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On Exact Account of Heavy Quark Thresholds in Hard Processes

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Abstract

We study the problem to accurately account for the heavy quark threshold effects in hard processes. We employ the direct perturbative Feynman diagram analysis and the Stueckelberg-Bogoliubov massive renormalization group formalism to show that both methods result in the same prescription for the "physical" mass-dependent QCD effective coupling as an argument of anomalous dimensions that determine the evolution of structure functions (parton distributions).

By considering the one-loop example, we demonstrate that an intrinsic ambiguity of the standard approach based on the notion of "effective number of quark flavours" may affect the α_s determination at the level of two-loop effects.

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1 Introduction

1.1 Prelude

"Perturbative QCD" in practice is a product of combined use of genuine perturbative calculations and of the renorm-group (RG) technique.

RG resummation algorithm employs the beta-function $\beta(g)$, anomalous dimensions $\gamma(g)$ and the effective coupling $\bar{g}(Q)$ calculated within some massless renormalization scheme (RS), *e.g.*, the popular $\overline{\mathrm{MS}}$ scheme. This practice seems rather natural for short-distance dominated phenomena. However in QCD it meets some specific trouble related to the fact that masses of t, b and c quarks are *parametrically* large as compared to the characteristic confinement scale of the order of few hundred MeV. The existing tool-kit of the Q-dependent effective number of quark flavours that has been invented to deal with the problem of heavy quark "threshold" effects is not free of ambiguities.

This paper is aimed at further developing a regular approach to the problem of an exact account of finite quark masses, in particular in hard processes. We argue that one has to employ the well known expression for the massive fermion loop directly in the "denominator" of the running coupling. When doing so, one avoids an ambiguity in fixing an integration constant that corresponds to effective change of the renormalization scheme in the course of passing the quark "threshold".

We demonstrate in Sec.2 how physical coupling that bears information about "Euclidean-reflected" heavy thresholds emerges in the evolution equations for parton distributions in hard processes. To this end we consider the non-singlet light-quark structure functions of deep inelastic lepton-hadron scattering (DIS) restricting ourselves for the sake of simplicity to the one-loop level. At the same time we point out that the parton evolution scheme that makes use of formal dispersion relations for inclusive parton interaction cross sections naturally embodies essential second loop effects including the subleading effect due to finite fermion masses.

In Sec.3 we show how the same solution can be found by means of the explicit one-loop solution of the massive RG equations.

We conclude in Sec.4 by making comparison with the standard approach that uses the notion of effective quark flavour number $n_f(Q)$.

1.2 Renormalized Coupling Constant

The problem of the appropriate definition of renormalized coupling constant g_{ren} is an essential part of the renormalization program. After cancellation of the ultra-violet (UV) divergences by counterterms (or the singular Dyson transformation), a finite degree of freedom remains that corresponds to the possibility of different definitions of the renormalized quantities. Usually this degree of freedom is treated in terms of a Renormalization Scheme (RS) and a reference-momentum scale μ ("scheme-scale" choice). Pragmatically it is preferable to define and use g_{ren} that allows a direct confrontation with experimental data.

It is noteworthy that during the first period of the development of QFT, which was dominated by the spectacular applications of QED, this issue remained "hidden" due to existence of the classical electrodynamical limit. This macroscopic heritage has provided QED with $\alpha(Q=0) \approx 1/137$ as a natural definition for the on-mass-shell g_{ren} .

The first detailed discussion of the problem was given by Källen [1] in 1954 who has compared the situation in QED and pion-nucleon Yukawa theory. The accent is on the ambiguity in defining the renormalized $g_{\pi NN}$ coupling. The most natural object for this definition, the renormalized vertex pion-nucleon on-mass-shell amplitude, is not available in physical processes for kinematical reasons.

Just in the mid-fifties due to the discovery of the pion-nucleon scattering dispersion relations and the first RG applications to the UV asymptotics in QED, it has been widely realized[2] that the specification of the RS with respect to g_{ren} (that is of the subtraction procedure for carrying out perturbative calculations) is an essential step in connecting theoretical results to experimental data. In fact one uses the notion of a renormalized coupling constant $g_{ren} = g$ in a more general context, that is to relate theoretical predictions for different physical observables.

Invariance of the observables, *e.g.*, of the matrix elements (and covariance of some other QFT objects) with respect to this degree of freedom leads to existence of the renormalization group (RG) and to a wide use of the notion of *effective coupling* $\bar{g}(Q)$. The latter is the functional generalization of g_{ren} and, in its turn, contains arbitrariness connected with a choice of RS. Generally speaking, $\bar{g}(Q)$ has no direct physical meaning.

An adequate definition (choice) of g_{ren} may improve approximation properties of perturbative series. The same is true for the RG-improved series expansions with $\bar{g}(Q)$ substituted for g that incorporate UV logarithms.

In QCD one meets some additional essential problems, an account of heavy quark masses being one of them. The point is that first three quarks u, d, s have rather small masses and together with the gluons they can be treated as "light", practically massless, particles. Their contributions are usually pure logarithms in the momentum transfer variable Q and can be analysed with a help of the massless RG formalism. This formalism based upon an infinite renormalization philosophy (see, *e.g.*, [3]) is widely used in the current literature. Historically it ascends to the Gell-Mann–Low paper [4]. We describe it briefly in the Appendix A.

However, the "GeV" domain contains heavy c and b quarks and their creation thresholds. To account for heavy masses within the massless RG formalism some artificial devices are used, such as an effective momentum-dependent number of quark flavours $n_f(Q)$ (see *e.g.* [5]).

The problem can be resolved quite simply in the framework of the original Stueckelberg-Petermann-Bogoliubov (St-P-B) RG formulation which does not contain any limitation connected with the massless approximation[3]. The corresponding RG formalism contains particle masses explicitly (see Appendix B and the chapter "Renormalization Group" of Ref. [2]).

To explain the difference between the massive St-P-B formulation and the more popular one let us remind the reader some of its basic features.

[2]See, *e.g.*, discussion on pp. 525 and 634 of the first edition of [2]

[3]unlike the one used for short-distance analysis by Gell-Mann and Low [4].

1.3 Finite Renormalization Approach to RG

Quite opposite to the standard GM-L formulation, the St-P-B approach from the very beginning has been based [6] upon the analysis of the finite arbitrariness in the renormalized, that is *finite results* of QFT calculations. Originally this RG algorithm was created [7] on the basis of finite Dyson transformations which evidently obey the group property.

In few words, instead of considering the singular Dyson relation for some renormalization scheme R_i

$$g_0 \rightarrow g_i = Z_{io} g_o$$

with infinite Z's one deals with non-singular renormalization transformations

$$g_i \rightarrow g_j = z_{ij} g_i \quad \left(z_{ij} = Z_{jo} Z_{io}^{-1} \right) \tag{1.1}$$

with finite z_{ij} factors. They describe transition between different RS's: $R_i \rightarrow R_j$ and obey the transitivity property

$$g_i \rightarrow g_k \rightarrow g_j \quad ; \quad z_{ji} = z_{jk} z_{ki} \, , \tag{1.2}$$

that is just the group composition law.

Here the central notion is that of the *invariant* (or *effective*) coupling $\bar{g}$ defined as a special product of the coupling constant g, the corresponding vertex Γ and of the scalar propagator amplitudes Δ_i for the lines joining the vertex. The product is composed in such a way that $\bar{g}$ does not acquire any z-factor under the Dyson transformation (1.1).

This group transformation corresponds now to a rescaling of the dimensional arguments of the coupling, namely, the Lorentz invariant combinations of particle momenta and their masses. As long as the squared momenta of participating fields are kept proportional to one another, the invariant coupling turns out to be a function of only one momentum squared variable, say Q^2, of the particle masses M, of a scale parameter μ and of the coupling constant g. Thus it can be written as a function of three dimensionless arguments:

$$\bar{g}\left(\frac{Q^2}{\mu^2}, \frac{M^2}{\mu^2}; g\right) \equiv \bar{g}(x, y, g) \, .$$

In the momentum-subtraction (MOM) scheme with Γ and Δ subtracted at the same point $Q^2 = \mu^2$ one can refer to $\bar{g}$ as the effective coupling. This mass-dependent function $\bar{g}(x, y; g)$ obeys the "normalization condition" $\bar{g}(x = 1) = g$ and satisfies the group functional equation

$$\bar{g}(x, y; g) = \bar{g}\left(\frac{x}{t} \, , \, \frac{y}{t} \, ; \; \bar{g}(t, y; \, g)\right) , \tag{1.3}$$

which is the evident generalization to the massive case of the corresponding massless Eq. (A.7). Quite analogously one can consider the mass-dependent functions $s(x, y; \, g)$, *e.g.*, scalar propagator amplitudes or structure function moments. Such objects we shall call hereafter "one-argument" functions bearing in mind a single argument of the Q^2 type. For $\bar{g}$ and s one can use partial differential equations analogous to (A.4) that include differentiation with respect to the "mass" argument y, The important feature of the St-P-B approach is that it deals with finite propagators, vertices and matrix elements calculated in some definite RS and depending on renormalized, *i.e.*, physical coupling(s) and mass(es).

