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APPLICATION OF NONPARAMETRIC STATISTICS TO MATERIAL STRENGTH/RELIABILITY ASSESSMENT

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Abstract: An advanced material technology requires data base on a wide variety of material behavior which need to be established experimentally. It may often happen that experiments are practically limited in terms of reproducibility or a range of test parameters. Statistical methods can be applied to understanding uncertainties in such a quantitative manner as required from the reliability point of view. Statistical assessment involves determinations of a most probable value and the maximum and/or minimum value as one-sided or two-sided confidence limit. A scatter of test data can be approximated by a theoretical distribution only if the goodness of fit satisfies a test criterion. Alternatively, nonparametric statistics(NPS) or distribution-free statistics can be applied. Mathematical procedures by NPS are well established for dealing with most reliability problems. They handle only order statistics of a sample. Mathematical formulas and some applications to engineering assessments are described. They include confidence limits of median, population coverage of a sample, required minimum number of a sample, and confidence limits of fracture probability. These applications demonstrate that a nonparametric statistical estimation is useful in logical decision making in the case a large uncertainty exists.

Keywords: Experimental data, Uncertainty, Nonparametric statistics, Statistical estimation, Beta distribution, Confidence limit, Material strength, Design minimum strength, Fracture probability, Structural reliability.

1. INTRODUCTION

A statistical or probabilistic theory is an effective engineering methodology for predicting a degree of certainty of observed events accompanying with apparent uncertainties. These uncertainties may result often either from a limited number of observations or from too large number of observations. Statistical estimation can be applied to understanding the uncertainties in such a quantitative manner as required from the reliability point of view. That is, it provides quantities related to a set of observations with an arbitrarily specified level of confidence.

As it is well known, an advanced material technology for nuclear engineering [1] requires data base on a wide variety of material behavior which must be established experimentally. It may however often happen that for some practical reasons experiments are limited in terms of reproducibility or a range of test parameters. In addition, uncertainties may result from an intrinsic nature of a property or from an experimental technique. Performance of nuclear components is associated always with so called reliability or safety principles because of a potential radiological hazard in nuclear energy systems.

Consequently, statistical understanding of experimental data is a logical prerequisite for accomplishing design improvements for better or higher performance of materials, components and structural systems in not only future but current nuclear facilities.

Concerning statistical estimation, there are two types of mathematical approaches: One is parametric statistics and the other nonparametric statistics. Both involve determinations of a most probable value and the maximum or minimum value as one-sided or two-sided confidence limit which is frequently called tolerance limit in manufacturing engineering.

The present paper is concerned with nonparametric statistics and their applications with special relevance to material strength, fracture probability. Fig. 1 shows a scope of statistical estimations based on a given set of data, which will be discussed in the paper. A conventional procedure based on parametric statistics is also shown for comparison. Parametric statistics , being in wide use in many scientific research fields, employ an assumption that a measured value or a phenomenon deviates according to a theoretical distribution whose mathematical form is given explicitly. For example, a normal distribution function has two distribution parameters; mean and standard deviation. One for Weibull distribution has generally three parameters; Weibull modulus (shape parameter), scale parameter and location parameter.

2. FUNDAMENTALS OF NONPARAMETRIC STATISTICS

Nonparametric statistics treat with a random variable x sampled from a population whose distribution function F(x) is continuous but is unknown in mathematical expression. A set of observations i.e. a sample is rearranged in order to get order statistics, $x_1, x_2, \cdots, x_n, \cdots, x_N$, in ascending order of magnitude, where N is a sample size. x_1, x_N and x_n are the minimum, maximum and nth order statistic, respectively.

Nonparametric statistics and their application employ only order statistics of the sample. They are based on some mathematical principles called sampling theory of order statistics [2], which is described briefly in the following.

Suppose F(x) is a continuous distribution function. A variable y, defined by y = F(x), is also a random variable having the rectangular distribution with mean and range being 1/2 and 1, respectively, R(1/2, 1).

