THREE-DIMENSIONAL GROUNDWATER VELOCITY FIELD IN AN UNCONFINED AQUIFER UNDER IRRIGATION

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A method for three-dimensional flow velocity calculation has been developed to evaluate unconfined aquifer sensitivity to areal agricultural contamination of groundwater. The methodology by Polubarinova-Kochina (1962) is applied for an unconfined homogeneous compressible or incompressible anisotropic aquifer. It is based on a three-dimensional groundwater flow model with a boundary condition on a moving surface. Analytical solutions are obtained for a hydraulic head under the influence of areal sources of circular and rectangular shape using integral transforms. Two-dimensional Hantush (1967) formulas result from the vertical averaging of the three-dimensional solutions, and the asymptotic behavior of solutions is analyzed. Analytical expressions for flow velocity components are obtained from the gradient of the hydraulic head field. Areal and temporal variability of specific yield in groundwater recharge areas is also taken into account. As a consequence of linearization of the boundary condition, the operation of any irrigation system with respect to groundwater is represented by superposition of the operating wells and circular and rectangular source influences. Combining the obtained solutions with Dagan (1967a, 1967b) or Neuman (1972, 1974) well functions one can develop computer codes for the analytical computation of the three-dimensional groundwater hydraulic head and velocity component distributions. Methods for practical implementation are discussed.

INTRODUCTION

The interaction between water supply wells, irrigation systems and groundwater flows in aquifers create a complex velocity flow field. Until recently, most methods for groundwater flow simulation have focused on efficient and accurate computation of groundwater heads for one- and two-dimensional problems (National Research Council, 1990). For contaminant transport below the water table, a three-dimensional approach is necessary to delineate major contaminant pathways. The role of groundwater flow modeling is to provide an estimate of the flow velocities. Head predictions are of little direct interest. Velocity estimates, however, are usually based on hydraulic head differences and therefore are much more sensitive to numerical modeling errors than are estimates of the hydraulic head alone. Satisfactory predictions of transport often require that the velocity field be calculated on a fine spatial grid. Therefore analytical solutions have some advantages over numerical procedures in spatial case (Dillon,1989).

Flow velocity field of an aquifer under the influence of irrigation can be represented as a result of interaction of vertical line sinks-wells and horizontal areal sources-recharge spots on the groundwater table. It is a three-dimensional flow rather than a two-dimensional one, and transient rather than steady state. To achieve a reasonable description of irrigation influence on the aquifer, a general source-sink distribution is decomposed into three types: wells, circular and rectangular sources (Hantush, 1967).

For two-dimensional solutions it would be sufficient to apply a superposition principle for a Theis (1935) well function with Hantush (1967) formulas of drawdown for rectangular and circular recharging areas. Unfortunately such an approach provides only an averaged evaluation of contaminant transport, neglecting vertical components that are responsible for the downward movement of contaminants.

The unconfined aquifer of a finite thickness is the most widespread environment for an agricultural contaminant. For the three-dimensional case, distributions of hydraulic head for a single well were given by Dagan (1964, 1967a) and by Neuman (1972, 1974) for an incompressible and compressible aquifer, respectively. For the distributed sources, the only available solutions were for a circular source in an incompressible aquifer (Dagan, 1967b).

The groundwater flow field structure under the replenishment area gives a clue to understanding the groundwater quality formation and its changes. Similar processes are relevant to a problem of artificial groundwater recharge (Morel-Seytoux et al., 1990). The difference is that long term and intensive artificial recharge causes relatively high saturation in the unsaturated zone between the soil surface and a water table. This is not the case for irrigated sites where the best management practices tend to eliminate irrigation water losses below the root zone. Some water losses are inevitable and they induce groundwater recharge under a decrease of available pore space.

A statement of problem is given for a saturated zone only. According to Kroszinsky and Dagan (1975) the unsaturated zone may have some quantitative influence upon drawdown only in case of a very shallow rigid aquifer, or in the case of soils with fine structure and relatively short times of system operation.

In the subsequent development an attempt is made to develop an analytical, three-dimensional solution for the hydraulic head and velocity distribution in an unconfined compressible or incompressible homogeneous anisotropic aquifer under the influence of irrigation.

PROBLEM STATEMENT

Consider an unconfined aquifer of infinite lateral extent and finite thickness that rests on an impermeable horizontal layer such as that shown schematically in Fig. 1 in the vicinity of a single groundwater well, or in Fig. 2 in the vicinity of a single groundwater mound. On the plane view in Fig. 3 the main types of sources and sinks are combined. The aquifer material is uniform and anisotropic, with the principal permeabilities being oriented parallel to the coordinate axes. The i-th well discharging at the rate $Q_i(t)$ is open to inflow from depth d_i to depth l_i beneath the initially static water table. The net specific recharge $I_j(t)$ at the water table is induced by irrigation on the j-th site within area of the distributed source G_j.

It is assumed that water storage or release from an aquifer is controlled by compressibility parameters of the aquifer material and water and as well as the specific yield at the free surface.