1.4 Renormalization Group Method

The Renorm-Group Method (RGM), as formulated in [7], is devoted to regular improvement of approximation properties of renormalized QFT perturbation expansions. The prescriptive part of the RGM generally states that to achieve this goal one has to solve differential group equations with the group functions (generators) β, γ defined on the base of approximate **finite** perturbative solution under consideration[4] calculated within some definite renormalization scheme (RS).

We remind the reader that the very first results on the UV asymptotics of QED at one- and two-loop level and the IR behaviour of the electron propagator obtained in the mid-fifties had demonstrated that the RGM procedure restores the correct structure of singularities destroyed by the weak coupling approximation.

It should be noted here that in a wide modern use by RG procedure (RGP) a different algorithm is implied:

- First one calculates massless beta-function coefficients β_l (numbers) on the base of the counter-term structure, *i.e.*, of the **singular** part of Feynman diagram contributions[5].

- Then one solves the massless RG differential equation and expresses the result for, *e.g.*, matrix element in terms of running coupling $\bar{g}(Q)$, or rather its boundary value g_μ and μ.

This approach relates the final result (like improved expression for matrix element) with the finite part of UV asymptotics of the corresponding perturbative input only indirectly. Constant terms originally adjacent to UV logarithms are not reproduced. Moreover, as there is no direct relation between g_μ and some measurable quantity, the problem of relating g_μ and μ parameters to data appears. In the following we shall distinguish between RGMethod which is free of this trouble and popular RGProcedure.

Besides, generally speaking, the RG approach (either RGM or RGP) can be directly applied only to real-valued functions, related to finite renormalization factors z_i that are expressed in terms of vertices and propagator amplitudes taken at appropriate space-like arguments. An application of the RGM to complex-valued objects (propagators, vertices and matrix elements with some $q_i^2 > 0$) is not a straightforward procedure. It needs (see, *e.g.*, [11]) some additional theoretical means (like Källen – Lehmann or Mandelstam spectral representations) for analytic continuation from Euclidean to Minkowskian q_i^2 values.

The RG technique faces some additional troubles in treating objects depending on several kinematic variables q_i. Under the RG transformation all scalar invariants $(q_i q_j)$ have to be scaled simultaneously. This makes it formally impossible to apply the RGM in a straightforward way to analysis of many physically important cases, where some of q_i^2 are kept fixed (*e.g.*, on-mass-shell).

[4]See Appendix B2 or Chapter "Renormalization Group" in [2] for further details.

[5]Technically this is much simpler than to define β_l via UV asymptotics of Feynman diagrams. This enabled to find, *e.g.*, three-, four- and even five-loop contributions to beta-function for several important cases [8-10].

1.5 Heavy Quarks in Hard Processes

As long as hard processes are concerned, heavy quarks (HQ) reveal themselves is two ways. First, explicit HQ production cross sections, multiplicities, momentum distributions *etc.* may be under focus. The problem of heavy quark production in DIS processes and, in particular, in the important and theoretically interesting small-x region has been addressed in great details in Refs. [12–14]. There are other examples of this kind — recent studies of certain peculiarities in heavy quark production at large x in DIS and hadron scattering processes [15] and inside heavy quark initiated jets [16].

Secondly, heavy quarks implicitly affect the light parton evolution via the running coupling, provided sufficiently high momentum transfers $Q^2 \gtrsim M^2$ are involved.

In spite of several attacks undertaken during the past two decades, the last issue, we believe, deserves nowadays a special attention. Given the high precision measurements of the effective coupling at Z^0, future experimental studies of the top physics that proves to be particularly sensitive [17] to the QCD interaction strength at characteristic scales $Q^2 \sim (10\mathrm{GeV})^2$ will inevitably give rise to the problem of an accurate theoretical control of the running coupling evolution over the broad momentum range.

The standard analysis aimed at resummation of "collinear" short-distance logs in hard interactions is based upon the UV, "singular-Z" RG technique with the use of some massless RS, *e.g.*, the popular $\overline{\mathrm{MS}}$ scheme. This yields a specific problem in analysing hard cross sections in the kinematical region of moderate momentum transfer $Q^2 \sim M^2$. Formally speaking, an account of heavy quark threshold effects here would require summation of higher twist $(M^2/Q^2)^n$ terms in the operator product expansion[6].

2 Running Coupling in Parton Evolution

In this section we reconsider the perturbative approach to the calculation of the cross section of deep inelastic lepton-hadron scattering (DIS) to explicitly verify what kind of "running coupling" enters the perturbative QCD expressions for the cross sections of hard processes.

This general question has been directly addressed in the literature (for a review see [21] and references therein). The aim of our recollection is to check how the finite (large) quark mass affects the running coupling that enters the standard expressions for DIS structure functions as an argument of the anomalous dimension or, equivalently, as a measure of intensity of parton splitting if the language of the probabilistic parton evolution picture is applied [22–26].

Here we restrict ourselves to the *non-singlet* structure function (valence quark distribution) at the first-loop level. As long as the quark and the gluon contributions to α_s^{-1} are additive, one can make a further simplification by considering the Abelian model in which no complications due to the self-interaction of bosons emerge in the vector boson ("photon") propagator. In spite of such a significant simplification we see no difficulty in translating the result into the statement applicable to the really interesting QCD case.

[6]From this point of view the momentum subtraction schemes are better suited for keeping track of finite mass effects, see, *e.g.*, [18–20].

From the first sight the structure of the Feynman amplitudes that contribute to the DIS cross section looks very far from being suited for introducing the renormalized effective coupling in a "classical" way. Indeed, propagators of the fermion that experiences multiple bremsstrahlung before taking an impact correspond to space-like momenta with essentially different virtualities, while the radiated bosons are on mass-shell: the situation that is too remote from the Euclidean definition. Solution of the puzzle was given by V. Gribov and L. Lipatov [22] who have shown how the "Euclidean" kinematics gets effectively restored due to integration over (positive) virtual mass of unregistered final bosons. Within the QCD context similar conclusions have been arrived at in a list of publications, see, *e.g.*, [26–29].

In what follows in this section we carefully repeat the argumentation of Ref. [22] to explicitly verify that the coupling that emerges along this way is bound to bear in full extent the "mirror reflected" information about the heavy quark threshold.

2.1 Matrix Element

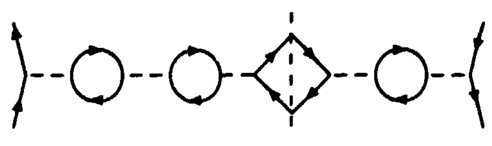
The cross section (parton splitting probability) in terms of Sudakov variables (see Appendix C) takes the form

$$g_0^2 \int \frac{dz d\alpha d^2\vec{k}_\perp}{(2\pi)^4} \frac{s}{2}\, dm^2\, (2\pi)\delta(m^2-(p-k)^2) \cdot A_\nu A_\mu^* = \frac{g_0^2}{16\pi^2} \int \frac{dz}{(1-z)} \frac{d^2\vec{k}_\perp}{\pi}\, dm^2 \cdot A_\nu A_\mu^* \quad (2.1)$$

with the matrix element squared, averaged over initial fermion polarizations,

$$A_\nu A_\mu^* = \tfrac{1}{2} \sum_{\sigma=1,2} \bar{u}^\sigma(p) \left[\gamma_\mu \hat{k} \ldots \hat{k} \gamma_\nu \right] u^\sigma(p) \cdot \frac{\rho(m^2)}{[\,k^2\,]^2}\,, \quad (2.2)$$

where we have neglected the incoming fermion mass. The spectral density of a boson with the time-like virtual mass m^2 in (2.2) is

$$\rho(m^2) = \mathcal{Z}(0)\delta(m^2) + \frac{1}{\pi} \sum_{loops} \sum_{cuts}$$  (2.3)

Sum over loops results in the analytic expression (C.10) for $\rho(m^2)$

$$\rho(m^2) = \mathcal{Z}(0)\delta(m^2) - \frac{1}{\pi} \frac{\mathrm{Im}\, \mathcal{Z}(m^2)}{m^2}\,. \quad (2.4)$$

Analysis of the matrix element (2.2) is performed in Appendix C3. For the parton interaction cross section averaged over initial olarisaations,

$$\mathcal{M}(p,q;\mu^2) \;=\; \tfrac{1}{2} \sum_{\sigma=1,2} \bar{u}^\sigma(p)\; |A|^2\; u^\sigma(p)\,, \quad (2.5)$$

the following evolution equation emerges,

$$\begin{aligned}\mathcal{M}(p,q;\mu^2) &= \int_x^1 \frac{dz}{z} \int_{\mu^2}^{Q^2} dk_\perp^2\, \mathcal{M}(zp,q;k_\perp^2)\, \frac{g_0^2}{8\pi^2} \\ &\cdot [C_F] \left\{ P(z) \left[\frac{\mathcal{Z}(0)}{k_\perp^2} - \frac{1}{\pi} \int \frac{dm^2}{m^2(k_\perp^2 + zm^2)} \cdot \operatorname{Im} \mathcal{Z}(m^2) \right] \right. \\ &\left. + \phi(z) \frac{1}{\pi} \int \frac{z \cdot dm^2}{(k_\perp^2 + zm^2)^2} \cdot \operatorname{Im} \mathcal{Z}(m^2) \right\}.\end{aligned} \tag{2.6}$$

The function P in (2.6) is the usual GLAP splitting probability of the *fermion* $\rightarrow$ *fermion* + *vector boson* transition [22–26] depending on the longitudinal parton momentum fraction z,

$$P(z) = \frac{1+z^2}{1-z}, \tag{2.7a}$$

while the z-dependent factor in the last term of Eq. (2.6) is

$$\phi(z) = (1-z). \tag{2.7b}$$

2.2 Integration over Virtual Boson Mass

Now we show how the integral over the boson mass in (2.6) makes the effective coupling in the parton decay cell run with $k_\perp^2$ [22].