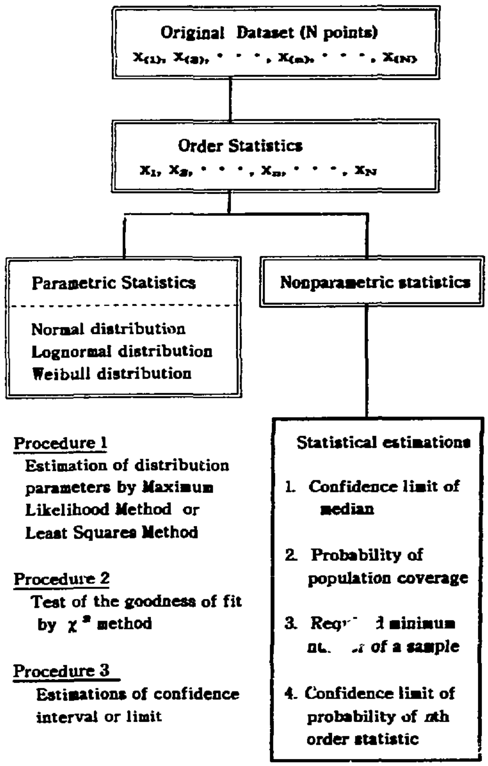

Fig. 1 Scope of applications of nonparametric statistics in the study as compared with parametric statistics.

THEOREM I : Let $y_n = F(x_n)$ (n=1, 2, $\cdots$, N) be the order statistics of the sample from R(1/2, 1). Then a random variable y_n has the beta distribution Be(n, N−n+1).

The probability density function B_y and the cumulative distribution function I_y of the beta distribution Be(n, N−n+1) are given by the following.

$$B_y(n, N-n+1) = \frac{\Gamma(N+1)}{\Gamma(n)\Gamma(N-n+1)} y^{n-1}(1-y)^{N-n}$$

$$= n\binom{N}{n} y^{n-1}(1-y)^{N-n}, \quad (1)$$

$$I_y(n, N-n+1) = \int_0^y B_y(n, N-n+1)\,dy$$

$$= \frac{\Gamma(N+1)}{\Gamma(n)\Gamma(N-n+1)} \int_0^y y^{n-1}(1-y)^{N-n}\,dy, \quad (2)$$

where $\Gamma(g)$ is the gamma function.

Examples of the function Eq.(1) are shown in Fig. 2 with n/N=0.05 and Fig. 3 with n/N=0.50 [3].

Equations (1) and (2) are of particular interest in deriving a formula for the experimental cumulative probability of nth order statistic $F(x_n)$. It is usually either n/(N+1) by mean ranking or (n−0.3)/(N+0.4) by median ranking, which is proved in Appendix.

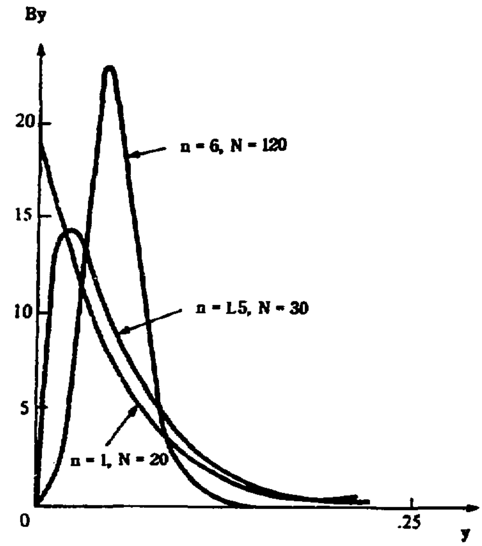

Fig. 2 Examples of beta distribution functions with n/N=0.05.

The second and third quantities derived from the order statics are sample blocks, $(-\infty, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_N, \infty)$, and the functions, $F(x_1)$, $F(x_2)-F(x_1)$, $\cdots$, $1-F(x_N)$ of these blocks. The latter are called coverages, u_1, u_2, $\cdots$, u_N, u_{N+1}.

