DRFC/SPPF/STPF EUR-CEA-FC 1483

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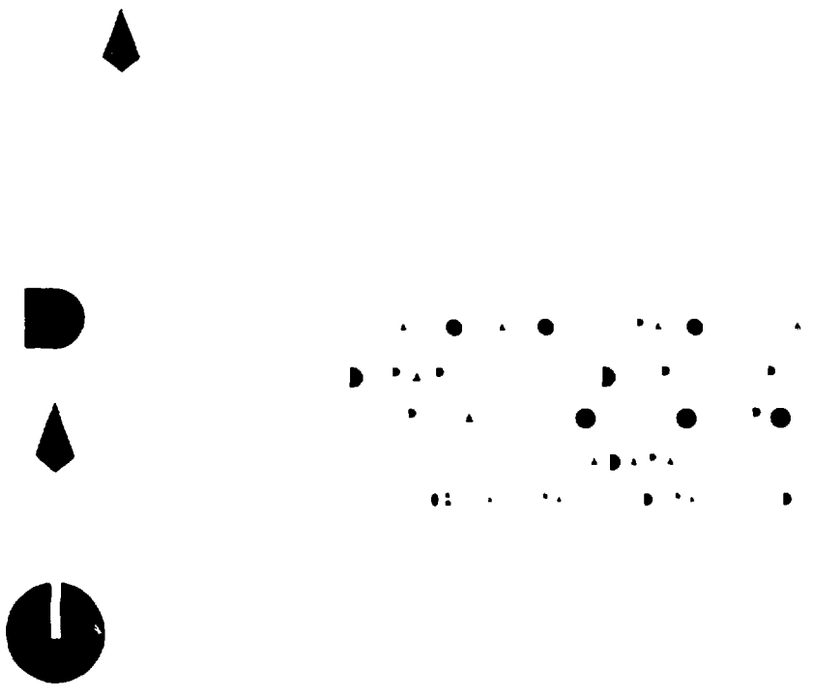
J.D. Reuss, M. Vlad*), J.H. Misguich, F. Spineanu*)

(February 1993)



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ABSTRACT

The Campbell process is a stationary random process which can have various correlation functions, according to the choice of an elementary response function. The statistical properties of this process are presented. A numerical algorithm and a subroutine for generating such a process is built up and tested, for the physically interesting case of a Campbell process with Gaussian correlations. The (non-Gaussian) probability distribution appears to be similar to the Gamma distribution.

1. INTRODUCTION

In various treatments of a rather large class of plasma phenomena described as stochastic processes, the mostly used assumption is that the basic random variable has a δ type auto-correlation function. In a rather large number of casses, this assumption is not experimentally sustained since finite width correlations have been measured, in time or in space. This observation has been included in some more elaborated models by considering a gaussian correlation of the random variable. We give here an exemple of a stochastic process which could have this property. In fact, practically any type of auto-correlation function can be obtained by means of this process. It was defined in 1909 by R.N.Campbell /1/ and is known as "Campbell's process". It is especially used in experimental physics for describing the response of a detecting system to a random flux of particles.

Although Campbell's process is very apropriate for specific problems of plasma theory, it seems not to have been used in this field. Moreover, this process has simple properties which allow one to perform analytical calculations. With the hope of inspiring interesting applications, we derive explicitely here interesting analytical properties of the Campbell's process. We also present an explicit algorithm for generating numerically such a process in view of future numerical simulations in plasma physics.

Section 2 contains the definition of the Campbell's process and a short review of the statistical notions on which it is based (random sets of dots, stochastic processes, Poissonian flow). A statistical average over an ensemble of realizations of this process yields simple analytical expressions for the average, the correlation, and all the higher order cumulants (see Section 3). The distribution function is obtained in Section 4, and in Section 5 we give a numerical algorithm and a subroutine for generating a random variable of Campbell's type. The conclusions and some suggestions for applications are summarized in Section 6.

2. DEFINITION OF THE CAMPBELL PROCESS

We first introduce the notion on which Campbell's process relies, namely the random sets of dots and the stochastic processes /2/.

2.A. Poisson flow.