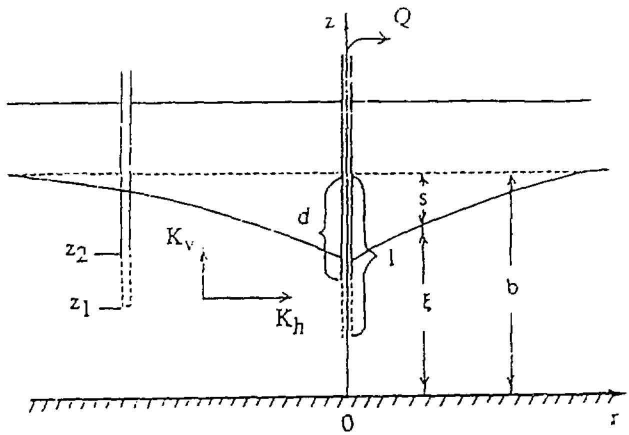

Fig.1. Schematic diagram of well in an unconfined aquifer

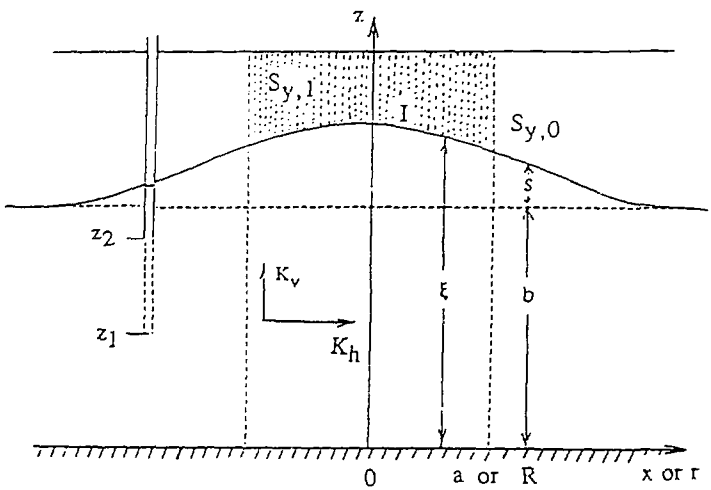

Fig.2. Schematic diagram of a groundwater recharge in an unconfined aquifer

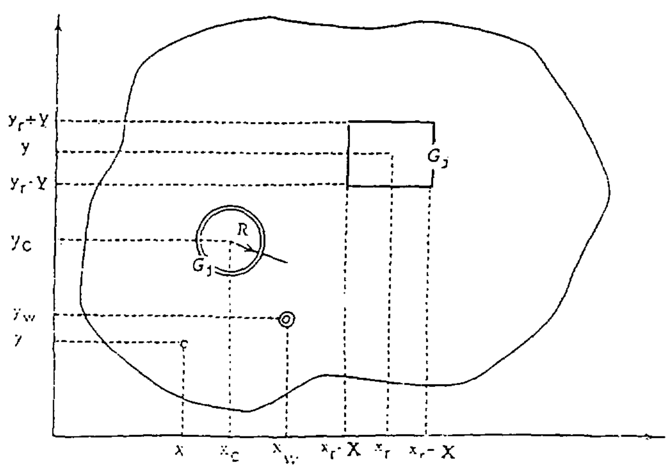

Fig.3. Schematic plan view for distribution of wells (coordinate index w), rectangular (r) and circular (c) sources.

In the analytical approach it is convenient to treat the well as a line sink with uniform sink density $Q_i/(l_i-d_i)$; i.e. we neglect the well storage and the presence of a seepage surface (Neuman, 1974).

For the groundwater mound the changes of specific yield over the recharged surface are not taken into account as a first approximation.

The governing equations for the hydraulic head drawdown s are

$$K_h\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}\right)+K_v\frac{\partial^2 s}{\partial z^2}=S_s\frac{\partial s}{\partial t}-W \tag{1}$$

$$s(x,y,z,0)=0 \tag{2}$$

$$\xi(x,y,z,0)=b \tag{3}$$

$$s(x,y,z,t)=0, \qquad x,y\to\pm\infty \tag{4}$$

$$\frac{\partial s(x,y,0,t)}{\partial z}=0 \tag{5}$$

$$\xi(x,y,t)=b-s(x,y,z,t), \qquad at \quad z=\xi(x,y,t) \tag{6}$$

$$K_h\left(\frac{\partial s}{\partial x}n_x+\frac{\partial s}{\partial y}n_y\right)+K_v\frac{\partial s}{\partial z}n_z=\left(S_y\frac{\partial \xi}{\partial t}-I\right)n_z, \qquad at \quad z=\xi(x,y,t) \tag{7}$$

where K_h and K_v are the horizontal and vertical permeabilities respectively, S_s the specific (elastic) storage, S_y the specific yield, b the initial saturated thickness of an aquifer, ξ the elevation of the water table above bottom of an aquifer; n_x, n_y, n_z are the components of an unit normal vector to the water table; the terms W and I in equations (1) and (7) are:

$$W(x,y,z,t)=\sum_{i=1}^{N_w}\frac{Q_i(t)}{l_i-d_i}\delta(x-x_i)\,\delta(y-y_i)\,\phi(z,b-l_i,b-d_i) \tag{8}$$

$$I(x,y,z,t)=\sum_{j=1}^{N_I}I_j(t)\,\gamma(x,y,G_j) \tag{9}$$

Here $\delta(x)$ is the Dirac delta function (Lavrentjev and Shabat, 1973), N_w the total number of wells, N_I the total number of areal sources, and

$$\phi(z,a,b)=\begin{cases}1, & z\in(a,e)\\ 0, & z\notin(a,e)\end{cases}, \qquad \gamma(x,y,G_j)=\begin{cases}1, & (x,y)\in G_j\\ 0, & (x,y)\notin G_j\end{cases} \tag{10}$$

where a and e are arbitrary constants ($0<a<e$), and G_j is the j-th distributed

area source.

Stemming from this general statement, particular problems were studied for a single well ($I=0$, $N_w=1$) in an incompressible aquifer ($S_s=0$) by Dagan (1964, 1967a) and in a compressible aquifer ($S_s>0$) by Neuman (1972, 1974). For a singular circular source ($W=0$, $N_I=1$) in incompressible aquifer the solution was given by Dagan (1967b).