Considering the integrals

$$\frac{1}{k_\perp^2} I_1(k_\perp^2) = \frac{g_0^2 \mathcal{Z}(0)}{k_\perp^2} - \frac{g_0^2}{\pi} \int \frac{dm^2}{m^2(k_\perp^2 + zm^2)} \cdot \operatorname{Im} \mathcal{Z}(m^2), \tag{2.8a}$$

$$\frac{1}{k_\perp^2} I_2(k_\perp^2) = \frac{g_0^2}{\pi} \int \frac{z \cdot dm^2}{(k_\perp^2 + zm^2)^2} \cdot \operatorname{Im} \mathcal{Z}(m^2), \tag{2.8b}$$

we first observe that I_α depend on the combination $\kappa^2 \equiv k_\perp^2/z$. Secondly, we note that although the mass integration is restricted in the DIS cross section by the inequality

$$m^2 < s = Q^2/x \sim Q^2,$$

one can extend the integration region up to ∞. Such an extension does not affect the anomalous dimension analysis since in the quasi-collinear kinematics, $k_\perp^2 \sim \kappa^2 \ll Q^2$, our integrals converge at $m^2 \sim \kappa^2$ long before the kinematical boundary is reached.

After such a modification we invoke the dispersion relation for the boson propagator,

$$\mathcal{Z}(p^2) = \frac{1}{\pi} \int_0^\infty dm^2\, \frac{\operatorname{Im} \mathcal{Z}(m^2)}{m^2 - p^2 - i\epsilon} = \mathcal{Z}(0) + \frac{p^2}{\pi} \int_0^\infty \frac{dm^2}{m^2} \frac{\operatorname{Im} \mathcal{Z}(m^2)}{m^2 - p^2 - i\epsilon}, \tag{2.9}$$

to be considered in the space-like momentum region, $p^2 = -\kappa^2 < 0$,

$$\mathcal{Z}(-\kappa^2) = \frac{1}{\pi} \int_0^\infty dm^2\, \frac{\operatorname{Im} \mathcal{Z}(m^2)}{m^2 + \kappa^2} = \mathcal{Z}(0) - \frac{\kappa^2}{\pi} \int_0^\infty \frac{dm^2}{m^2} \frac{\operatorname{Im} \mathcal{Z}(m^2)}{m^2 + \kappa^2}. \tag{2.10}$$

Comparing (2.8) with (2.10) one concludes that the mass integrals result in appearance of the running coupling $g^2(\kappa^2) \equiv g_0^2 Z(-\kappa^2)$ as a measure of parton interaction strength:

$$I_1(\kappa^2) = g_0^2 Z(-\kappa^2) \equiv g^2(\kappa^2), \tag{2.11a}$$

$$-I_2(\kappa^2) = g_0^2 \frac{d\,Z(-\kappa^2)}{d\ln\kappa^2} = \frac{d}{d\ln\kappa^2} g^2(\kappa^2). \tag{2.11b}$$

Finally,

$$\mathcal{M}(p,q;\mu^2) = \int_x^1 \frac{dz}{z} \int_{\mu^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \mathcal{M}(zp,q;k_\perp^2) \cdot \left\{ P(z)\frac{\alpha(\kappa^2)}{2\pi} - \phi(z)\frac{d}{d\ln\kappa^2}\frac{\alpha(\kappa^2)}{2\pi} \right\}, \tag{2.12a}$$

$$\kappa^2 \equiv k_\perp^2/z. \tag{2.12b}$$

2.3 Discussion

Some comments are in order.

1. In the derivation of (2.12) no approximations were made with respect to the structure of the boson spectral density $\mathrm{Im}\, Z$. In particular, the contributions from "heavy" fermions with masses $M_f^2 \ll Q^2$ affect significantly the behaviour of the "physical" coupling (2.11a) when the space-like evolution parameter κ^2 in (2.12) passes the $\kappa^2 \sim M_f^2$ region. This is a reflection image of a time-like threshold. As we shall discuss below, such a choice of coupling for parton evolution provides natural quantitative explanation for the notion of "the number of *active* quark flavours".

2. We are basically interested in the one-loop anomalous dimension. Therefore it would be perfectly legitimate to neglect the second term in curly brackets of (2.12) because it contributes to the *second* loop, $\mathcal{O}(\alpha^2)$. The precise argument of the coupling (2.12b) is also of the same order compared to, *e.g.*, the $\alpha(k_\perp^2)$ prescription. Making use of the expansion

$$\alpha' \approx \alpha \cdot \frac{\alpha}{2\pi} \cdot \beta_1 ; \qquad \beta_1 = [-\beta_g] + \beta_f = \left[-\frac{11}{6} C_A\right] + \frac{2}{3} n_f \, [T_R] \,,$$

where the terms in square brackets apply to the non-Abelian case ($C_A = 3, T_R = \frac{1}{2}$), one may write

$$\frac{\alpha(k_\perp^2/z)}{2\pi} \approx \frac{\alpha(k_\perp^2)}{2\pi} + \ln\frac{1}{z} \cdot \frac{d}{d\ln k_\perp^2}\frac{\alpha(k_\perp^2)}{2\pi} = \frac{\alpha(k_\perp^2)}{2\pi} - \ln z \cdot \beta_1 \left(\frac{\alpha(k_\perp^2)}{2\pi}\right)^2 + \mathcal{O}\left(\alpha^3\right). \tag{2.13}$$

Since the GLAP splitting function (2.7a) is regular at $z = 0$, the $\mathcal{O}(\alpha^2)$ correction is uniformly small.

3. At the same time (2.12b) differs significantly from the case when the parton *virtuality* $|k^2|$ is chosen as an argument of α. Such prescription naturally emerges within the formal approach based upon the Wilson operator product expansion in which the RG technique is applied to x-moments of parton distributions [30]. The difference between these two prescriptions (see (C.12)),

$$\alpha(-k^2) - \alpha(k_\perp^2) = \alpha\left(k_\perp^2/(1-z)\right) - \alpha(k_\perp^2) \approx -\ln(1-z)\cdot\alpha'(k_\perp^2),$$

gets enhanced when the region $z \to 1$ becomes important. This happens when DIS structure functions in the quasi-elastic kinematics $(1-x) \ll 1$ are concerned, which corresponds to large moments $N \gg 1$. As a result in higher orders perturbative expansion for the anomalous dimension γ_N blows up. Symbolically,

$$\gamma_N(\alpha) = \sum_{k=1}^{\infty} C_k(N)\,\alpha^k, \qquad C_k(N) \sim (\ln N)^k \gg 1.$$

Turning back to the "physical" coupling $\alpha(k_\perp^2)$ restores reasonable approximating property of the perturbative expansion [21,28,31].

4. Making use of the expansion (2.13) one may express the evolution kernel of (2.12) in terms of the next-to-leading order expansion for the anomalous dimension,

$$\begin{aligned}\gamma(x,\alpha) &= \frac{\alpha}{2\pi}\,[C_F]\left\{ P(x) + \frac{\alpha}{2\pi}\,\gamma_2(x) + \ldots\right\}; \\ \gamma_2^{\rm DS}(x) &= -\beta_1\cdot\{P(x)\ln x \;+\; (1-x)\},\end{aligned} \tag{2.14a}$$

where the superscript DS refers to the *Dispersion Scheme* for the running coupling.

It is instructive to compare this result to the well known second-loop non-singlet anomalous dimension [32,33]. As long as we are dealing with the quasi-Abelian corrections due to fermion loops, the AD correction (2.14a) has to be compared to the corresponding term in $\gamma_2^{\overline{\rm MS}}$ proportional to the number of fermion flavours n_f:

$$\gamma_{2(f)}^{\overline{\rm MS}}(x) = \; -\beta_f\cdot\left\{P(x)\left(\ln x + \frac{5}{3}\right) + (1-x)\right\} = \frac{\beta_f}{\beta_1}\cdot\gamma_2^{\rm DS} - \beta_f\cdot P(x)\,\frac{5}{3}. \tag{2.14b}$$

This is the $\overline{\rm MS}$ expression of Ref. [33] derived in the framework of the Jet Calculus motivated scheme. The original result of Refs. [32] slightly differs from (2.14b). It contains a factor 2 in front of the $(1-x)$ term. This extra contribution cancels with corresponding correction to the coefficient function C in the scheme-invariant combination

$$\mathcal{P}(x,\alpha) \equiv \frac{dC}{d\ln Q^2} + \gamma \;=\; \frac{\alpha}{2\pi}\,[C_F]\left(P \;+\; \frac{\alpha}{2\pi}\;[\beta_1\cdot C_2(x) + \gamma_2(x)] \;+\ldots\right)$$

Eqs. (2.14) make it possible to relate the coupling constants of the two schemes as

$$\alpha_{\rm DS} = \alpha_{\overline{\rm MS}}\left(1 - \frac{\alpha}{2\pi}\cdot\frac{5}{3}\beta_f \;+\ldots\right). \tag{2.15}$$

The origin of the mismatch is rather simple. The point is that under the name "$\overline{\rm MS}$ scheme" two different algorithms may be implemented.