THEOREM II: Note that y_n is the sum of n coverages, i.e. $u_1+u_2+\cdots+u_n$. It follows that the sum of any n of the coverages is again a random variable having the beta distribution Be(n, N-n+1).

In the succeeding, nonparametric estimations can be made using the beta distribution Be(n, N-n+1) as given in Eqs.(1) and (2). Calculus of Eq.(2) may be done directly using the numerical tables of a general form of the incomplete beta function below.

$$I_p(\nu_1, \nu_2) = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^p x^{\nu_1-1}(1-x)^{\nu_2-1}\,dx \qquad (3)$$

However, tables of a binomial distribution can be utilized more simply by applying the following relationship. The distribution function of the binomial distributioo E(N, r, p) is given as follows.

$$E(N, r, p) = \sum_{i=r}^{N} \binom{N}{i} p^i (1-p)^{N-i} \qquad (4)$$

Differentiating E(N, r, p) with N and r being fixed, we have

$$\frac{dE}{dp} = r\binom{N}{r} p^{r-1}(1-p)^{N-r}$$

$$= \frac{\Gamma(N+1)}{\Gamma(r)\Gamma(N-r+1)} p^{r-1}(1-p)^{N-r} \qquad (5)$$

Then, it follows,

$$E(N, r, p) = \frac{\Gamma(N+1)}{\Gamma(r)\Gamma(N-r+1)} \int_0^p x^{r-1}(1-x)^{N-1}\,dx \qquad (6)$$

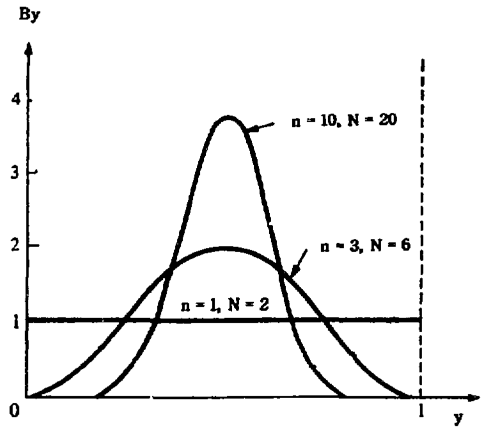

Fig. 3 Examples of beta distributions with n/N=0.5.

Comparing Eq.(3) and Eq.(6), we have

$$I_p(\nu_1, \nu_2) = E(\nu_1+\nu_2-1, \nu_1, p) \qquad (7)$$

or

$$I_p(N, N-r+1) = E(N, r, p). \qquad (8)$$

Thus, the value of the incomplete beta function is determined using tables of the binomial probabilities. If p>0.5 the following relation may be applied.

$$E(N, r, p) = 1 - E(N, N-r+1, 1-p). \qquad (9)$$

3. SOME APPLICATIONS TO STATISTICAL ESTIMATIONS

3.1 Confidence Irterval or Limits of Median

3.1.1 Mathematical Formula

For any number of p on the interval (0, 1), one can find a quantity θ_p which satisfies the equation below.

$$F(\theta_p) = p, \qquad (10)$$

where F(x) is a cumulative distribution function of the continuous random variable x. θ_p is called the pth quantile or fractile (100pth percentile). In particular, $x_{0.5}$ is the median of x.

Our subject to determine a confidence interval of $x_{0.5}$ is formulated more generally in the following.