In all these cases the equations were linearized using a perturbation technique similar to that presented by Dagan (1964), provided the aquifer is thick enough and the drawdown is much smaller than the average saturated thickness of the aquifer. Using this technique a first-order approximation is obtained by shifting the boundary condition from the free surface to the horizontal plane $z=b$ in equation (6), and then neglecting the second-order terms in equations (6) and (7). This eliminates ξ from equations (1)-(7) and we obtain

$$\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}+K_d\frac{\partial^2 s}{\partial z^2}=\frac{1}{\alpha_s}\frac{\partial s}{\partial t}-\frac{W}{K_h}, \quad K_d=\frac{K_v}{K_h}, \quad \alpha_s=\frac{K_h}{S_s} \tag{11}$$

$$s(x,y,z,0)=0 \tag{12}$$

$$s(x,y,z,t)=0, \qquad x,y\to\pm\infty \tag{13}$$

$$\frac{\partial s(x,y,0,t)}{\partial z}=0 \tag{14}$$

$$\frac{\partial s(x,y,b,t)}{\partial z}+\frac{1}{\alpha_y}\frac{\partial s(x,y,b,t)}{\partial t}=-\frac{I}{K_v}, \quad \alpha_y=\frac{K_v}{S_y} \tag{15}$$

Since the problem (11)-(15) with the conditions (8)-(10) is linear one can use the principle of superposition to obtain a general solution incorporating any number of wells and distributed sources. The main results given below have been obtained for a single circular source in an compressible aquifer and a rectangular source in both compressible and incompressible aquifers. The problem was solved for $W=0$, $N_I=1$, and G_j a circular or rectangular domain. The new solutions provide complete basis for analytical determination of three-dimensional groundwater flow velocities in unconfined aquifers of finite thickness. The solutions for the transient well production and the groundwater recharge rate are calculated by the standard methods for the linear initial value problems (Streltsova, 1988).

CIRCULAR SOURCES

The flow underneath a circular uniformly recharging area can be approximated by the system

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)+K_d\frac{\partial^2 s}{\partial z^2}=\frac{1}{\alpha_s}\frac{\partial s}{\partial t}, \quad r>0, \quad t>0 \tag{16}$$

$$s(r,z,0)=0 \tag{17}$$

$$s(r,z,t)=0, \qquad r\to\infty \tag{18}$$

$$\frac{\partial s(0,z,t)}{\partial r}=0 \tag{19}$$

$$\frac{\partial s(r,0,t)}{\partial z}=0 \tag{20}$$

$$\frac{\partial s(r,b,t)}{\partial z}+\frac{1}{\alpha_y}\frac{\partial s(r,b,t)}{\partial t}=-\frac{I}{K_v}, \quad I(r,t)=\begin{cases} I_0, & r<R, \\ 0, & r\geq R \end{cases} \tag{21}$$

The system arises from the initial boundary value problem (11)-(15) given in cylindrical coordinates r, z.

Applying Laplace and Hankel transforms to (16)-(21) and inverting the results by the method similar to Neuman (1972), one obtains a first order approximation to the original problem. The mathematical calculations are outlined in the Appendix A. The solution is expressed in terms of 5 dimensionless parameters τ, β, z_d, σ and β_R as

$$s(r,z,t)=-I_c\int_0^\infty J_0(y\beta^{\frac{1}{2}})\,J_1(y\beta_R^{\frac{1}{2}})\sum_{n=0}^{\infty}\omega_n(\tau,z_d,y)\,dy, \quad I_c=\frac{2\,I_0 R\sigma}{(K_h K_v)^{\frac{1}{2}}} \tag{22}$$

$$\omega_0(\tau,z_d,y)=\Omega_0(\tau,y)\;\frac{\cosh(\gamma_0 z_d)}{\cosh\gamma_0} \tag{23}$$

$$\Omega_0(\tau,y)=\frac{1-\exp[-\tau(y^2-\gamma_0^2)]}{(y^2/\gamma_0^2+1)\;[\gamma_0^2(1+\sigma)+\gamma^2-(\gamma_0^2-y^2)^2/\sigma]} \tag{24}$$

$$\omega_n(\tau,z_d,y)=\Omega_n(\tau,y)\;\frac{\cos(\gamma_n z_d)}{\cos\gamma_n} \tag{25}$$

$$\Omega_n(\tau,y)=\frac{1-\exp[-\tau(y^2+\gamma_n^2)]}{(y^2/\gamma_n^2)\left[\gamma_n^2(1+\sigma)-y^2+(\gamma_n^2+y^2)^2/\sigma\right]} \tag{26}$$

where $\gamma_n(\sigma,y)$ are implicit functions defined as roots of the equations

$$\sigma\gamma_0\sinh\gamma_0-(y^2-\gamma_0^2)\cosh\gamma_0=0,\quad \gamma_0<y, \tag{27}$$

$$\sigma\gamma_n\sin\gamma_n-(y^2+\gamma_n^2)\cos\gamma_n=0,\quad (n-\frac{1}{2})\pi<\gamma_n<n\pi,\ n\geq 1 \tag{28}$$

The other parameters are given by

$$z_d=\frac{z}{b},\ \tau=\frac{K_z t}{S_s b^2},\ \sigma=\frac{S_s b}{S_y},\ \beta=K_d(\frac{r}{b})^2,\ \beta_R=K_d(\frac{R}{b})^2 \tag{29}$$

where J_0 and J_1 are the Bessel functions of zero and first order. The drawdown is negative in the case of the groundwater recharge.

The averaged drawdown in an observation well that is perforated between the elevations z_1 and z_2 (Fig. 2) is the average of (22) over the vertical distance and is given by the formula

$$\langle s\rangle_{z_1,z_2}(r,t)=(z_2-z_1)^{-1}\int_{z_1}^{z_2}s(r,z,t)\,dz \tag{30}$$