(a) The first (standard one) treats fermions as purely massless particles and deals only with logarithms $\ell = \ln(Q/\mu)$ which corresponds to UV calculation in the popular RGProcedure (see Subsection 1.4). It is implied here that μ is just the normalization point:

$$\alpha(Q^2) = \alpha_{\overline{\text{MS}}} + \frac{\alpha^2}{\pi}\beta_f \ell + \ldots , \qquad \alpha(\mu^2) = \alpha_{\overline{\text{MS}}} .$$

(b) The other algorithm keeps track of the *finite* contribution to the one-loop fermion polarization diagram defined (UV-regularized) in the $\overline{\text{MS}}$ scheme[7], *i.e.* it deals with the UV limit of massive perturbative $\overline{\text{MS}}$ calculation. In this case the effective coupling reads

$$\alpha(Q^2) = \alpha_{\overline{\text{MS}}} + \frac{\alpha^2}{\pi}\beta_f \left(\ell - \frac{5}{6}\right) + \ldots$$

Comparing with (2.15) we conlude that the first realization of the "$\overline{\text{MS}}$ ideology" misses some essential subleading effect in the vacuum polarization due to the occurence of fermions.

5. It is worthwhile to notice that the very fact that the Dispersion Scheme (2.12) would "suggest" slightly different couplings for different functional z-dependence seems to be meaningful. Indeed, rewriting the elementary parton splitting probability as

$$\propto 2[C_F] \left\{ \frac{2z}{(1-z)} \cdot \alpha(\frac{k_\perp^2}{z}) + (1-z) \cdot \left[\alpha(\frac{k_\perp^2}{z}) - \beta(\frac{k_\perp^2}{z}) \right] \right\} , \tag{2.16}$$

one can notice that two terms here correspond to physically different contributions to the radiation cross section. The first one originates from the universal "soft" bremsstrahlung which is independent of the nature of the incoming parton and corresponds to the boson polarization in the scattering plane, (p, q). The second contribution is due to "hard" emission. It depends on initial parton spin and consists of an equal mixture of longitudinal and transverse physical polarizations.

3 Effective Coupling in a Massive Case

3.1 Definition of the Effective Coupling

In the massive RG formalism an invariant (or effective) coupling $\bar{\alpha}(Q)$ is defined as a specific product of the expansion parameter α and several finite Dyson renormalization factors $z_i(Q)$ that could be related to the scalar propagator and vertex amplitudes considered at some Euclidean momentum-squared value $k^2 = -Q^2 < 0$. This product is invariant under the finite Dyson renormalization transformation. For example, in QED $\bar{\alpha}(Q)$ is just the product of α and z_d — the transverse photon propagator renormalization factor:

$$\alpha(Q) = \alpha\, z_d(-Q^2) . \tag{3.1}$$

[7]See, *e.g.*, Eq.(3.21e) of Ref. [34].

On the other hand in the QCD case the appropriate $\bar{\alpha}_s(Q)$ could be defined as a product of some vertex Γ and corresponding propagator renormalization factors. If one chooses the quark-gluon interaction vertex, $\Gamma = \Gamma_{qqg}$, then

$$\bar{\alpha}_s(Q) = \alpha\, z_\Gamma^2(-Q^2)\, z_S^2(-Q^2)\, z_d(-Q^2)\,. \tag{3.2}$$

Here z_d is the transverse vector boson (gluon) propagator amplitude as defined in (C.7), $z_S \equiv s$ is the appropriate light quark propagator factor,

$$S_q(p) = \frac{s(-p^2)}{m(-p^2) - \hat{p}}\,,$$

and z_Γ can be expressed in terms of scalar constituent of the quark-gluon vertex,

$$\Gamma_{qqg} \sim (\Gamma_\mu)^a_{ij}\,(p,q,k) = \gamma_\mu\,(t^a)_{ij} \cdot \Gamma(p^2,q^2,k^2)\,+\,\ldots$$

under the special kinematical restriction

$$\mathcal{Z}_\Gamma(Q^2) = \Gamma(-\xi_1 Q^2, -\xi_2 Q^2, -\xi_3 Q^2)$$

with ξ_i fixed finite numbers.

It is essential that all factors z_i are expressed here in terms of finite mass dependent scalar amplitudes $s(-Q^2)$, $\Gamma(-Q^2)$ calculated under some definite renormalization prescription. For technical simplicity we shall assume in the following the momentum subtraction (MOM) scheme with the scale parameter μ playing a role of the subtraction point.

3.2 RG summation

As it is well known from the mid-fifties [35] the first perturbative contribution to the effective coupling

$$\alpha_{\rm PTh}(Q) = \alpha\,-\,\alpha^2 A_1(Q,\ldots)$$

"goes into denominator",

$$\bar{\alpha}^{(1)}_{\rm RG}(Q)\;=\;\frac{\alpha}{1+\alpha\, A_1(Q)}\,, \tag{3.3}$$

just in the form as it appears in perturbation theory from one-loop Feynman diagrams. Thus (3.3) is the exact solution of the massive RG equations.

About 10 years ago the analogous approximate RG-summed expressions have been obtained [36] for a more advanced case including $\bar{\alpha}(Q)$ at two- and three-loop level and *one-argument RG function* $s(Q)$ like a propagator amplitude or a structure function moment at the one- and two-loop levels. In particular,

$$s^{(1)}_{PT}\;=\;1+\bar{\alpha}\, s_1(Q)+\mathcal{O}\left(\bar{\alpha}^2\right)\;; \tag{3.4a}$$

$$s^{(1)}_{RG}\;=\;\left(\frac{\alpha}{\bar{\alpha}^{(1)}_{RG}(Q)}\right)^{\nu_1(Q)}\;;\quad \nu_1(Q) = \frac{s_1(Q,M,\mu)}{A_1(Q,M,\mu)}. \tag{3.4b}$$

It is quite remarkable that the final expressions (3.3), (3.4b) are build up of the perturbative bricks A_l, s_l *etc.* . and contain no memory about the intermediate RG entities such as the beta-function.

3.3 Heavy Quark Threshold

In our particular physical situation the one-loop contributions consist of two terms

$$A_1(Q) = \beta_1 \ell + \Delta\beta\, I_1(Q,M,\mu)\,; \qquad \ell \equiv \ln(Q/\mu)\,; \tag{3.5a}$$

$$s_1(Q) = \gamma_1 \ell + \Delta\gamma\, J_1(Q,M,\mu)\,, \tag{3.5b}$$

where the numerical coefficients β_1, γ_1 correspond to the radiative corrections due to light particles and I_1, J_1 stand for the heavy loop contributior

As it was mentioned above, the subtraction prescription has to be defined **before** the RG is used, *i.e.*, on the stage of perturbative calculations.

The most natural possibility is to subtract those functions at the zero value of Q,

$$I(Q) \equiv I_0(Q/M) \quad \text{with} \quad I_0(0) = 0, \quad \text{and} \quad I_0(Q/M) \sim \mathcal{O}\big(Q^2/M^2\big) \ \text{at} \ Q \ll M\,, \tag{3.6}$$

which choice provides the physically natural low-energy decoupling. Here far *below* the heavy quark threshold we have a simple MOM scheme

$$A_1 = \beta_1 \ln\frac{Q}{\mu} - \Delta\beta\, I_0 \;\rightarrow\; \beta_1 \ln\frac{Q}{\mu}\,, \tag{3.7}$$

with μ parameter treated as a subtraction point.