Let F(x) be the distribution function of the population from which the order statistics, X_n (n=1~N), are defined. Since F(x) is a monotonically increasing function of x the following probability relation holds for $1 \leq s < r \leq N$ [4].

$$P_r\{x_s < \theta_p < x_r\} = P_r\{F(x_s) < p < F(x_r)\}. \qquad (11)$$

Recalling that $F(x_n)$ has the beta distribution Be(n, N-n+1), we have

$$P_r\{F(x_s) < p < F(x_r)\} = 1 - P_r\{F(x_s) > p\} - P_r\{F(x_r) < p\}$$

$$= I_p(s, N-s+1) - I_p(r, N-r+1). \qquad (12)$$

Consequently,

$$P_r\{x_s < \theta_p < x_r\} = I_p(s, N-s+1) - I_p(r, N-r+1). \qquad (13)$$

Then, let us take a confidence interval (x_s, x_r) of the pth fractile θ_p of the population at a confidence level of 100γ %. By definition,

$$P_r\{x_s < \theta_p < x_r\} = \gamma \qquad (14)$$

Eq.(14) gives the following statistical statement. In repeated sampling from the same population these limits, x_s and x_r, will vary about the population confidence limits. And for some samples the limits will include less than 100γ % of the population and for some other samples more than 100γ %. To be reasonably sure that at least 100γ % of the population lie between the sample confidence limits we can find the number s and r such that there is a high probability that the interval (x_s, x_r) will include at least 100γ % of the *p*th fractile in the population.

Comparing Eq.(13) with Eq.(14) leads to the following formula.

$$\gamma = I_p(s, N-s+1) - I_p(r, N-r+1). \tag{15}$$

In the special case of the median $x_{0.5}$, it would be desirable to choose s and r so that they are apart equally from the minimum and the maximum, respectively, i.e.

$$r = N - (s-1). \tag{16}$$

It follows that

$$\begin{aligned} P_r\{x_s < \theta_{0.5} < x_{N-s+1}\} &= I_{0.5}(s, N-s+1) - I_{0.5}(N-s+1, s) \\ &= 1 - 2I_{0.5}(N-s+1, s) \\ &= \gamma. \end{aligned} \tag{17}$$

Accordingly, the two-sided confidence interval (x_s, x_{N-s+1}) can be determined to find the maximum integer s satisfying

$$1 - 2I_{0.5}(N-s+1, s) \ge \gamma \tag{18}$$

If we consider a one-side confidence interval (x_s, ∞), the similar argument results in the following formulas for finding the corresponding integer s.

$$\begin{aligned} P_r\{x_s < \theta_{0.5}\} &= P_r\{F(x_s) < 0.5\} \\ &= I_{0.5}(s, N-s+1) \\ &= \gamma \end{aligned} \tag{19}$$

$$I_{0.5}(s, N-s+1) = E(N, s, 0.5) \ge \gamma \tag{20}$$

3.1.2 Calculated Results

The solution of Eq.(20) is listed in Table 1 and illustrated in Fig. 4 for typical ranges of sample size N and confidence level γ. For instance, the lower confidence limits of median of the sample N=40 are x_{20} and x_{13} for the confidence level of 50% and 99%, respectively.

3.2 Population Coverage and Required Number of Sample

3.2.1 Mathematical Formula

We are now interested in evaluating a fraction of the observed sample range (x_1, x_N) within the population. Using $u_2+u_3+\cdots+u_N = F(x_N) - F(x_1)$, the problem is formulated as follows.

$$P_r\{F(x_N) - F(x_1) > p\} = \gamma \quad (0 < p, \gamma < 1) \tag{21}$$

In engineering statistics X_N and x_1 are referred to as 100p% distribution-free tolerance limits at probability level γ.

Table 1 The sth order statistic as one-side lower confidence limit of median of the sample.

Sample size (N)	Confidence level (100γ %)				
	50	80	90	99	99.9
5	3	2	1	1	
10	5	4	3	1	1
15	8	6	5	3	2
20	10	8	7	5	3
25	13	10	9	7	5
30	15	13	11	9	7
35	18	15	14	11	8
40	20	17	16	13	10
45	23	20	18	15	12
50	25	22	20	17	14