A **random set of dots** on a line represents the values of a stochastic variable with the following characteristics :

a) the sample space is formed of states each consisting of a nonnegative integer $s = 0, 1, ...$ and of a set of s real numbers τ_σ obeying:

$$-\infty < \tau_1 < \tau_2 < ... < \tau_s < \infty \tag{1}$$

b) the probability distribution over these states is given by an infinite sequence of non-negative functions:

$$Q_0, Q_1(\tau_1), Q_2(\tau_1, \tau_2), ..., Q_s(\tau_1, \tau_2, ..., \tau_s), ...$$

which are normalized according to:

$$Q_0 + \int_{-\infty}^{\infty} d\tau_1 Q_1(\tau_1) + \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 Q_2(\tau_1, \tau_2) + ... = 1 \tag{2}$$

In most applications it is convenient to eliminate the ordering (1) of τ_s by considering that the $s!$ sets $\{\tau_1, \tau_2, ..., \tau_s\}$, that are the same apart from a permutation, represent one and the same state, and that the functions $Q_s(\tau_1, \tau_2, ..., \tau_s)$ are symmetric functions of their variables. In this case the normalization condition (2) may be written:

$$Q_0 + \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 ... d\tau_s Q_s(\tau_1, \tau_2, ..., \tau_s) = 1 \tag{3}$$

The averages are defined for functions on the sample space for this random variable, i.e. for functions of the type

$$\{A_0, A_1(\tau_1), A_2(\tau_1, \tau_2), ...\}$$

and are given by :

$\langle A \rangle = A_0 Q_0 +$

$$+ \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} A_s(\tau_1, \tau_2, ..., \tau_s) Q_s(\tau_1, \tau_2, ..., \tau_s) d\tau_1 ... d\tau_s \tag{4}$$

A random set of dots are called **independent** when the functions Q_s factorize in the form :

$$Q_s(\tau_1, \tau_2, ..., \tau_s) = e^{-\nu} q(\tau_1) q(\tau_2) ... q(\tau_s), \qquad Q_0 = e^{-\nu} \tag{5}$$

where $q(\tau)$ is a non-negative integrable function and

$$\nu = \int_{-\infty}^{\infty} q(\tau) d\tau \tag{6}$$

This parameter ν will be proved to represent the total number of dots. It can be easily checked that the normalization condition (3) is verified by the functions (5).

We mention here only one example which illustrates the procedure of calculating the averages according to (4). It is particularly interesting for the random sets of dots since it represents the statistics of the number of dots which fall in a given time interval. We first show that, for a set of independent dots, the average number of dots in an interval $[t_a, t_b]$ is given by:

$$\langle N \rangle = \int_{t_a}^{t_b} q(\tau) d\tau \tag{7}$$

This can be proved by defining a function on the sample space of the random set of dots which "counts" the number of dots in the given interval for each realisation :

$$N = \{0, \chi(\tau_1), \chi(\tau_1) + \chi(\tau_2), ...\} \tag{8}$$

where

$$\chi(t) = \begin{cases} 1 \text{ for } t \in [t_a, t_b] \\ 0 \text{ for } t \notin [t_a, t_b] \end{cases}$$

Then, the average of N can be calculated according to (4):

$$\langle N \rangle = \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} d\tau_1 ... d\tau_s Q_s(\tau_1 ... \tau_s) \sum_{r=1}^{s} \chi(\tau_r)$$

$$= e^{-\nu} \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} d\tau_1 ... d\tau_s q(\tau_1) .... q(\tau_s) \sum_{r=1}^{s} \chi(\tau_r) \tag{9}$$

One of the s integrals on τ_i has the integrand $q\,\chi$ and it is reduced at an integral on the finite interval $[t_a, t_b]$ while the others $(s-1)$ are of the type (6) and give ν^{s-1}. Thus:

$$\langle N \rangle = \int_{t_a}^{t_b} d\tau_1\, q(\tau_1)\, e^{-\nu} \sum_{s=1}^{\infty} \frac{1}{(s-1)!} \nu^{s-1} = \int_{t_a}^{t_b} d\tau_1\, q(\tau_1) \tag{10}$$

The average of the square of the nomber of dots in the interval $[t_a, t_b]$ is given by:

$$\langle N^2 \rangle = \langle N \rangle^2 + \langle N \rangle \tag{11}$$

This can be proved by observing that

$$\left(\sum_{r=1}^{s} \chi(\tau_r) \right)^2 = \sum_{r=1}^{s} \chi(\tau_r) + \sum_{r,r'=1}^{s} \chi(\tau_r) \chi(\tau_{r'})$$

since $\chi^2 = \chi$. The first term corresponds to N and its average gives $\langle N \rangle$ and the average of the second term is:

$$\sum_{s=2}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} d\tau_1 ... d\tau_s \, e^{-\nu} q(\tau_1) ... q(\tau_s) \sum_{\substack{r, r' = 1 \\ r \neq r'}}^{s} \chi(\tau_r) \chi(\tau_{r'}) \tag{12}$$