However the one-loop massive polarization operator thus defined behaves in the high energy limit $Q \gg M$ like

$$I_0\left(\frac{Q}{M}\right) \rightarrow \ln\frac{Q}{M} - \frac{5}{6} + \ldots = \ln\frac{Q}{M^*} + \mathcal{O}\left(\frac{M^2}{Q^2}\right)\,, \tag{3.8}$$

with the effective threshold mass

$$M^* = M\,\exp(5/6) \approx 2.302\,M\,. \tag{3.9}$$

This means that far *above* the threshold we have no longer a simple subtraction at $Q = \mu$:

$$A_1 \rightarrow \tilde{\beta}_1 \ln\frac{Q}{\mu} - C_1\,; \qquad \tilde{\beta}_1 \equiv \beta_1 + \Delta\beta\,, \quad C_1 = \Delta\beta\, \ln\frac{\mu}{M^*}\,. \tag{3.6}$$

Thus in the mass-dependent case the transition across the quark threshold (as it was recently emphasized in [37]) *changes effectively the subtraction scheme.* Note also that this RS change can be attributed to the modification of the scale which at the one-loop level has the form

$$\mu \rightarrow \mu^* = \mu \exp(C_1) = \mu \left[\frac{\mu}{M^*}\right]^{\Delta\beta/\tilde{\beta}_1}\,. \tag{3.7}$$

Let us mention another possible way to define I that consists of subtracting its value at some finite scale, $Q = \mu$, in the form [8]

$$I \equiv I_\mu = I_0(Q/M) \; - \; I_0(\mu/M) \; . \tag{3.8}$$

3.4 Effective Number of Quark Flavours

It is instructive to compare these results with the practice to account for the mass effects in the framework of the usual massless schemes. We have in mind here the notion of the effective flavour number $n_f(Q)$.

Within the standard approach it is not a genuine Feynman mass-dependent contribution like, *e.g.*, $A_1(Q, M, \mu)$ but rather its logarithmic derivative

$$\dot{A}_1 = \left.\frac{dA_1}{d\ell}\right|_{\ell=0} = \beta_1\left(\frac{Q}{M}\right) \tag{3.9}$$

that is dealt with as the main object of the analysis. The mass-dependent $\beta_k(z)$ coefficients (with the first loop coefficient β_1 defined by (3.9)) are then substituted for the usual β_k to construct the mass-dependent beta-function as

$$\beta(z,\alpha) = \beta_1(z)\,\alpha^2 + \beta_2(z)\,\alpha^3 + \ldots \,, \qquad z \equiv Q/M \; .$$

In the case under consideration β_1 has the structure

$$\beta_1(z) = \beta_1 + \Delta\beta \cdot n_1(z)\,; \quad \Delta\beta = -\frac{4T_R}{3} = -\frac{2}{3}\,. \tag{3.10a}$$

The z-dependent factor n_1

$$n_1(z) \equiv \dot{I}_0(z) \tag{3.10b}$$

is usually considered as a measure of intensity of the heavy flavour. To be more precise, it should be treated as the *heavy quark intensity for the beta-function.*

It can be approximated as [20]

$$n_1(z) \approx \frac{z^2}{z^2+5} \begin{cases} \to 0 & \text{at } z \to 0\,, \\ \to 1 & \text{at } z \to \infty\,. \end{cases} \tag{3.11}$$

Eq. (3.10) corresponds to the popular procedure of the mass account, discussed , e.g., by Edwards and Gottschalk [38] and advocated by Yndurain [6] . This expression is well suited for

[8] In that case we would have a simple structure, the MOM-scheme, as long as large momenta $Q, \mu \gg M$ are considered: $A_1 \to \tilde{\beta}_1 \ln Q/\mu$. At the same time to restore the decoupling property within this scheme below the heavy threshold $Q \ll M$ where

$$A_1 \to \beta_1 \ln\frac{Q}{\mu} + \Delta\beta\, I_0\left(\frac{\mu}{M}\right) ,$$

we would have to perform the change of RS which is reciprocal to the previous one.

keeping track of mass effects in the beta-function. At the same time, it should not be directly used in the explicit expression for the coupling.

Indeed, let us consider the UV asymptotic properties of the one-loop coupling (3.3). To this end we represent A_1 in the following form

$$A_1 = [\,\beta_1 + \Delta\beta \cdot N_1(Q,M,\mu)\,]\,\ell\,, \qquad (3.12a)$$

where

$$N_1(Q,M,\mu) = I_1/\ell = I_0\left(\frac{Q}{M}\right)\Big/\ln\frac{Q}{\mu} \qquad (3.12b)$$

can be referred to as the *heavy quark intensity for the effective coupling.*

The essential difference between N_1 and n_1 is that in the UV limit A_1 contains the constant term $\sim \ln\mu/M^*$. This μ-dependent term (changing effectively the RS, sic!) gets lost after differentiation, that is in course of the transition from N_1 to n_1.

As long as within the standard practice one would employ n_1 in the final expressions for the effective coupling, the abovementioned mismatch between n_1 and the "true" N_1 responsible for passing the heavy threshold in $\bar{\alpha}$ has to be taken care of to accurately account for finite mass effects.

4 Numerical results and Conclusions

Our main result is the explicit mass-dependent expression for the QCD effective coupling

$$\bar{\alpha}^{(1)}(Q) = \frac{\alpha_\mu}{1+\alpha_\mu\cdot[\,\beta_\ell\ln Q/\mu + A(Q,\mu,M,\ldots)]} \qquad (4.1)$$

with $\beta_\ell = (33-2n_\ell)/6\pi$ the light particle one-loop coefficient and

$$A = -\frac{1}{3\pi}\sum_h\left\{I_o(\frac{Q}{M_h}) - I_o(\frac{\mu}{M_h})\right\} \equiv -\frac{1}{3\pi}H \qquad (4.2)$$

the heavy quark polarization contribution[9].

The function I_o, subtracted at $Q=0$,

$$I_o\left(\frac{Q}{M}\right) = \frac{v(3-v^2)}{4}\ln\frac{v+1}{v-1} + \frac{v^2-1}{2} - \frac{5}{6}\;; \qquad v=\sqrt{1+4r^2} = \sqrt{1+\frac{4M^2}{Q^2}}\,, \qquad (4.3)$$

we expressed here in terms of v, the "mirror reflection" of the heavy quark velocity in the $Q\overline{Q}$ c.m.s. For the reader's convenience we indicated also its relation to the variable r used in [19]. We remind that I_o has the following simple asymptotes,

$$I_o \;\dot{\sim}\; \begin{cases} \ln\dfrac{Q}{M} - \dfrac{5}{6} \equiv \ln\dfrac{Q}{M_*}\,, & Q \gg M\,; \\ Q^2/5M^2\,, & Q \ll M\,. \end{cases}$$

[9]Quite recently an interesting paper by A. Petermann [41] has appeared that supports our exact expression for $\bar{\alpha}_s$ given by Eqs.(4.1), (4.3).

Let us stress once again that no analytic continuation is involved in applying the "Euclidean" (4.1) to the inclusive cross sections which physically occur in the Minkowskian region (see Sec. 2).

The expression (4.1) can be compared with the standard (massless) one

$$\bar{\alpha} = \frac{\alpha_\mu}{1 + \alpha_\mu \cdot \beta_1[n_f] \ln \frac{Q}{\mu}} \ , \qquad \beta_1[n_f] = \beta_\ell - \sum_h \frac{n_h(Q)}{3\pi} \ . \tag{4.4}$$

The "effective quark number n_h" [38] could be taken in the DeRujula-Georgi form [20] Eq.(3.11),

$$n_h(Q) = n_{DG}(\frac{Q}{M_h}) = \frac{Q^2}{Q^2 + 5M_h^2} \ ,$$

or as a step function [5,39]

$$n_{st}(Q) = \theta(Q - kM_h) \ , \quad k \simeq 2 \ . \tag{4.5}$$

To make the comparison more definite and realistic we choose $\bar{\alpha}_s$ at the CERN Z-factory as a reference point and consider the momentum scale Q around charm and beauty thresholds, $M_c \lesssim Q \lesssim M_b$. To this end we put

$$\mu = 91 \text{ Gev} \quad ; \quad \alpha_\mu = 0.117 \ .$$

For such a case the ratios μ/M_c and μ/M_b are large enough to use the logarithmic asymptotes for the subtraction constants in (4.2). We have to compare the exact expression

$$H_{ex}(Q) = I_o(\frac{Q}{M_b}) + I_o(\frac{Q}{M_c}) - \ln\frac{\mu}{M_b} - \ln\frac{\mu}{M_c} + \frac{5}{3} \tag{4.6}$$

with[10]

$$H_{DG}(Q) = \left\{ \frac{Q^2}{Q^2 + 5M_c^2} + \frac{Q^2}{Q^2 + 5M_b^2} \right\} \ln\frac{Q}{\mu} \ . \tag{4.7}$$

Another choice would be the "step-like" expression,

$$H_{st}(Q) = \{\theta(Q - M_c^*) + \theta(Q - M_b^*)\} \ln\frac{Q}{\mu} \ ; \quad M^* = kM \ , \tag{4.8}$$

that, after being supplemented by the $\bar{\alpha}_s$ continuity condition at the junction points $Q = M_h^*$, leads to the *spline-type* behaviour for $\bar{\alpha}_s$ and to the popular "junction-type" relation [40] for the QCD scale parameter Λ, namely,

$$\frac{\Lambda(n+1)}{\Lambda(n)} = \left(\frac{M_{n+1}^*}{\Lambda(n+1)} \right)^{\frac{\beta_{n+1} - \beta_n}{\beta_n}} \ .$$

[10] Discussed, *e.g.*, in [38]

Note that (4.8) with continuity conditions is equivalent to another continuous spline-type representation for H, namely,

$$H_{spl}(Q;k) = \theta(Q - M_c^*)\ln\frac{Q}{M_c^*} + \theta(Q - M_b^*)\ln\frac{Q}{M_b^*} + \ln\frac{M_c^* M_b^*}{\mu^2} . \qquad (4.9)$$

As it can be easily seen, all three expressions for H coincide in the vicinity of the reference point μ and above it, $Q > \mu$. At the same time they differ considerably down to $Q \ll \mu$, *i.e.*, around quark thresholds. For instance,

$$H_{ex}(M_b) = -4.86 \;;\; H_{DG}(M_b) = -2.44 \;;\; H_{spl}(M_b;1) = -6.02 \;;$$
$$H_{ex}(M_c) = -5.36 \;;\; H_{DG}(M_c) = -0,77 \;;\; H_{spl}(M_c;1) = -7.12 \,.$$

The origin of the observed difference is evident: In the low-energy range our exact expression H_{ex} has a finite limit that equals to the sum of three last terms in the r.h.s. of Eq.(4.6). In our case it is equal to -5.45. The existence of this constant follows directly from (4.1) and corresponds to the effect of *changing the subtraction scheme under threshold crossing* [37].