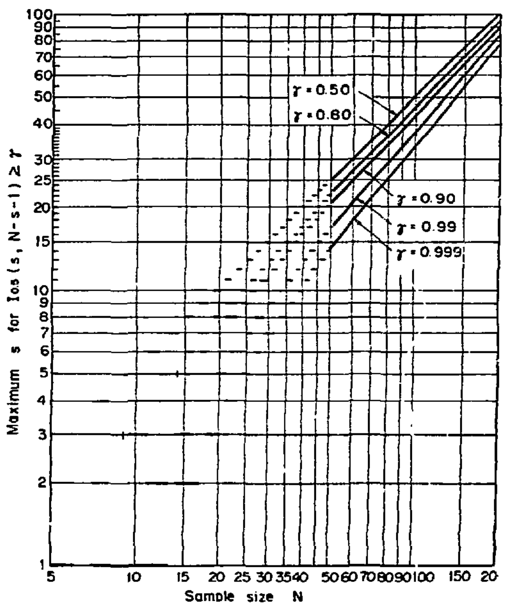

Fig. 4 Lower confidence limit of median as a function of sample size and confidence level.

Applying THEOREM II, it follows that

$$Pr\{F(x_N) - F(x_1) > p\} = \int_p^1 B_y\ (N-1,\ 2)\,dy$$

$$= 1 - I_p\ (N-1,\ 2), \qquad (22)$$

leading to the formula,

$$I_p\ (N-1,\ 2) = E(N,\ N-1,\ p) = 1-\gamma \qquad (23)$$

Meanwhile, Eq.(21) may be applied to another practical problem in that we want to know the minimum number of the sample required for a particular experiment in a sense of confidence or reliability. This problem is represented by the following equation for finding N, which depends on p and γ.

$$I_p\ (N-1,\ 2) = E\ (N,\ N-1,\ p) \geq 1 - \gamma. \qquad (24)$$

3.2.2 Calculated Results

Equation (24) is solved to form a relationship between p and γ as a function of sample size N. The results are shown in Fig. 5. From the figure we are able to find the minimum number of sample for individual reliability requirements. For instance, if we should be sure that the sample contains at least 95% of the population at 95 probability, we need at least a sample size of 95.

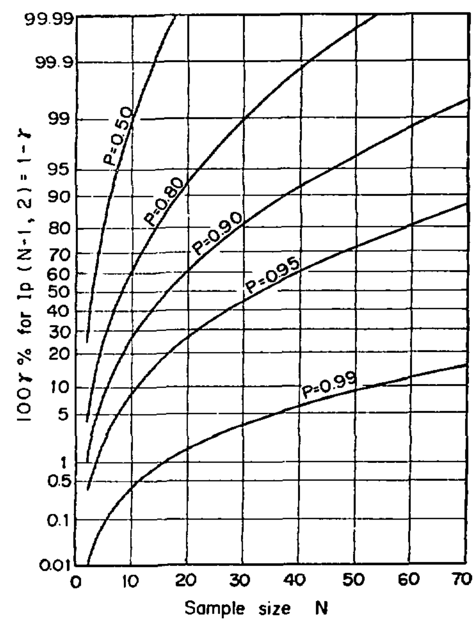

Fig. 5(a) Calculated population coverage of the sample having a sample size less than 70.

3.3 Confidence Limit of Probability of nth Order Statistics

3.3.1 Mathematical Formula

As the third application of nonparametric statistics to engineering problems, we discuss a one-sided upper confidence limit p_n, which is a cumulative probability $F(x_n)$ corresponding to the nth order statistic of the sample.

The problem can be described by the equation below.

$$P_r\{F(x_n) < p_n\} = \gamma \qquad (25)$$

This is related to a failure (fracture) probability of strength x and its confidence limit. Those are bases to define a specified minimum ultimate strength for design on the basis of strength tests. Applying THEOREM I to Eq.(25) leads to the equation;

$$P_r\{F(x_n) < p_n\} = I_{p_n}\ (n,\ N-n+1) = \gamma \qquad (26)$$

3.2.2 Calculated Results

Representative solutions of Eq.(26) are listed in Table 2 for the minimum x_1 in the samples of N=5~25.