As an consequence of the structure of (22) one can obtain the averaged value by redefinition of the expressions

$$\langle s\rangle_{z_1,z_2}=-I_c\int_0^{\infty}J_0(y\beta^{\frac{1}{2}})J_1(y\beta_R^{\frac{1}{2}})\sum_{n=0}^{\infty}u_n(y)\,dy \tag{31}$$

$$u_0(y)=\Omega_0(\tau,y)\,\frac{\sinh(\gamma_0 z_{2,d})-\sinh(\gamma_0 z_{1,d})}{(z_{2,d}-z_{1,d})\gamma_0\cosh\gamma_0} \tag{32}$$

$$u_n(y)=\Omega_n(\tau,y)\,\frac{\sin(\gamma_n z_{2,d})-\sin(\gamma_n z_{1,d})}{(z_{2,d}-z_{1,d})\gamma_n\cos\gamma_n} \tag{33}$$

where all parameters were defined in (27)-(29). If there is no gravity drainage ($S_y=0$, $\alpha_y\to\infty$) the vertically averaged equations (16)-(21) between elevations $z_1=0$ and $z_2=b$ can be transformed into Hantush's (1967) problem for the averaged overall saturated thickness drawdown

$$\langle s\rangle(r,t)=\langle s\rangle_{0,b}(r,t) \tag{34}$$

Indeed, after the integration of (16) and application of (20) and (21), one obtains

$$\frac{K_h}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\langle s\rangle}{\partial r}\right)-I=\frac{\partial\langle s\rangle}{\partial t}, \quad r>0, \ t>0 \tag{35}$$

$$\langle s\rangle(r,0)=0, \quad \frac{\partial\langle s\rangle(0,t)}{\partial r}=0, \quad \langle s\rangle(\infty,t)=0 \tag{36}$$

The same statement results from the averaged solution (31). To prove this, one obtains from (28) and (33), for $z_1=0$, $z_2=b$, and $S_y\to 0$

$$\lim_{\sigma\to\infty}\gamma_n=n\pi, \quad \lim_{\sigma\to\infty}u_n(y)=0, \quad n\geq 1 \tag{37}$$

Also, from (27) and (32)

$$\lim_{\sigma\to\infty}\gamma_0=0, \quad \lim_{\sigma\to\infty}\sigma\gamma_0^2=y^2 \tag{38}$$

$$\lim_{\sigma\to\infty}u_0(y)\,I_c=\frac{I_0 R}{(K_h K_v)^{\frac{1}{2}}}\,\frac{1-\exp(-\tau y^2)}{y^2} \tag{39}$$

After substitution of (37)-(39) to (31) and redefinition of the variables one obtains

$$\langle s\rangle(r,t)=-\frac{I_0 R^2}{bK_h}\int_0^\infty(1-e^{-t_R\eta^2})\,J_0\!\left(\eta\frac{r}{R}\right)J_1(\eta)\,\frac{d\eta}{\eta^2} \tag{40}$$

$$t_R=\frac{Tt}{SR^2}, \quad T=K_h b, \quad S=S_s b \tag{41}$$

Here the drawdown depends on the horizontal permeability only.

The asymptotic drawdown for large values of time is given by the formula (see Appendix B)

$$s(r,z,t)_{t\to\infty}=-I_{c,1}\int_0^\infty dy\,\frac{\cosh(yz_d)}{y\sinh y}\,J_0(y\beta^{\frac{1}{2}})\,J_1(y\beta_R^{\frac{1}{2}})\,(1-e^{-\tau y\tanh y}) \tag{42}$$

$$I_{c,1}=\frac{I_0 R}{(K_h K_v)^{\frac{1}{2}}}, \quad \tau=\frac{tK_v}{S_y b} \tag{43}$$

It is Dagan's formula (1967b) for an incompressible aquifer after the generalization for the anisotropy. After sufficiently large times the significance of compressibility of an aquifer becomes negligible. The growth of a mound tends asymptotically to a logarithmic function of time.

The asymptotic drawdown for small values of time is given by infinite series (see Appendix C)

$$s(r,z,t) \approx \frac{I_0 b\sigma}{K_v}\eta(R-r)E(z_d,t), \quad t\to 0 \tag{44}$$

$$\eta(R-r) = \begin{cases} 1, & r<R, \\ 1/2, & r=R, \\ 0, & r>R \end{cases} \tag{45}$$

$$E(z_d,t) = \theta t - \frac{2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n[1-e^{-\theta(n+\frac{1}{2})^2\pi^2 t}]}{(n+\frac{1}{2})^3}\cos[(n+\frac{1}{2})z_d], \quad \theta = \frac{K_v}{S_s b^2} \tag{46}$$

From this formula it is apparent that drawdown does not depend on the horizontal permeability or the radial distance within the recharge area. The drawdown growth occurs on the free surface faster than on the bottom of an unconfined aquifer.

Upon truncation of the infinite series one obtains a uniform linear growth of the mound elevation and the hydraulic head within recharged area of aquifer

$$s(r,z,t) \approx -\frac{I_0 t}{S_y}\eta(R-r), \quad t\to\infty \tag{47}$$

RECTANGULAR SOURCES

The flow underneath a rectangular recharging area with length $2X$ and width $2Y$ can be approximated by the initial boundary value problem (11)-(15) with the distributed source

$$I(x,y,t) = I_0\ \phi(x,-X,X)\ \phi(y,-Y,Y), \quad W=0 \tag{48}$$

All the involved functions were described in (8)-(10).

Applying the Laplace transform on time and the double Fourier cosine transforms on the horizontal cartesian coordinates one obtains formulas similar to those given above for circular sources. Omitting the tedious calculations, we have

$$s(x,y,z,t) = -I_R\int_0^\infty\int_0^\infty \Pi(u_1,u_2,x,y)\sum_{n=0}^{\infty}\omega_n(\tau,z_d,u)\frac{du_1 du_2}{u_1 u_2}, \quad I_R = \frac{8 I_0 b\sigma}{\pi^2 K_v} \tag{49}$$

$$\Pi(u_1,u_2,x,y) = \cos(u_1\beta_x^{\frac{1}{2}})\cos(u_2\beta_y^{\frac{1}{2}})\sin(u_1\beta_X^{\frac{1}{2}})\sin(u_2\beta_Y^{\frac{1}{2}}) \tag{50}$$

$$u^2=u_1^2+u_2^2,\ \beta_x=K_d(\frac{x}{b})^2,\ \beta_y=K_d(\frac{y}{b})^2,\ \beta_X=K_d(\frac{X}{b})^2,\ \beta_Y=K_d(\frac{Y}{b})^2 \tag{51}$$

The notation for the functions $\omega_n(\tau,z_d,u)$ was defined by (23)-(26).