On the other hand, $H_{DG} \to 0$ with $Q \to 0$, while H_{spl} has a k-dependent finite limit. We have chosen above $k = 1$ in accordance with usual practice (see, *e.g.*, the popular paper [40]).

Substituting the above figures into the expression for $\bar{\alpha}$ (with $n_\ell = 3, \beta_\ell = 9/2\pi$),

$$\frac{\alpha_\mu}{\bar{\alpha}} = 1 + \frac{\alpha_\mu}{2\pi}\left[9\ln\frac{Q}{\mu} - \frac{2}{3}H(Q)\right] , \qquad (4.10)$$

one obtains

$$\bar{\alpha}^{(1)}_{ex}(M_b) = 0.214 \;; \qquad \bar{\alpha}^{(1)}_{ex}(M_c) = 0.316$$

and

$$\bar{\alpha}_{DG}(M_b) = 0.226 = 1.06 \cdot \bar{\alpha}^{(1)}_{ex}(M_b) \;; \quad \bar{\alpha}_{DG}(M_c) = 0.374 = 1.18 \cdot \bar{\alpha}^{(1)}_{ex}(M_c) \;;$$
$$\bar{\alpha}_{spl}(M_b;1) = 0.208 = 0.97 \cdot \bar{\alpha}^{(1)}_{ex}(M_b) \;; \quad \bar{\alpha}_{spl}(M_c;1) = 0.299 = 0.94 \cdot \bar{\alpha}^{(1)}_{ex}(M_c) \,.$$

We see that the errors of the "DeRujula-Georgi" and popular "spline" approximations, roughly speaking, are of the same size as the one connected with neglect of the next-to-leading log effects. Such a rather large difference in our one-loop example demonstrates the quantitative importance of an accurate account of heavy masses in the coupling constant evolution.

We have taken for demonstrative comparison the one-loop spline with $k = 1$. From (4.6), (4.9) it follows, however, that the optimum value of the adjustable parameter k is $k_* = \exp(5/6) = 2.301$. Note that just this value has been empirically found in [38]. Making use of k_* we have $H_{spl}(M_b;\ k_*) = -5.18$; $H_{spl}(M_c;\ k_*) = -5.45$, and get a close correspondence with the exact expression:

$$\bar{\alpha}_{spl}(M_b;\ k_*) = 0.212 = 0.99\,\bar{\alpha}^{(1)}_{ex}(M_b) \;;\quad \bar{\alpha}_{spl}(M_c;\ k_*) = 0.316 = \bar{\alpha}^{(1)}_{ex}(M_c) \,,$$

This example explicitly demonstrates the importance of an adequate account of the constant k_* that effectively shifts position of the threshold as it was recently emphasized in [37].

The more detailed analysis reveals that the spline-type approximation, even with the best-fit value $k = k_*$, still distorts the shape of the $\bar{\alpha}_s$ behaviour in the close vicinity of the threshold $(2M_{c,b})^2$. This means that around the threshold one should use the exact expression and not the spline to avoid a few percent error.

Final remarks:

- In a more realistic application of our technique to DIS it is necessary to treat the problem on the two-loop level. Such a program can be realized by the light-cone expansion method combined with the Stueckelberg-Bogoliubov massive RG. The full two-loop mass-dependent analysis would lead, besides $\bar{\alpha}^{(2)}$, to a certain modification of the second loop parton splitting kernels of the Master evolution equation and of the first loop coefficient functions.

- To study evolution of singlet structure function one has to account for the class of diagrams with HQ interacting with electromagnetic current. These diagrams involving gluon structure function contain specific effective coupling $\bar{\alpha}_{HQ}$ based on the HQ-gluon vertex. In the massive case this second effective coupling is generally *different from* $\bar{\alpha}_s$ (defined above in Section 3) *within the same subtraction scheme.*

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Appendix A : Renormalization-Invariance Folklore

The standard approach is based upon singular Z factors entering the infinite Dyson renormalization transformation and exploits the property of renormalization invariance. It can be associated (see, *e.g.*, the QCD texts [3]) with a possibility to present any physical quantity, $F(Q^2, g)$, calculated under definite renormalization prescription in a form $F(Q...; \mu, g_\mu)$, *i.e.*, as a function of a renormalized coupling constant g_μ and some momentum scale μ (besides other natural arguments, like momenta Q_i and renormalized masses m_j).

F is first calculated perturbatively in a form of the series expansion in powers of a "bare" coupling constant (CC) g_o,

$$F(Q, ...; g_o) = F_0\{1 + g_o F_1 + g_o^2 F_2 + \ldots\}, \tag{A.1}$$

were F_0 is a tree approximation and the expression in curly brackets contain radiative corrections with divergent coefficients F_ℓ.

In the framework of a renormalizable QFT model one can represent F in the form

$$F(...; g_o) = Z_F(\mu; g_\mu)\, f(...; \mu, g_\mu) \tag{A.2}$$

with renormalized $f = F_{ren}$ given by analogous series in powers of the renormalized CC g_μ with *finite* expansion coefficients. g_μ, in turn, is related to g_o by the CC renormalization equation

$$g_o = g_\mu Z_g^{-1} = G(\mu;\ g_\mu) = g_\mu + g_\mu^2 G_1 + g_\mu^3 G_2 + \ldots\ . \tag{A.3}$$

The differential RG equation is usually said to be driven from the condition that F does not depend on the choice of μ. This, according to (A.2), yields

$$\mu\frac{d\ln F}{d\mu} = 0 \rightarrow \left(\mu\frac{\partial}{\partial\mu} - \beta(g_\mu)\frac{\partial}{\partial g} + \gamma_f(g_\mu)\right) f(...; \mu, g_\mu) = 0\,. \tag{A.4}$$

with

$$\beta(g_\mu) = \mu\frac{\partial g_\mu}{\partial\mu} \quad \text{and} \quad \gamma_f(g) = \mu\frac{\partial\ln Z_F}{\partial\mu} \tag{4.5}$$

known as an anomalous dimension of F.

The β-function can be explicitly extracted from (A.3) by its appropriate differentiating:

$$\frac{dg_o}{d\mu} = 0 \quad \Rightarrow \quad \beta(g) = \frac{\partial G(\mu,\, g)}{\partial\mu} \Big/ \frac{\partial G}{\partial g}\,. \tag{A.6}$$

The beta-function definition (4.5) is usually considered as a characteristic equation for a specific function $\bar{g}(\mu)$ known as an *effective coupling* (sometimes — *effective coupling constant*):

$$g_\mu \equiv \bar{g}(\mu)\,.$$

Given the beta-function on the basis of Eq. (A.6), the dependence of g_μ on the momentum scale μ (*i.e.*, $\bar{g}(\mu)$) can be derived from Eq. (4.5).

An important feature of this approach is that it is tightly connected with UV singularities — here β and γ are to be defined via singular Z-factors. These factors, as a rule, correspond to *massless* counter-terms. Due to this one automatically arrives at the massless RG formulation.

The second point to be mentioned here is that the group structure of the results obtained along these lines is not evident from the first glance[11]. In a sense it is hidden under the notion of renormalization invariance. Note also that in the pioneering Gell-Mann-Low paper results were formulated in terms of functional equations. The main one for $\bar{g}$ in modern notation can be presented in a form

$$\bar{g}(x,g) = \bar{g}\left(\frac{x}{t}, \bar{g}(t,g)\right) \ ; \quad x = \frac{Q}{\mu}\ , \quad g = g_\mu \ . \tag{A.7}$$

Appendix B : Massive RG Formalism

B1. Functional and Differential Equations

Our starting point is two Functional Equations (FEqs) for the *invariant coupling* $\bar{g}$

$$\bar{g}(x,y;g) = \bar{g}\left(\frac{x}{t}, \frac{y}{t}; \bar{g}(t,y;g)\right) \tag{B.1}$$

$$x = Q^2/\mu^2 \ , \ y = M^2/\mu^2 \ , \quad g - \text{the coupling constant} \ ,$$

and for the *one-argument function* s (like a propagator amplitude or a structure function moment),

$$s(x,y;g) = s(t,y;g) \ s\left(\frac{x}{t}, \frac{y}{t}; \bar{g}(t,y;g)\right) . \tag{B.2}$$

Their derivation can be found in the Chapter "Renormalization Group" of the monograph [2].