In this case, two of the s integrals on τ have the integrand $q\,\chi$ and thus reduce to integrals of q over the finite interval(t_a, t_b) (which, according to (7), gives $\langle N \rangle$). The other $(s-2)$ integrals are of the type (6) and give ν^{s-2}. Thus, the expression (12) can be written as: $e^{-\nu} \left(\sum_{s=2}^{\infty} \frac{1}{(s-2)!} \nu^{s-2} \right) \langle N \rangle^2$ and it gives $\langle N \rangle^2$ (which proves equation (11)).

The number of dots in a given time interval can be described not only as a function of a random variable (namely the set of independent dots) but also as a random variable by determining its own distribution function. It is rather easy to show, by means of the characteristic function, that *the probability that N of the independent random dots fall in the chosen time interval is given by the Poisson distribution:*

$$P_N = \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle} \tag{13}$$

The distribution function is determined by only one parameter $\langle N \rangle$ which is the average number of dots in the chosen interval.

The function $q(\tau)$ which determines the probability distribution of the dots (5) can be constant on a finite time interval:

$$q(\tau) = \begin{cases} \rho & \tau \in [-T, T] \\ 0 & \tau \notin [-T, T] \end{cases} \tag{14}$$

where $\rho = \frac{\nu}{2T}$ is the average number of events in the unit time interval (or average density of events). At the limit of very large T ($T \to \infty$ and $\nu \to \infty$ with finite ρ) the random set of dots is **stationary**.

The parameters characterizing this Poisson flow are summarized in Table I:

Total number of dots on $[-T, T]$:	=	ν
Time of arrival :	=	τ_i
Density of dots on $[-T, T]$:	=	$\rho = \frac{\nu}{2T}$

Table I: *Parameters of the Poisson flow.*

2.B. Campbell's stochastic process.

We consider **a stochastic process** defined by a function of two variables: one is a real number (the time) and the other is a stochastic variable X:

$$Y_X(t) = f(X,t) \tag{15}$$

The realizations of the stochastic process are obtained by replacing the stochastic variable X by its possible values x: $Y_x = f(x,t)$. The average of the stochastic process is a function of t given by:

$$\langle Y(t) \rangle = \int Y_x(t) P_X(x) dx \tag{16}$$

where $P_X(x)$ is the probability density for the stochastic variable X.

The **auto-correlation function** for the stochastic process is defined by:

$$K(t_1,t_2) \equiv \langle (Y(t_1)Y(t_2)) \rangle = \langle [Y(t_1) - \langle Y(t_1) \rangle] [Y(t_2) - \langle Y(t_2) \rangle] \rangle =$$

$$= \langle Y(t_1)Y(t_2) \rangle - \langle Y(t_1) \rangle \langle Y(t_2) \rangle \tag{17}$$

For a stationary process, the moments are invariant with respect to a shift in time and the correlation function depends only on the time difference:

$$K(t_1,t_2) = f(|t_1 - t_2|) \tag{18}$$

We can give now the definition of the Campbell process.

Definition :

The Campbell process is a stochastic process depending on an independent and stationary random set of dots $X \equiv \{s; \tau_1, \tau_2, ..., \tau_s\}$ and defined by:

$$\boxed{Y(t) = \sum_{\sigma=1}^{s} \Psi(t - \tau_\sigma)} \tag{19}$$

where Ψ is a given "response" function with finite width. The Campbell process describes the response of a detecting system to a random flux of particles, knowing that its elementary response to a single particle is $\Psi(t)$. It can also be used for describing various random perturbations of particle trajectories (or of magnetic field lines).

