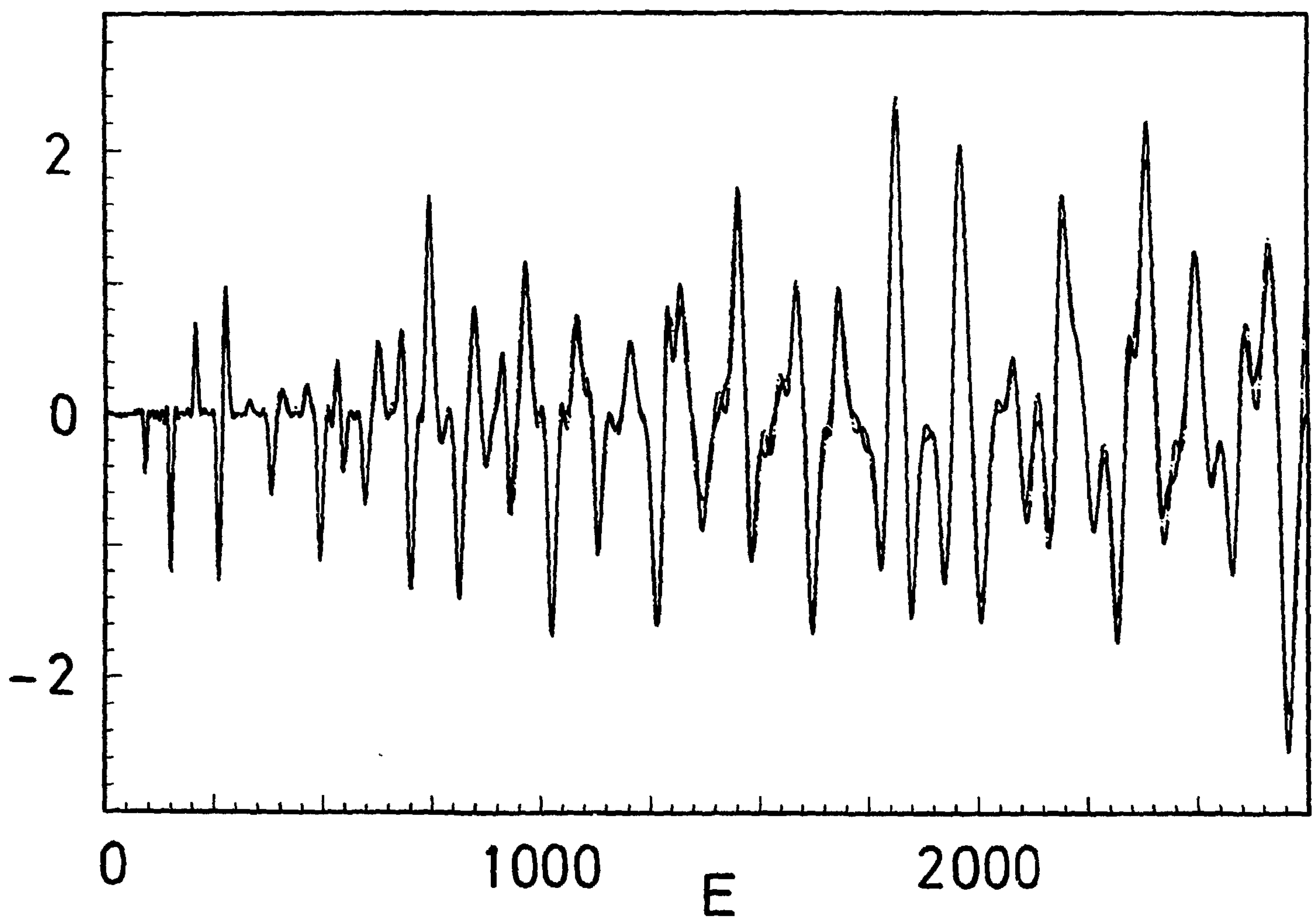

Figure 2