The drawdown in the observation well perforated between elevations z_1 and z_2 (Fig. 2) is the average of drawdown in (49) over a vertical distance. This is similar to (30) and (31)

$$s_{z_1,z_2}(x,y,t)=I_R\int_0^\infty\int_0^\infty \Pi(u_1,u_2,x,y)\sum_{n=0}^{\infty}\frac{u_n(u)\,du_1du_2}{u_1u_2} \tag{52}$$

where functions $u_n(u)$ were defined by equations (32)-(33).

By averaging across the saturated thickness drawdown (49), one can derive Hantush's (1967) solution for a rectangular recharging area under the absence of gravity drainage. To demonstrate such a relationship it is necessary to evaluate the average drawdown (52) under the conditions

$$z_1=0,\quad z_2=b,\quad S_s\to 0 \tag{53}$$

After the reduction of $u_n(u)$, $n\geq 0$, to the same expressions (37)-(39), and applying double Fourier cosine transforms on the horizontal space variables and Laplace transform on time, one can establish the identity of both formulas.

The asymptotic behavior for large values of time for (49) can be obtained from the Laplace transform in the asymptotic limit $p\to 0$, p being the Laplace transform parameter. We obtain

$$s(x,y,z,t)\approx -I_{c,1}\int_0^\infty\int_0^\infty \Pi(u_1,u_2,x,y)\,\Phi(\tau,z_d,u)\,\frac{du_1du_2}{u_1u_2} \tag{54}$$

$$I_{c,1}=\frac{4I_0b}{\pi^2K_v},\quad \Phi(\tau,z_d,u)=(1-e^{-\tau u\tanh u})\,\frac{\cosh(uz_d)}{u\sinh u},\quad \tau=\frac{K_vt}{S_yb} \tag{55}$$

These quantities do not depend on S_s; i.e., after sufficiently large times the significance of a compressibility becomes negligible. The same result stems from (11)-(15) for $S_s=0$. The growth of a hydraulic head tends asymptotically to a logarithmic function of time. The formula (54) is an exact solution for an incompressible aquifer of a finite thickness.

For small values of time or for large values of Laplace parameter p, the hydraulic head distribution is given by the formula

$$s(x,y,x,t)=-\frac{I_0b\sigma}{K_v}\eta(X-x)\eta(Y-y)E(z_d,t),\quad t\to 0 \tag{56}$$

where the functions η and E were defined in (45)-(46). After truncating the infinite series for $t\to 0$ one obtains

$$s(x,y,z,t) \approx -\frac{I_0 t}{S_y} \eta(X-x)\eta(Y-y), \quad t \to 0 \tag{57}$$

DRAWDOWN FOR VARIABLE SPECIFIC YIELD

Percolation of irrigation water through the unsaturated zone underneath area G_j reduces the specific yield by the value of moisture content $S_{y,1}$, i.e.,

$$S_y(x,y,z,t) = S_{y,0}(x,y,z,t) - S_{y,1}\gamma(x,y,G_j) \tag{58}$$

$S_{y,0}$ is the specific yield considered earlier for an undisturbed, unsaturated zone. In this case a change of head and velocity distribution occurs in an unconfined aquifer, and

$$s(x,y,z,t) = s_0(x,y,z,t) + s_1(x,y,z,t) \tag{59}$$

where s_0 is the solution for the undisturbed, unsaturated zone, and s_1 is the solution correction for the specific yield variation. The analysis of specific yield changes must be performed by soil physics methods (Marsily, 1988) and is beyond the scope of the paper.

The recharge of an unconfined aquifer under irrigation is supposed to be minimal for widespread agricultural practices based on the irrigation scheduling. For this case the increase of saturation in an unsaturated zone is a small value,

$$S_{y,1} \ll S_{y,0} \tag{60}$$

As a consequence, the corrections in the hydraulic head and velocity distributions are small values also as well, i.e.,

$$|s_1| \ll |s_0| \tag{61}$$

Hence a perturbation technique is applicable. The solutions given above without the correction for the saturation variability are valid as a first approximation

$$S_{y,1} = 0, \quad s_1 = 0 \tag{62}$$

For intensive aquifer recharge the changes in saturation are comparable with the specific yield and condition (61) is not valid. Furthermore, the assumption of small water table slopes and the consequent transition from nonlinear conditions (6)-(7) to linear condition (15) is not valid. Such a situation occurs for artificial groundwater recharge and an analytical approach for the three-dimensional processes is not straightforward (Morel-Seytoux et al., 1990).

To obtain necessary corrections s_1 for variable saturation one must substitute (58)-(59) to the linearized condition (15) and retain first-order terms in all expansions. Since s_0 satisfies an initial boundary value problem (11)-(15) the correction s_1 must satisfy the same equations, where equation (15) is modified by

$$K_v \frac{\partial s_1}{\partial z} + S_{y,0} \frac{\partial s_1}{\partial t} = S_{y,1} \gamma (x, y, G_j) \frac{\partial s_0}{\partial t} \tag{63}$$

The right-hand term is negative in a recharge area since the time derivative for drawdown is negative.

The calculation of the first-order terms shows that the influence of specific yield variability may be treated like an increase of recharge rate per unit area. The correction to the recharge rate is equivalent to the amount of water which is stored in the unsaturated zone within distance of the water table transition per unit time.

To obtain the correction s_1 one can apply the technique given above for (11)-(15) using (63) instead of (15). The procedure does not require any modification and is based on the same sequence of the integral transforms. The main point in the calculations is to invert the Laplace transforms. From Appendix A it is apparent that the transform for s_1 has the same singularities as for s_0, although the structure of the final expression will be more complicated. To evaluate the increase of complexity for the problem with saturation correction one can compare even more simple solutions of the analogous two-dimensional problem given by Carslaw and Jaeger, p. 347 (1959) with and without cylindrical nonhomogeneity in the heat conduction.