An essential technical point is the normalization conditions

$$\bar{g}(1,y;g) = g \ , \qquad s(1,y;g) = 1 \ , \tag{B.3}$$

built in given FEqs. They correspond to the MOM renormalization prescription with μ being the subtraction momentum value. We shall turn to the more general case later on. Note here, that in this treatment ascending to the original Bogoliubov's RG formulation [7] the m argument is just the propagator pole, *i.e.* the physical renormalized mass[12].

The differential equations can be obtained from the functional ones in two different ways. Differentiating (B.1) over x and putting $t = x$ afterwards, one obtains

$$x\frac{\partial \bar{g}(x,y,g)}{\partial x} = \beta\left(\frac{y}{x}, \bar{g}(x,y;g)\right) \ , \tag{B.4}$$

[11]It is curious that there is no word *group* in the famous Gell-Mann–Low paper.

[12]In a broader sense m can be treated as a parameter that violates an homogeneity under momentum scale transformation, $Q \to \kappa \cdot Q$, see [42] for a general discussion.

with the infinitesimal response or the *group generator*

$$\beta(y,g) = \left.\frac{\partial\bar{g}(t,y;g)}{\partial t}\right|_{t=1} . \tag{B.5}$$

The nonlinear equation (B.4) can be considered as a "massive" generalization of the standard massless equation

$$x\,\frac{\partial\bar{g}(x,g)}{\partial x} = \beta(\bar{g}(x,g)) .$$

On the other hand, one can differentiate (B.1) with respect to t at the point $t = 1$, which yields

$$\left\{ x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(y,g)\frac{\partial}{\partial g} \right\} \bar{g}(x,y;g) = 0 , \tag{B.6}$$

— a linear partial differential equation.

Analogous operations applied to (B.2) lead to

$$\frac{\partial s(x,y;g)}{\partial \ln x} = \gamma\left(\frac{y}{x}, g(x,y;g)\right) s(x,y;g) \tag{B.7}$$

and

$$\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(y,g)\frac{\partial}{\partial g} - \gamma(y,g)\right\} s(x,y;g) = 0 , \tag{B.8}$$

with

$$\gamma(y,g) = \left.\frac{\partial s(t,y;g)}{\partial t}\right|_{t=1} , \tag{B.9}$$

— the so called "anomalous dimension" of s. For an observable quantity M this dimension is zero. The corresponding partial differential equation takes the form

$$\left\{\sum_i x_i\frac{\partial}{\partial x_i} - y\frac{\partial}{\partial y} - \beta(y,g)\frac{\partial}{\partial g}\right\} M(\ldots x_i \ldots, y; g) = 0 . \tag{B.10}$$

Linear equations (B.6), (B.8), (B.10) express an independence of corresponding quantities ($\bar{g}$, s, M) on the variable t, *i.e.* a mutual compensation of t dependencies in all (three or more) arguments, see (B.1), (B.2). These equations can be called therefore *compensational*, to distinguish them from nonlinear equations (B.4), (B.7) that could be referred to as the *evolution* group equations.

Let us notice that the compensational and the evolution differential Eqs., taken together with normalization (boundary) conditions (B.3), are equivalent to functional Eqs. and to each other. Given the generators β and γ, the evolution Lie equations turn out to be more convenient for practical construction of the solution.

B2. Renormalization Group Method

The Renormalization Group Method (RGM) solves the problem of improving the approximative properties of power series QFT solutions usually obtained with help of renormalized perturbation theory.

The RGM recipe says that to improve some approximate expression s_{appr} for the function s one has to take the corresponding approximate expression for the effective (invariant) coupling $\bar{g}_{appr}$ and then

1. to define the group generators β and γ via these expressions, *i.e.*

$$\beta(y,g) =_{def} \left.\frac{\partial \bar{g}_{appr}(t,y;g)}{\partial t}\right|_{t=1} \quad \text{and} \quad \gamma(y,g) =_{def} \left.\frac{\partial s_{appr}(t,y;g)}{\partial t}\right|_{t=1} ; \tag{4.11}$$

2. to solve Eq. (B.4) and find its solution $\bar{g}_{rg}(x,y;g)$;

3. to solve Eq. (B.7) and obtain the RG-improved solution $s_{rg}(x,y;g)$.

To illustrate this program consider the one-loop approximation for effective coupling,

$$\bar{\alpha}^{(1)}_{pt} = \alpha - \alpha^2 A_1(Q,M,\mu) + O(\alpha^3) \quad \text{with} \quad A_1 = I(Q/M) - I(\mu/M) . \tag{4.12}$$

Using (4.11) we obtain

$$\beta^{(1)}(Q/M,\alpha) = \alpha^2 \dot{I}(Q/M) \tag{B.13}$$

with the upper dot standing for the logarithmic derivative. The simple structure of this expression allows one to separate the variables in the Lie differential equation,

$$Q\frac{\partial \bar{\alpha}}{\partial Q} = -\bar{\alpha}^2 \dot{I}(\frac{Q}{M}) ,$$

and to integrate it explicitly:

$$d\left(\frac{1}{\bar{\alpha}}\right) = d\,I\left(\frac{Q}{M}\right) ; \qquad \frac{1}{\bar{\alpha}} = I\left(\frac{Q}{M}\right) + C .$$

Defining the integration constant C from the boundary condition $\bar{\alpha}(Q=\mu) \equiv \alpha_\mu$, one finally arrives at the *exact* one-loop solution

$$\bar{\alpha}^{(1)}_{rg} = \frac{\alpha_\mu}{1 + \alpha_\mu A_1(Q,M,\mu)} . \tag{B.14}$$

This solution precisely satisfies the massive group functional equation (B.1).

Generally speaking, similarly to the massless case, it is impossible to obtain the exact explicit solution of the massive RG equations. Nevertheless, in several important cases, like at two- and three-loop level for $\bar{\alpha}$ and one- and two-loop level for one-argument function (see, *e.g.*, Eq.(3.8b)) sufficiently accurate approximate solutions have been explicitly found [36,43].

Appendix C : DIS Ladder Cell.

C1. Structure of the Boson Propagator

Let us write the bare virtual photon propagator in the following form

$$D_0^{\mu\nu}(p) = \left[-g^{\mu\nu} + (1+d_\ell)\,\frac{p^\mu p^\nu}{p^2} \right] \cdot \frac{1}{-p^2 - i\epsilon} \tag{C.1}$$

with d_ℓ the gauge parameter. In this notation the residue in the photon pole $p^2 = 0$ on the physical photon states

$$(e_\mu(p))^2 = -1\,, \qquad (e_\mu(p) \cdot p^\mu) = 0\,, \tag{C.2}$$

is plus "unity" since

$$\begin{aligned} e_\mu(p) \cdot \left[-g^{\mu\nu} + (1+d_\ell)\,\frac{p^\mu p^\nu}{p^2} \right] \cdot e_\nu(p) &= 1 \\ \frac{1}{\pi}\,\mathrm{Im}\,\frac{1}{-p^2 - i\epsilon} \qquad &= \delta(p^2)\,. \end{aligned} \tag{C.3}$$

With account of fermion loops the full propagator we find by solving the Schwinger-Dyson equation

$$D^{\mu\nu}(p) = D_0^{\mu\nu}(p) + D_0^{\mu\lambda}(p) \cdot \Sigma_{\lambda\lambda'}(p) \cdot D^{\lambda'\nu}(p) \tag{C.4}$$

with the "self-energy" function

$$\Sigma_{\lambda\lambda'}(p) = \left[-g^{\lambda\lambda'} \cdot p^2 + p^\lambda p^{\lambda'} \right] \cdot \Sigma(p^2)\,. \tag{C.5}$$

The sign in (C.5) is chosen such that the scalar dimensionless function Σ has *positive* imaginary part at $p^2 > 4M^2$ which follows from unitarity consideration:

$$\mathrm{Im}\left\{ e^\lambda \cdot \Sigma_{\lambda\lambda'}(p) \cdot e^{\lambda'} \right\} \propto \sigma\left(\gamma^* \to e^+ e^-\right) > 0\,. \tag{C.6}$$

Writing the full propagator as

$$D^{\mu\nu}(p) = \left[-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2} \right] \cdot \frac{\mathcal{Z}(p^2)}{-p^2 - i\epsilon} \; + \; d_\ell \frac{p^\mu p^\nu}{p^2} \cdot \frac{1}{-p^2 - i\epsilon} \tag{C.7}$$

we solve (C.4) to obtain

$$\mathcal{Z}(p^2) = \frac{1}{1 + \Sigma(p^2)}\,. \tag{C.8}$$

The full propagator projected onto physical states (C.2),

$$e_\mu(p) D^{\mu\nu}(p) e_\nu(p) \; = \; \frac{\mathcal{Z}(p^2)}{-p^2 - i\epsilon}\,, \tag{C.9}$$