3. STATISTICAL PROPERTIES OF THE CAMPBELL PROCESS

The Campbell process is characterized by simple expressions for its cumulants and for its correlation function : they all depend only on the function Ψ. These properties are known as **Campbell's theorem** which states that, for a Campbell's process $Y(t)$ (19), the average $\langle Y \rangle$, the correlation function $K(\tau)$, and the cumulants K_m are given respectively by:

$$\langle Y \rangle = \rho \int_{-\infty}^{\infty} \Psi(t) dt \tag{20}$$

$$K(\tau) \equiv \langle\langle Y(t) Y(t+\tau) \rangle\rangle = \rho \int_{-\infty}^{\infty} \Psi(t) \Psi(t+\tau) dt \tag{21}$$

$$K_m \equiv \langle\langle (Y(t))^m \rangle\rangle = \rho \int_{-\infty}^{\infty} (\Psi(t))^m dt \tag{22}$$

where ρ is the density of the random dots. This theorem is mentioned in /2/, without proof, as an exercice.

Proof of the theorem:

The average of $Y(t)$ is, according to (4):

$$\langle Y(t) \rangle = e^{-\nu} \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} d\tau_1 .... d\tau_s q(\tau_1) .... q(\tau_s) \sum_{\sigma=1}^{s} \Psi(t - \tau_\sigma) \tag{23}$$

One of the τ integrals in (23) has the integrand $q\,\Psi$ and, since the random set of dots is stationary, it yields $\rho \int_{-\infty}^{\infty} \Psi(t) dt$. The other $(s-1)$ integrals over τ are of the type (6) and give ν^{s-1} so that the summation over s is simply the series expansion of e^{ν}. Thus, equation (20) is obtained.

In a similar way, one can calculate the average of $Y(t)\,Y(t+\tau)$:

$$\langle Y(t) Y(t+\tau) \rangle = e^{-\nu} \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} d\tau_1 ... d\tau_s \, q(\tau_1) ... q(\tau_s) \sum_{\sigma=1}^{s} \sum_{\sigma'=1}^{s} \Psi(t - \tau_\sigma) \Psi(t + \tau - \tau_{\sigma'})$$

The diagonal part of the double summation over σ and σ' can be treated as (23): one of the s integrals over τ is

$$\int_{-\infty}^{\infty} d\tau_1 q(\tau_1) \Psi(t - \tau_1) \Psi(t + \tau - \tau_1) = \rho \int_{-\infty}^{\infty} \Psi(\tau') \Psi(\tau' + \tau) d\tau' \tag{25}$$

and all the others $(s-1)$ are of the type (6) and give ν^{s-1} which, summed over s, reproduce the exponential e^{ν} and compensates the first factor in (24). The nondiagonal part of the double sum contains two time integrals with the integrand $q\Psi$ (which finally give $\langle Y\rangle^2$) and $(s-2)$ integrals which again lead to e^{ν}. Thus:

$$\langle Y(t)Y(t+\tau)\rangle = \rho\int_{-\infty}^{\infty}\Psi(\tau')\Psi(\tau'+\tau)d\tau' + \left(\rho\int_{-\infty}^{\infty}\Psi(\tau')d\tau'\right)^2 \tag{26}$$

which is equivalent with (21).

The higher order cumulants K_m (22) are analogously determined. They result from the diagonal terms in the m sum:

$$Y^m = \sum_{\sigma_1,\ldots,\sigma_m}^{s}\Psi(t-\tau_{\sigma_1})...\Psi(t-\tau_{\sigma_m})$$

The expression (21) for the correlation function shows that practically any function $K(\tau)$ can be obtained by an apppropriate choice of the function Ψ. In particular, if the function $\Psi(t)$ is a Gaussian, then the correlation is also Gaussian:

$$\Psi(t) = e^{-\alpha t^2} \Longrightarrow K(\tau) = \rho\sqrt{\frac{\pi}{2\alpha}}e^{-\frac{\alpha}{2}\tau^2} \tag{27}$$

Such correlations are often considered in simple theoretical models of statistical physics. This important property of random Campbell's processes motivates us to practically build a numerical algorithm to generate such a process. For this reason, this property is further considered in Section (5) in order to check the accuracy of the numerical algorithm we present.