For simplicity one can sacrifice the universality of a final expression to the practical convenience to calculate correction s_1 and to study it for small time values. According to (47) and (57) the time derivative in right-hand expression in (63) does not depend on time for its small values; hence

$$\frac{\partial s_0}{\partial t} = \frac{I_0}{S_y} \gamma (x, y, G_j) \tag{64}$$

After substitution of (64) into (63) one obtains

$$K_v \frac{\partial s_1}{\partial z} + S_{y,0} \frac{\partial s_1}{\partial t} = \frac{S_{y,1}}{S_{y,0}} I_0 \; \gamma (x, y, G_j) \tag{65}$$

Upon comparison of this equation with (15) the following rule is apparent. Because of the variable specific yield, one can obtain corrected formulas for the drawdown or velocity components by multiplication the initial formulas by factor k_f

$$s(x,y,z,t) = s_0(x,y,z,t) \; k_f, \quad k_f = 1 + \frac{S_{y,1}}{S_{y,0}} \tag{66}$$

It is important to note that such factor is applicable for the recharge processes only. Therefore the hydraulic processes are noninvariant under a transition from discharge to recharge processes in an unconfined aquifer. This feature is unique for an unconfined aquifer due an influence of the downward irreversible infiltration processes in the unsaturated zone.

DERIVATION OF GROUNDWATER VELOCITY FORMULAS

The velocity vector is obtained from the drawdown formulas by multiplication of the permeability tensor $\boldsymbol{K}$ by a gradient for the above integrals:

$$\vec{V} = \boldsymbol{K}\,\nabla s \tag{67}$$

The gradients, of course, depend upon the coordinate system. To derive explicit expressions for the velocity components one has to redefine some factors in the integrands for those integrals.

For the circular sources the replacements in formulas (22)-(29) are

$$s(r,z,t) \rightarrow V_r(r,z,t): \qquad J_0(y\beta^{\frac{1}{2}}) \rightarrow -K_h J_1(y\beta^{\frac{1}{2}})\, K_d^{\frac{1}{2}} \frac{y}{b} \tag{68}$$

$$s(\ldots) \rightarrow V_z(\ldots): \cosh(\gamma_0 z_d) \rightarrow K_v \sinh(\gamma_0 z_d)\frac{\gamma_0}{b},\ \cos(\gamma_n z_d) \rightarrow -K_v \sin(\gamma_n z_d)\frac{\gamma_n}{b} \tag{69}$$

For rectangular sources the replacements in (49)-(51) and (23)-(29) are

$$s(x,y,z,t) \rightarrow V_x(x,y,z,t): \cos(u_1\beta_x^{\frac{1}{2}}) \rightarrow -K_h \sin(u_1\beta_x^{\frac{1}{2}})\, K_d^{\frac{1}{2}} \frac{u_1}{b} \tag{70}$$

$$s(x,y,z,t) \rightarrow V_y(x,y,z,t): \quad \cos(u_2\beta_y^{\frac{1}{2}}) \rightarrow -K_h \sin(u_2\beta_y^{\frac{1}{2}})\, K_d^{\frac{1}{2}} \frac{u_2}{b} \tag{71}$$

$$s(\ldots) \rightarrow V_z(\ldots): \cosh(\gamma_0 z_d) \rightarrow K_v \sinh(\gamma_0 z_d)\frac{\gamma_0}{b},\ \cos(\gamma_n z_d) \rightarrow -K_v \sin(\gamma_n z_d)\frac{\gamma_n}{b} \tag{72}$$

The convergence of the improper integrals representing the horizontal groundwater velocity components in an aquifer is slower than the drawdown since the integrand vanishes slower for large horizontal distances from a source.

Transformations similar to (68)-(72) are used to obtain the velocities in an unconfined compressible aquifer in a vicinity of the penetrating or nonpenetrating well from Neuman's solution (1974).

The formulas for velocities in an unconfined, incompressible aquifer involve the same transformations.

METHODS OF NUMERICAL EVALUATION FOR INTEGRALS

The first numerical calculations for these types of an integrals were performed by Dagan (1967a, 1967b) and slightly modified by Neuman (1972).

Based on the straightforward application of Simpson's rule, they inserted up to 30 nodes between adjacent roots of Bessel functions to match the oscillating behavior of integrands. Such approach is sufficient for a individual well or a recharge source behavior evaluation. For a group of wells and distributed sources the approach becomes impractical because of time consumption restrictions. Especially severe limitations arise for the velocity component calculations which converge slower than hydraulic head due to the larger exponent of the singularity in the infinity for horizontal coordinates for all integrals.

The simplest way to reduce multiple repeated calculations is by introduction of a table for each kind of a source or sink. The tables have to be prepared and stored in computer memory before starting the calculation for the every special case study. With the appropriate parameters for an aquifer the tables have to be developed for a set of discrete coordinates and times only. The application of a nonlinear interpolation scheme may provide sufficient accuracy under the relatively sparse tables (Hamming, 1963).

Additional computer time reduction may be achieved by taking into account the special nature of the integrands under a table construction. The Filon method permits the reduction of the number of integration nodes at least by an order of magnitude in any spatial dimension for the oscillating integrands (Tranter, 1966). However, its application is not straightforward for two oscillating functions and multidimensional integrands.

An additional resource in computer time reduction exists for the solutions for a compressible aquifer. Since the exact formula of Laplace inversion leads to infinite series and the evaluation of implicit functions, one can eliminate the resulting consequences by applying numerical inversion of Laplace transforms (Talbot, 1979).

CONCLUSIONS

To obtain a self-consistent, three-dimensional groundwater flow velocity model for an unconfined aquifer of a finite thickness the analytical method of integral transforms was applied. The results are new solutions for the drawdown and velocity components for distributed sources with circular and rectangular shapes, taking into account the compressibility properties of the aquifer. The calculations of the velocity for an arbitrary combination of wells and areal sources can be performed using the superposition principle. For the incompressible aquifer, the solutions given above for a circular or rectangular source can be combined with Neuman's well function (1974). For the incompressible aquifer the appropriate solutions should be combined with Dagan's solutions (1967a, 1967b). No additional aquifer parameters are required for this complex model. The parameters are obtainable from the same pumping tests as in these references.