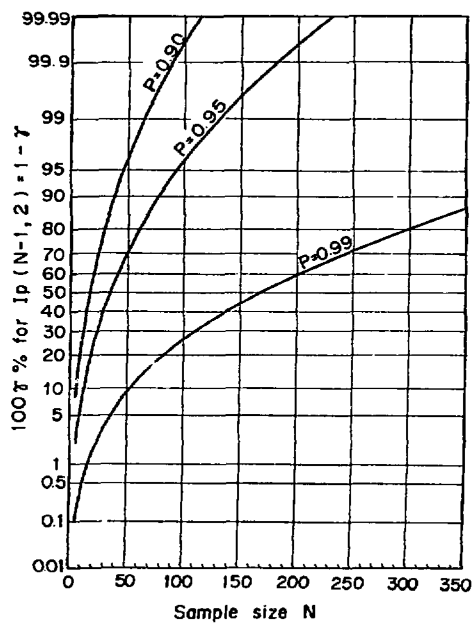

Fig. 5(b) Calculated population coverage of the sample having a sample size less than 350.

As an example of the statistical assessment, a set of tensile strength data on a grade of nuclear graphite is analyzed on the basis of normal, lognormal, 2-parameter Weibull distributions as well as nonparametric statistics. Fig. 6 shows plots of observed values and theoretical distributions fitted to the data. Fig. 7 depicts the upper confidence limit of failure probabilities predicted at a 90% probability. It is noted that the confidence limits based on nonparametric statistics are mostly higher than those predicted on parametric statistics. This means nonparametric statistical estimation provides a safer or conservative judgement.

Table 2 One-sided upper confidence limit of probability of the minimum $F(x_1)$.

Sample size (N)	Confidence level (100γ %)				
	50	90	95	99	99.9
5	.1295	.3690	.4507	.6019	.7488
10	.0670	.2057	.2589	.3690	.4988
20	.0341	.1088	.1391	.2058	.2921
30	.0228	.0739	.0950	.1423	.2057
40	.0172	.0559	.0722	.1088	.1586
50	.0138	.0450	.0582	.0880	.1290

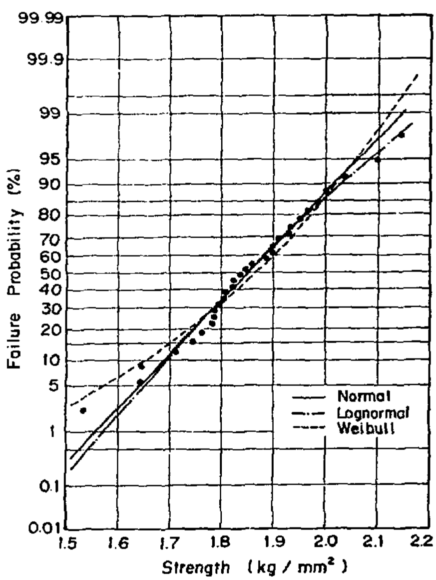

Fig. 6 Plots of tensile strength data of a nuclear graphite and best-fitted theoretical distributions.

4. CONCLUDING REMARKS

Conventional statistical assessments involve a theoretical distribution function under the assumptions that a set of data can be approximated by it, which is justified by satisfying a test of the goodness of fit. The procedure is sometimes not appropriate because a sample size is too limited to predict a confidence interval or limits with a high level of confidence.

Alternatively, nonparametric statistics are desirable for them to be predicted on the basis of the order statistics of the sample. Mathematical bases and calculated results have been given in the study on some engineering assessments: confidence limits of medial, population coverage of a sample, required number of sample, and confidence limit of probability of mth order statistics. The procedure in nonparametric statistics is mathematically simple. Its application may be favorable in the statistical assessments with a higher level of conservatism.

Finally, it should be mentioned that nonparametric statistics can handle a set of observed values as well as a set of calculated values. The latter is often encountered in mechanical engineering (e.g. structural reliability prediction [5]) and reactor safety engineering (e.g. probabilistic safety assessment PSA [6]). They are particularly effective when we are determined to make a decision in the case where a phenomenon involves a large uncertainty and the decision needs to be free from a risk.