4. THE DISTRIBUTION FUNCTION

It is usually possible to represent a stationary stochastic process as a stochastic variable by determining its own probability distribution function. This can be done by means of the characteristic function $G(k)$ which is determined by the set of cumulants K_m:

$$\log G(k) = \sum_{m=1}^{\infty}\frac{(ik)^m}{m!}K_m \tag{28}$$

and is related to the probability distribution function $P(Y)$ by the Fourier transform:

$$P(Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}G(k)\,e^{-ikY}dk \tag{29}$$

In the case of the Campbell process, the cumulants have very simple expressions (22) and the characteristic function $G(k)$ can easily be determined as:

$$\log G(k) = \rho \int_{-\infty}^{\infty} \left(e^{ik\Psi(\tau)} - 1 \right) d\tau \tag{30}$$

The probability distribution function is:

$$P(Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} exp \left(\sum_{m=1}^{\infty} \frac{(ik)^m}{m!} K_m - ikY \right) dk =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} exp \left(\rho \int_{-\infty}^{\infty} \left(e^{ik\Psi(\tau)} - 1 \right) d\tau - ikY \right) dk \tag{31}$$

For a Gaussian function $\Psi(t)$, the cumulants are:

$$\Psi(t) = e^{-\alpha t^2} \quad \Longrightarrow \quad K_m = \rho \sqrt{\frac{\pi}{\alpha m}} \equiv \frac{c}{\sqrt{m}} \tag{32}$$

where

$$c = \rho \sqrt{\pi / \alpha}$$

is the average value of Y (equal to the average density of dots multiplied by the integral of Ψ, i.e. to the average number of dots on the width of the response function Ψ). The probability distribution function for the values of Y is:

$$P(Y) = 2 \int_0^{\infty} \rho(k) cos[k(\varphi(k) - Y)] dk \tag{33}$$

where:

$$\rho(k) = \rho(-k) = \frac{1}{2\pi} exp \left[c \sum_{n=1}^{\infty} \frac{(-k^2)^n}{(2n)!} \frac{1}{\sqrt{2n}} \right] \tag{34}$$

$$\varphi(k) = \varphi(-k) = c \sum_{n=0}^{\infty} \frac{(-k^2)^n}{(2n+1)!} \frac{1}{\sqrt{2n+1}} \tag{35}$$

The probability $P(Y)$ is ploted in Fig.1 for several values of c. It has a maximum which is close to c, the average value of Y. The width of the probability distribution increases with the increase of c. We notice that $P(Y) = 0$ for $Y < 0$.

A rather unexpected similitude links the probability distribution $P(Y)$ with the Gamma distribution $f_\Gamma(Y)$ which is defined by /3/:

$$f_\Gamma(Y) = \begin{cases} \frac{1}{\beta\,\Gamma(\alpha+1)} \left(\frac{Y}{\beta}\right)^{\alpha} e^{-Y/\beta} & Y \geq 0 \\ 0 & Y < 0 \end{cases} \tag{36}$$

where $\beta = \frac{K_2}{K_1}, \alpha = \frac{K_1^2}{K_2} - 1$ and $\Gamma(\alpha+1) = \int_0^\infty e^{-t} t^\alpha dt$. For the values (32) of the cumulants, the parameters of the Gamma distribution are : $\beta = \frac{1}{\sqrt{2}}$ and $\alpha = c\sqrt{2} - 1$. Although their expressions are different, the probability distributions $P(Y)$ and $f_\Gamma(Y)$ are found to be rather similar. This can be seen by comparing Fig.1 with Fig.2 where the $f_\Gamma(Y)$ for the same values of c are plotted. Thus, the probability distribution function for Campbell's process with Gaussian Ψ could be approximated by the simpler f_Γ distribution.

In Appendix 1, we present another example which could be useful in applications. It is the Campbell's process with a constant Ψ function on a constant interval.

5. NUMERICAL ALGORITHM FOR CAMPELL PROCESSES.

The numerical method we present in this Section is equivalent to computing the response of a fictious (non causal) counter apparatus to a Poissonian flow of incoming particles.

This method is based on the use of a generator for (pseudo) random numbers ("white noise generator"), the precision of which will fix the precision of the obtained results. This generator is defined in Section 5.A. In Section 5.B. we describe the simple method used to build up, from this white noise, a Poissonian flow, i.e. a time series of random dots which can represent, for instance, the time of arrival of the particles on the detector. In Section 5.C, the time intervals given by the Poissonian flow are used to build a continuous random process, the Campbell process. In order to compare the numerical results of the algorithm with the statistical properties derived in the previous Sections, we indicate in Section 5.D. how an ensemble of realizations of the Campbell process has been built. The results of the numerical tests are summarized in Section 5.E.

5.A. Choosing a sampled white noise generator.

Because of its practical importance, the problem of generating random numbers has been widely studied (see e.g. Knuth /4/). The algorithm we use here is the RAN1 one, which can be found in /5/.