For sufficiently large times the compressibility effect becomes negligible. Therefore the choice of an appropriate model for aquifer properties should be consistent with the time scale of the simulated processes.

One new feature of the calculation is the consideration of the variable saturation between the areal infiltration source and groundwater table. Formulas for drawdown and groundwater recharge corrections were given in order to evaluate the irreversibility of groundwater movement under transition from a pumping to a recharge, caused by partial saturation in the zone above water table. For small variability of the specific yield, essentially the same structure of the formulas is retained essentially.

The computational problems arising from the complex expressions were discussed with regard to computer time consumption. The main sources for the reduction of computer time are: special integration procedures with the treatment of the oscillating integrands, numerical inversion of Laplace transforms, application of the pre-calculation for sparce tables, and nonlinear interpolation.

The new results are intended mainly for the simulation of the agricultural contamination by the fertilizers and pesticides. They are also applicable for the nuclear areal contamination of groundwater, similar to that in Chernobyl and in other cases of intensive areal groundwater recharge and the interaction with well fields.

APPENDIX A

To obtain Green's function for the problem under investigation, the integral transforms were applied (Lavrentjev and Shabat, 1973). The main stages of derivation given below are similar to Neuman (1972, 1974). Applying the Laplace transform on time to the function $s(r,z,t)$ gives

$$\bar{s}(r,z,p) = L[s(r,z,t)] = \int_0^\infty e^{-pt} s(r,z,t)\,dt \tag{A1}$$

Consequently, the Hankel transform on r of the function $\bar{s}(r,z,t)$ gives

$$\bar{\bar{s}}(a,z,p) = H[\bar{s}(r,z,p)] = \int_0^\infty r J_0(ra)\,\bar{s}(r,z,p)\,dr \tag{A2}$$

For the initial boundary value problem (1)-(7) one obtains a boundary value problem for ordinary differential equation

$$\frac{d^2\bar{\bar{s}}}{dz^2} - \eta^2 \bar{\bar{s}} = 0 \tag{A3}$$

$$\frac{d\bar{\bar{s}}(a,0,p)}{dz} = 0, \quad \frac{d\bar{\bar{s}}(a,b,p)}{dz} + \frac{p}{\alpha_y}\bar{\bar{s}}(a,b,p) = -\tilde{I} \tag{A4}$$

$$\eta^2 = \frac{1}{K_d}\left(\frac{p}{\alpha_s} + a^2\right), \quad \tilde{I}(a,p) = \frac{I_0 R}{apK_v} J_1(Ra) \tag{A5}$$

The relationships based on Bessel function properties

$$H\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{s}}{\partial r}\right)\right] = -a^2\bar{\bar{s}} - \lim_{r\to\infty}\left(r\frac{\partial \bar{s}}{\partial r}\right), \quad J_0'(r) = -J_1(r) \tag{A6}$$

were used for this procedure. The solution of the boundary value problem is

$$\bar{\bar{s}}(a,z,p)=-\frac{I_0R}{apK_z}J_1(Ra)\frac{\cosh(\eta z)}{\eta\sinh(\eta b)+\frac{p}{\alpha_y}\cosh(\eta b)} \tag{A7}$$

Using inverse Hankel transform on the dependent variable

$$\bar{s}(r,z,p)=H^{-1}[\bar{\bar{s}}(a,z,p)]=\int_0^\infty \bar{\bar{s}}(a,z,p)\,aJ_0(ra)\,da \tag{A8}$$

one obtains after the introduction of the new variables

$$\xi^2=b^2\eta^2=y^2+\frac{pr^2}{\alpha_s\beta},\quad z_d=\frac{z}{b},\quad \beta=K_d\left(\frac{r}{b}\right)^2,\quad \beta_R=K_d\left(\frac{R}{b}\right)^2,\quad \sigma=\frac{S_sb}{S_y} \tag{A9}$$

an expression for Laplace transform of required function:

$$\bar{s}(r,z,p)=-I_1\int_0^\infty dy\frac{J_0(y\beta^{\frac{1}{2}})\,J_1(y\beta_R^{\frac{1}{2}})\cosh(\xi z_d)}{[\sigma\xi\sinh\xi+(\xi^2-y^2)\cosh\xi]\,p},\quad I_1=\frac{I_0R\sigma}{(K_hK_v)^{\frac{1}{2}}} \tag{A10}$$

To obtain the drawdown formula one can apply the inversion theorem for the Laplace transformation,

$$s(r,z,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\bar{s}(r,z,\lambda)\,e^{\lambda t}d\lambda \tag{A11}$$

where the integration is along a line in complex plane and γ is so large that all singularities of integrand s lie to the left of the line $(\gamma-i\infty,\gamma+i\infty)$. λ is written in place of p in (A11) to emphasize the fact that in (A11) we are considering the behavior of integrand regarded as a function of a complex variable. Applying the initial condition (17) one can modify this formula to

$$s(r,z,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}s(r,z,\lambda)\,(e^{\lambda t}-1)\,d\lambda \tag{A12}$$

The expression for an immediate inversion is

$$s(r,z,t)=-I_1\int_0^\infty J_0(y\beta^{\frac{1}{2}})\,J_1(y\beta_R^{\frac{1}{2}})\,dy\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\bar{g}(\lambda)}{\pi i}d\lambda \tag{A13}$$

$$\bar{g}=\frac{\phi(\lambda,\xi)}{\psi(\xi)},\quad \xi^2=y^2+\lambda\frac{r^2}{\alpha_s\beta} \tag{A14}$$

$$\phi(\lambda,\xi)=\frac{\exp(\lambda t)-1}{\lambda}\cosh(\xi z_d),\ \psi(\xi)=\sigma\xi\sinh\xi+(\xi^2-y^2)\cosh\xi \quad \text{(A15)}$$

where $g(\lambda)$ is a single-valued function of λ with infinite many of simple poles along the negative real axis. For convenience, of the removable singularity at $\lambda=0$ formula (A12) was selected rather than standard Mellin's formula (A11). The singularities of g are the roots of equation

$$\psi(\xi)=0 \quad \text{(A16)}$$

which was studied by Neuman (1972). This equation has the one real root, and an infinite number of the purely imaginary roots that can be determined numerically from the equations

$$\xi_0=\gamma_0, \quad \xi_n=i\gamma_n, \ n\geq 0 \quad \text{(A17)}$$

$$\sigma\gamma_0\sinh\gamma_0+(\gamma_0^2-y^2)\cosh\gamma_0=0, \quad \gamma_0<y \quad \text{(A18)}$$

$$\sigma\gamma_n\sinh\gamma_n+(\gamma_n^2+y^2)\cos\gamma_n=0, \quad (n-\frac{1}{2})\pi<\gamma_n<n\pi, \ n\geq 1 \quad \text{(A19)}$$