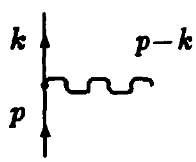
has a positive imaginary part. From (C.8) we get

$$\frac{1}{\pi}\,\mathrm{Im}\left\{\frac{\mathcal{Z}(p^2)}{-p^2-i\epsilon}\right\} = \mathcal{Z}(0)\cdot\delta(p^2) + \frac{1}{\pi}\,\frac{\mathrm{Im}\,\Sigma(p^2)}{|1+\Sigma(p^2)|^2}\cdot\frac{1}{p^2}\,. \tag{C.10}$$

Notice that at the same time $\mathrm{Im}\,\mathcal{Z}$ is *negative.*

C2. Sudakov variables and kinematics

Let us express the parton momenta involved in terms of the light-cone components (Sudakov decomposition)



$$\begin{aligned} q_\mu &= \;-x\,p_\mu \;+\; q'_\mu\,, \\ k_\mu &= \;\;z\,p_\mu \;\;+\alpha\,q'_\mu + (k_\perp)_\mu\,, \\ (p-k)_\mu &= (1-z)\,p_\mu - \alpha\,q'_\mu - (k_\perp)_\mu\,, \end{aligned} \tag{C.11}$$

with $p^2 = q'^2 = 0$ and z, α and $k_\perp$ as new independent variables. We have

$$s \equiv 2(pq') = 2(pq) = -\frac{q^2}{x}\,; \qquad d^4k \;=\; d\alpha\,dz\,d^2\vec{k}_\perp\cdot\frac{s}{2}\,.$$

Virtuality of parton k we express in terms of longitudinal momentum fraction z, transverse momentum $k_\perp$ and the virtual photon mass m^2:

$$\begin{aligned} (p-k)^2 = -\alpha(1-z)\,s - k_\perp^2 \;=\; m^2\,, &\quad -\alpha s = \frac{k_\perp^2+m^2}{(1-z)}\,; \\ k^2 = \alpha z\,s - k_\perp^2 \;=\; -\frac{1}{(1-z)}\left[\,k_\perp^2 + z\,m^2\,\right]\,. & \end{aligned} \tag{C.12}$$

C3. Analysis of the matrix element (2.2)

To study the *anomalous dimension* one has to pick up contributions logarithmic in $k_\perp^2$ in the kinematical region

$$-k^2 \;\sim\; k_\perp^2 \;\ll\; -q^2 \sim s\,.$$

According to (2.1) collinear logarithm originates from

$$A_\mu\,A_\nu^*\cdot D^{\mu\nu}(p-k) \sim \frac{1}{k_\perp^2}\,.$$

Such an enhancement in the quasi-collinear kinematics originates from the diagrams with initial parton splitting. So we single out from the full radiation amplitude the contribution with vector boson emission off the incoming fermion to write

$$\begin{aligned} A_\mu &= A_\mu^{(S)} \;+\; A_\mu^{(R)}\,; \\ A_\mu^{(S)} &= \bar{u}(p)\,\gamma_\mu\frac{\hat{k}}{k^2}\cdot B \end{aligned} \tag{C.13}$$

with $A_\mu^{(R)}(k)$ being regular in the collinear limit $k_\perp \to 0$.

Substituting Sudakov decomposition (C.11) for $\hat{k}$, we make use of the Dirac equation, $\bar{u}(p)\,\hat{p} = \hat{p}\,u(p) = 0$, to obtain

$$k^2\,A_\mu^{(S)} = \bar{u}(p)\,\gamma_\mu\hat{k}\cdot B = \bar{u}(p)\,\Big(2zp_\mu + \gamma_\mu\hat{k}_\perp + \mathcal{O}\big(k_\perp^2\big)\Big)\cdot B\,. \tag{C.14}$$

Then let us represent the leading $\mathcal{O}(1/k_\perp^2)$ term proportional to p_μ as (see (C.11))

$$p_\mu \;=\; \frac{(p-k)_\mu}{(1-z)} \;+\; \frac{(k_\perp)_\mu}{(1-z)} \;+\; q'_\mu\cdot\mathcal{O}\big(k_\perp^2\big) \tag{C.15}$$

Now we rewrite the total amplitude (C.13) in terms of the *redefined* leading part,

$$\begin{aligned} A_\mu^{(S)} &= \frac{2z(p-k)_\mu}{(1-z)k^2}\cdot\bar{u}(p)B \;+\; \tilde{A}_\mu^{(S)}\,; \\ \tilde{A}_\mu^{(S)} &= \bar{u}(p)\,\frac{1}{(1-z)k^2}\,\Big[2zk_{\perp\mu} + (1-z)\gamma_\mu\hat{k}_\perp\Big]\cdot B \;=\; \mathcal{O}(1/k_\perp)\,, \end{aligned} \tag{C.16a}$$

with the $\mathcal{O}(k_\perp^2)$ contributions from (C.14), (C.15) absorbed into the regular piece $A^{(R)} = \mathcal{O}(1)$. Corresponding expression for the amplitude conjugated reads

$$\begin{aligned} A_\nu^{(S)*} &= \frac{2z(p-k)_\nu}{(1-z)k^2}\cdot B^*u(p) \;+\; \tilde{A}_\nu^{(S)*}\,; \\ \tilde{A}_\nu^{(S)*} &= B^*\cdot\frac{1}{(1-z)k^2}\,\Big[2zk_{\perp\nu} + (1-z)\hat{k}_\perp\gamma_\nu\Big]\;u(p)\,. \end{aligned} \tag{C.16b}$$

We have to construct the product

$$A_\mu\,A_\nu^*\cdot D^{\mu\nu}(p-k) \;=\; -g^{\mu\nu}\,A_\mu\,A_\nu^*\,, \tag{C.17}$$

where one can drop the longitudinal components of the vector propagator (C.7) due to the current conservation condition,

$$(p-k)^\mu\,A_\mu \;=\; (p-k)^\nu\,A_\nu^* \;=0\,. \tag{C.18}$$

The same property allows us to get rid of the longitudinal $(p-k)_\mu$ components in the amplitudes (C.16) as well, leading to

$$-\left|A_\mu^{(S)} + A_\mu^{(R)}\right|^2 = -\left|\tilde{A}_\mu^{(S)} + A_\mu^{(R)}\right|^2 \;+\; \left[\frac{2z(p-k)_\mu}{(1-z)k^2}\right]^2\cdot\left[\bar{u}(p)BB^*u(p)\right]\,. \tag{C.19}$$

From (2.1)–(2.4) we conclude that with an account of the virtual boson mass one has to keep track of the contributions *linear* in $k_\perp^2$ and m^2 only,

$$\int dk_\perp^2 \int \frac{dm^2}{m^2}\cdot\frac{C_1k_\perp^2 + C_2m^2}{\left[k_\perp^2 + z\,m^2\right]^2}\,. \tag{C.20}$$

Therefore the regular part of the amplitude, $A^{(R)}$ does not affect the anomalous dimension and can be omitted.

To calculate the first term of (C.19) we take an average over initial fermion polarizations,

$$\frac{1}{2}\sum_{\sigma=1,2}\bar{u}^\sigma(p)\,M\,u^\sigma(p) \;=\; \frac{1}{2}\mathrm{Tr}\,[\,\hat{p}\,M\,]\;,$$

with use of the relations

$$-k_{\perp\mu}k_\perp{}^\mu = -\hat{k}_\perp\hat{k}_\perp \;=\; k_\perp^2 \;>\; 0\,,$$
$$\gamma_\mu\hat{p}\gamma^\mu \;=\; -2\,\hat{p}\,, \qquad \hat{k}_\perp\hat{p} = -\hat{p}\,\hat{k}_\perp\,.$$

This results in

$$\begin{aligned}\left|(1-z)k^2\cdot\tilde{A}_\mu^{(S)}\right|^2 &= -\tfrac{1}{2}\mathrm{Tr}\left[\hat{p}\cdot\left(2zk_{\perp\mu}+(1-z)\gamma_\mu\hat{k}_\perp\right)\,BB^*\,\left(2zk_{\perp\mu}+(1-z)\hat{k}_\perp\gamma_\mu\right)\right]\\ &= k_\perp^2\cdot\left[\,(2z)^2+2\cdot2z(1-z)+2(1-z)^2\,\right]\cdot\tfrac{1}{2}\mathrm{Tr}\,[\hat{p}\,BB^*]\\ &= k_\perp^2\cdot2\left[1+z^2\right]\cdot\tfrac{1}{2}\mathrm{Tr}\,[\hat{p}\,BB^*]\end{aligned}\qquad\text{(C.21)}$$

The second term in (C.19) leads to an extra contribution proportional to the virtual boson mass squared,

$$\begin{aligned}k_\perp^2\cdot2\left[1+z^2\right] &\to k_\perp^2\cdot2\left[1+z^2\right]\;+\;m^2\cdot4z^2\\ &= (k_\perp^2+zm^2)\cdot2\left[1+z^2\right]\;-\;m^2\cdot2z(1-z)^2\end{aligned}\qquad\text{(C.22)}$$

Collecting the phase space factors and expressing the parton virtuality k^2 via transverse momentum, (C.12), one arrives at the final expression (2.6) for the parton cross section (2.5).

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