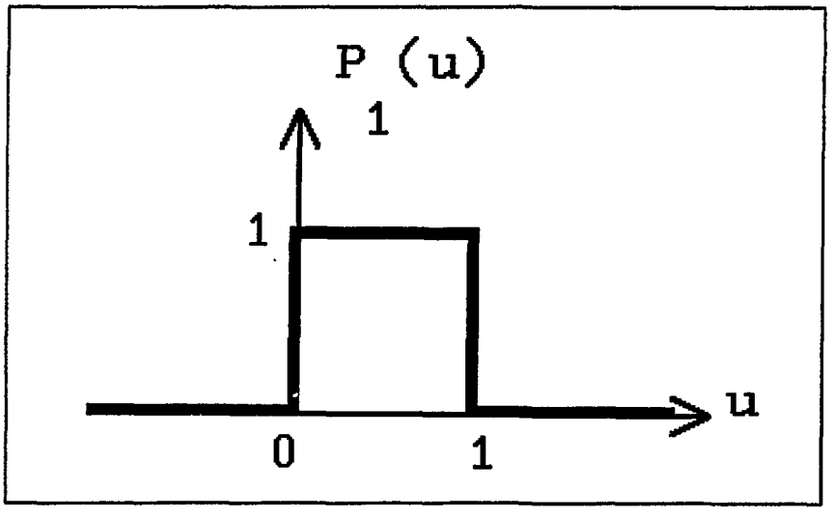
In order to illustrate the principle let us remind that a very simple (but not very precise) algorithm is described by the following iteration procedure ($k = 1, 2, \ldots \infty$) which yield random values for u_k:

$$\left.\begin{aligned} I_k &= \{n_a\, I_{k-1} + n_c\} \text{ modulo } m \\ u_k &= \frac{I_k}{m} \end{aligned}\right\} \tag{37}$$

Here n_a, n_c and m are three integer constants which should be chosen carefully, taking into account the limitations due to the format of the integers in the computer system. The initial value I_0 is also named the *seed*.

Using the language of signal theory, one can say that the series of values of u_k, defined by (37) is (approximately) a *sampled white noise*, with average value 1/2, with variance 1/12 and with a distribution function

$$\left.\begin{aligned} P_1(u) &= 1 \quad \text{for} \quad 0 \le u < 1 \\ P_1(u) &= 0 \quad \text{for} \quad u < 0 \ \text{ or } \ u \ge 1 \end{aligned}\right\} \tag{38}$$



White noise distribution $P_1(u)$.

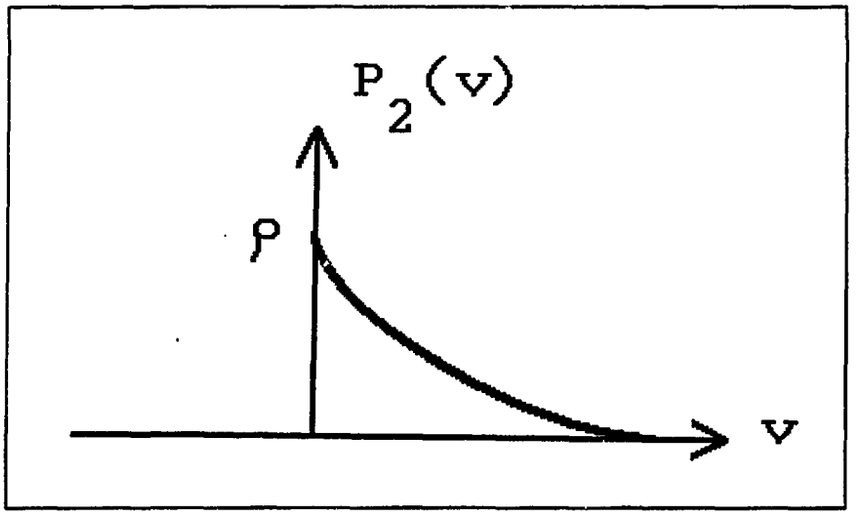
5.B. Poissonian flow: time series as a random set of dots.

From the above uniformly distributed noise u, let us perform the following transformation:

$$v_k = -\frac{1}{\rho} \ln u_k \tag{39}$$

where ρ is a constant representing here the inverse of a time. The process v_k is a sampled (white) noise with average value $1/\rho$, with variance $1/\rho^2$, and with the following distribution function:

$$\left.\begin{aligned} P_2(v) &= \rho e^{-\rho v} \quad \text{for} \quad v \geq 0 \\ P_2(v) &= 0 \quad \text{for} \quad v < 0 \end{aligned}\right\} \tag{40}$$



Gamma distribution $P_2(v)$.