The roots in the complex plane are shown in a Fig. 4. After completing the

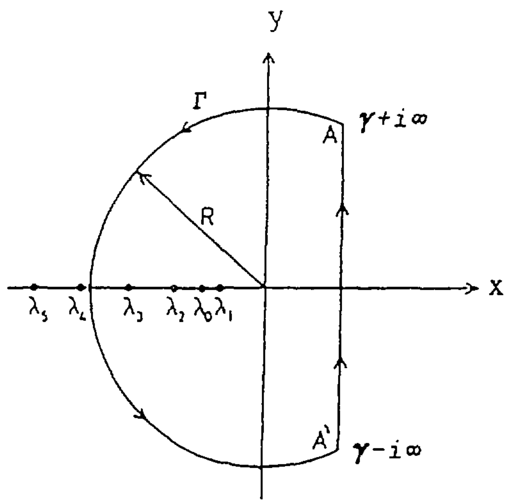

Fig. 4. Contour of integration used with Laplace inversion formula.

contour $(\gamma-i\infty,\gamma+i\infty)$ by a large circle Γ of radius R passing through the points A and A', and avoiding any pole of the integrand, the value of integral (A13) is not changed for any $R\to\infty$. According to the residue theorem one has

$$\int_{\gamma+i\infty}^{\gamma-i\infty} \frac{\bar{g}(\lambda)}{\pi i} d\lambda = -\lim_{R\to\infty} \int_{A}^{A'} \frac{\bar{g}(\lambda)}{\pi i} d\lambda + 2\sum_{n=0}^{\infty} Res[\bar{g}(\lambda), \lambda_n] \tag{A20}$$

where $Res\ [\bar{g}\ (\lambda), \lambda_n]$ is the residue of g at the pole λ_n. The right-hand integral vanishes for $R\to\infty$. The residues of the poles given by (A17)-(A19) can be obtained from the formula (Lavrentjev and Shabat, 1973)

$$Res[\bar{g}(\lambda), \lambda_n] = [\phi / \frac{d\psi}{d\lambda}]_{\lambda=\lambda_n} \tag{A21}$$

After the substitution of (A21) and (A20) into (A13) one immediately arrives at (22)-(29).

APPENDIX B

To obtain the asymptotic formulas for a hydraulic head for large times one has to invert the Laplace transform taken for asymptotic values $p\to 0$. Since

$$\xi = y + O(p), \quad p\to 0 \tag{B1}$$

one finds from (A11)

$$s(r,z,p) = -I_2 \int_0^{\infty} dy \frac{J_0(y\beta^{\frac{1}{2}}) J_1(y\beta_R^{\frac{1}{2}})}{p(p+\alpha_y \frac{y}{b} \tanh y)} \frac{\cosh(yz_d)}{\cosh y}, \quad I_2 = \frac{I_0 R K_d^{\frac{1}{2}}}{K_v b} \tag{B2}$$

The order term $O(...)$ is understood to mean that

$$F(x) = O[G(x)] \quad \to \quad \lim_{x\to\infty} \left| \frac{F(x)}{G(x)} \right| = A, \quad A = const$$

Noting the Laplace transform

$$L[1-e^{-at}] = \frac{a}{p(p+a)}$$

one immediately comes to (42).

APPENDIX C

To obtain the asymptotic drawdown behavior for small times one has to invert the Laplace transform (A10) for large asymptotic values of p. We have

$$\xi = \sqrt{\frac{p}{\theta}} + O(p^{-\frac{1}{2}}), \quad \theta = \frac{K_z}{S_s b^2}, \quad p\to\infty \tag{C1}$$

From this expansion the Laplace transform can be decomposed into the multiplication of two factors. One of them depends only on p

$$\bar{s}(r,z,p) = -I_1 F(\beta, \beta_R) \bar{E}(z_d, p) \tag{C2}$$

$$F(\beta,\beta_R)=\int_0^{\infty} J_0(y\beta^{\frac{1}{2}})\,J_1(\beta_R^{\frac{1}{2}})\,dy=\frac{1}{\beta_R^{\frac{1}{2}}}\,\eta(R-r)\,,\ \eta(x)=\begin{cases}1, & x>0\\ \frac{1}{2}, & x=0\\ 0, & x<0\end{cases} \qquad (C3)$$

$$\bar{E}(z_d,p)=\frac{\cosh\left(z_d\sqrt{\frac{p}{\theta}}\right)}{\frac{p^2}{\theta}\cosh\sqrt{\frac{p}{\theta}}} \qquad (C4)$$

The integration in (C3) is accomplished via formula (6.512.3) in Gradshteyn and Ryzhik (1980). The Laplace transform inversion of (C4) uses formula (6) on p.313 from Carslaw and Jaeger (1959). Then

$$\bar{E}(z_d,t)=\theta\int_0^{\infty}\left[1-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}e^{-\theta(n+\frac{1}{2})^2\pi^2 t}\cos\left[(n+\frac{1}{2})\pi z_d\right]\right]dt \qquad (C5)$$

After term-by-term integration, we obtain equation (46).

ACKNOWLEDGMENTS

Support for this research was provided by the University of Nebraska-Lincoln, Water Center with funds from the Nebraska Research Initiative. I also appreciate the assistance provided by Dr. Roy Spalding in initiating this project.

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