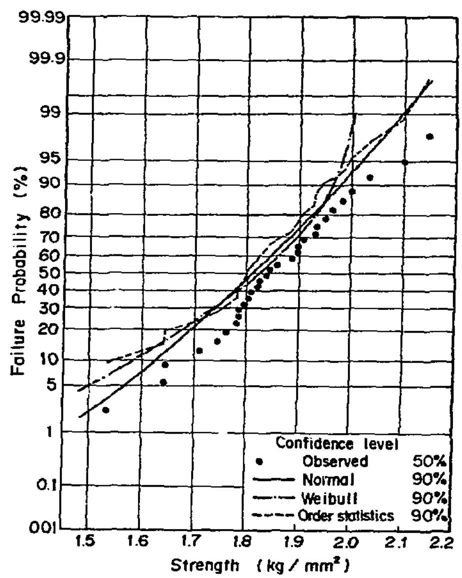

Fig. 7 Upper confidence limits of failure probabilities predicted nonparametric and parametric statistics.

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APPENDIX DERIVATION OF EXPERIMENTAL PROBABILITY OF nth ORDER STATISTIC

In statistical data analyses an experimental probability of nth order statistics, $F(x_n)$, is normally taken as

$$F(x_n) = \frac{n}{N+1} \tag{A1}$$

or

$$F(x_n) = \frac{n-0.3}{N+0.4} \tag{A2}$$

These formulas can be derived in the following.

According to THEOREM I, $F(x_n)$ has the beta distribution. The rth moment of $F(x_n)$, μ_r, is written as follows.

$$\begin{aligned}\mu_r &= \frac{\Gamma(N+1)}{\Gamma(n)\Gamma(N-n+1)}\int_0^1 y^{n+r-1}(1-y)^{N-n}\,dy \\ &= \frac{\Gamma(N+1)}{\Gamma(n)\Gamma(N-n+1)}\cdot\frac{\Gamma(n+r)\Gamma(N-n+1)}{\Gamma(N+r+1)} \\ &= \frac{\Gamma(n+r)\Gamma(N+1)}{\Gamma(n)\Gamma(N+r+1)} = \frac{(n+r-1)!\,N!}{(n-1)!\,(N+r)!}\end{aligned} \tag{A3}$$

Hence, the mean and variance of $F(x_n)$, $E\{F(x_n)\}$ ($r=1$ in Eq.(A1)) and $V\{F(x_n)\}$ ($r=2$), are given below.

$$E\{F(x_n)\} = \frac{n!\,N!}{(n-1)!\,(N+r)!} = \frac{n}{N+1} \tag{A4}$$

$$\begin{aligned}V\{F(x_n)\} &= E\{(F(x_n))^2\} - [E\{F(x_n)\}]^2 \\ &= \frac{n(N-n+1)}{(N+1)^2(N+2)}\end{aligned} \tag{A5}$$

From Eq.(A4) it follows that the experimental probability of the nth order statistic, $n/(N+1)$, corresponds strictly to the mean of the population distribution. That is, Eq.(1) is according to mean ranking.

Next, a one-sided upper confidence limit of $F(x_n)$ has given by Eq.(26), which is

$$P_r\{F(x_n) < p_n\} = I_{p_n}(n, N-n+1) = \gamma \tag{A6}$$

Using Eq.(3) we have

$$\int_0^{p_n} y^{n-1}(1-y)^{N-n}\,dy = \frac{\Gamma(n)\Gamma(N-n+1)}{\Gamma(N+1)}\cdot\gamma \tag{A7}$$

An assumption of $\gamma=0.5$ in Eq.(A7) gives an experimental probability by median ranking. The solution p_n of Eq.(A7) for $\gamma=0.5$ can be obtained by use of tables of the beta function or binomial probabilities. It may be approximated by the following

$$\tilde{p}_n = \frac{n-0.3}{N+0.4} \tag{A8}$$