This distribution $P_2(v)$ is found to be a gamma distribution (see 36):

$$P_2(v) = P_\Gamma(v) \quad \text{with} \quad \alpha = 0 \quad \textit{and} \quad \beta = 1/\rho \tag{41}$$

This process actually define a *"Poissonian flow"* /6/, the value of v representing the time intervals between two successive events of the flow. The average time interval is $1/\rho$.

5.C. The Campbell process with Gaussian response function.

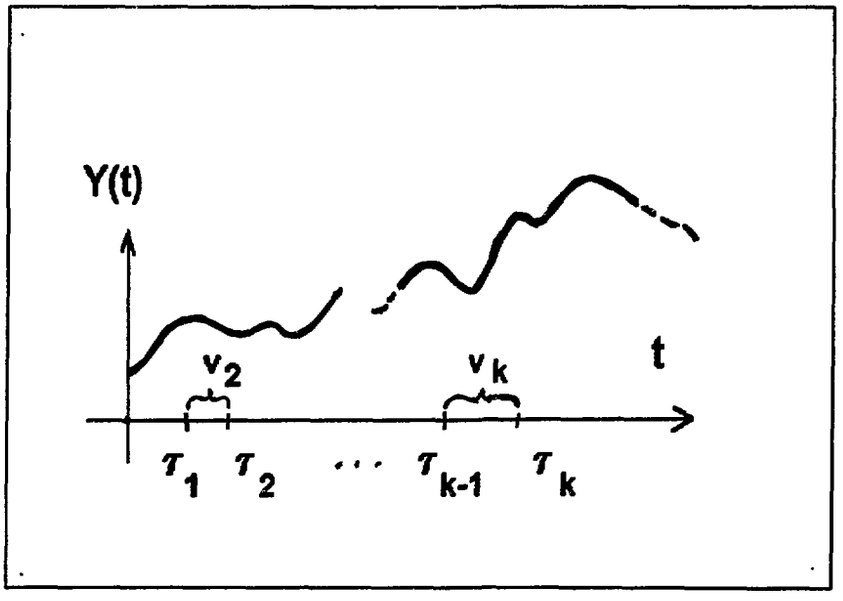
From the above noise describing time intervals with distribution $P_\Gamma(\alpha = 0, \beta = 1/\rho)$, we may simulate a Poissonian flow from the knowledge of the *"arrival times"* τ :

$$\tau_k = v_k + \tau_{k-1} \tag{42}$$

If we now introduce the elementary response function $\Psi(t)$ with a Gaussian form $\Psi(t) = e^{-\alpha t^2}$, we can now generate, from (19), the (continuous) signal of a Campbell process:

$$\boxed{Y(t) = \sum_p e^{-\alpha (t - \tau_p)^2}} \tag{43}$$

Actually the summation over the events of the Poissonian flow should not be performed on the whole time axis from $-\infty$ to $+\infty$. The summation can advantageously be restricted to all p–values such that (for instance) $\alpha (t - \tau_p)^2 < 20$, since $e^{-20} \simeq 2\,10^{-9}$. This implies $(t - \tau_p) < \Delta_{eff.} \equiv \sqrt{\frac{20}{\alpha}}$.

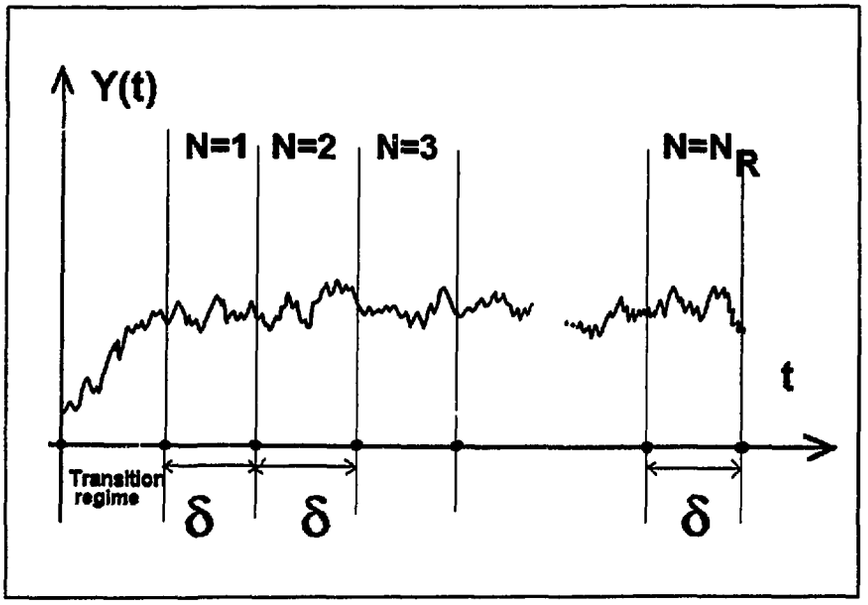


Arrival times τ_k deduced from the Poissonian flow v_k.

5.D. Algorithm for an ensemble of realizations of Campbell processes.

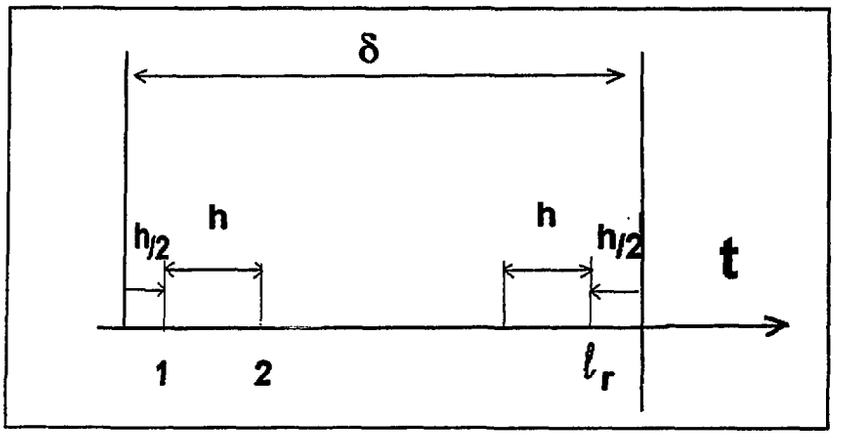
The algorithm we present here actually uses a more efficient random noise generator than the one defined in Eqs. (37), in order to simulate a Poissonian flow from Eqs. (39), (42). From the signal obtained in (43), one can compute characteristic values like the average value, the variance and the distribution function.

In order to numerically test the process, we have to build an *ensemble of realizations* of this process. In this aim, we take N_R partitions in the stationary regime of the obtained signal $Y(t)$: we obtain in this way N_R realizations of length δ.



Partition of the Campbell signal into N_R realizations of length δ.

Moreover, the continuous Campbell process is sampled in time with a time step $h = \delta/\ell_r$, which implies that each realization involves ℓ_r discrete measured values of $Y(t)$.



Time discretization with time step h in each realization.

In summary, we have to chose six parameters as input values for the Campbell generator:

ρ = the average density of Poisson events (per time unit);

$\alpha^{-1/2}$ = temporal width of the elementary Gaussian response $\Psi(t)$ of the counter apparatus (see 43);

h = sampling time step;

germe = value of the initial seed for the white noise generator;

δ = temporal width (duration) of one realization of the Campbell process.

N_R = number of realizations in the ensemble.

We present in Appendix 2 a practical version of the subroutine we propose (in Pascal). This main part of the program could be explained by the following comments.

The subroutine for the generation of the Campbell process involves the following main steps:

- generation of the Poissonian flow $\tau_1, \tau_2, \ldots$;
- computation of the signal $Y(t)$, using (43), on the points of the equally spaced mesh: $t_k \equiv k\,h$ (= uht in Appendix 2). The response functions Ψ are summed only on a finite width $\geq \Delta_{eff.}$ in time (= $duref$ in the subroutine of Appendix 2). The arrival time of the last event at the advanced time $t_k + \Delta_{eff.}$ is denoted by τ_1, and the previous ones within $\Delta_1 + \Delta_2 \geq 2\Delta_{eff.}$ are denoted by $\tau_2, \ldots, \tau_n$.

- splitting of the stationary signal $Y(t)$ in N_R realizations, and computation of the statistical characteristics of the stochastic process.

This procedure is illustrated in Table II. We mention that the algorithm presented has an initial transition regime, meaning that the generated signal $Y(t)$ becomes stationary only after this transient stage.

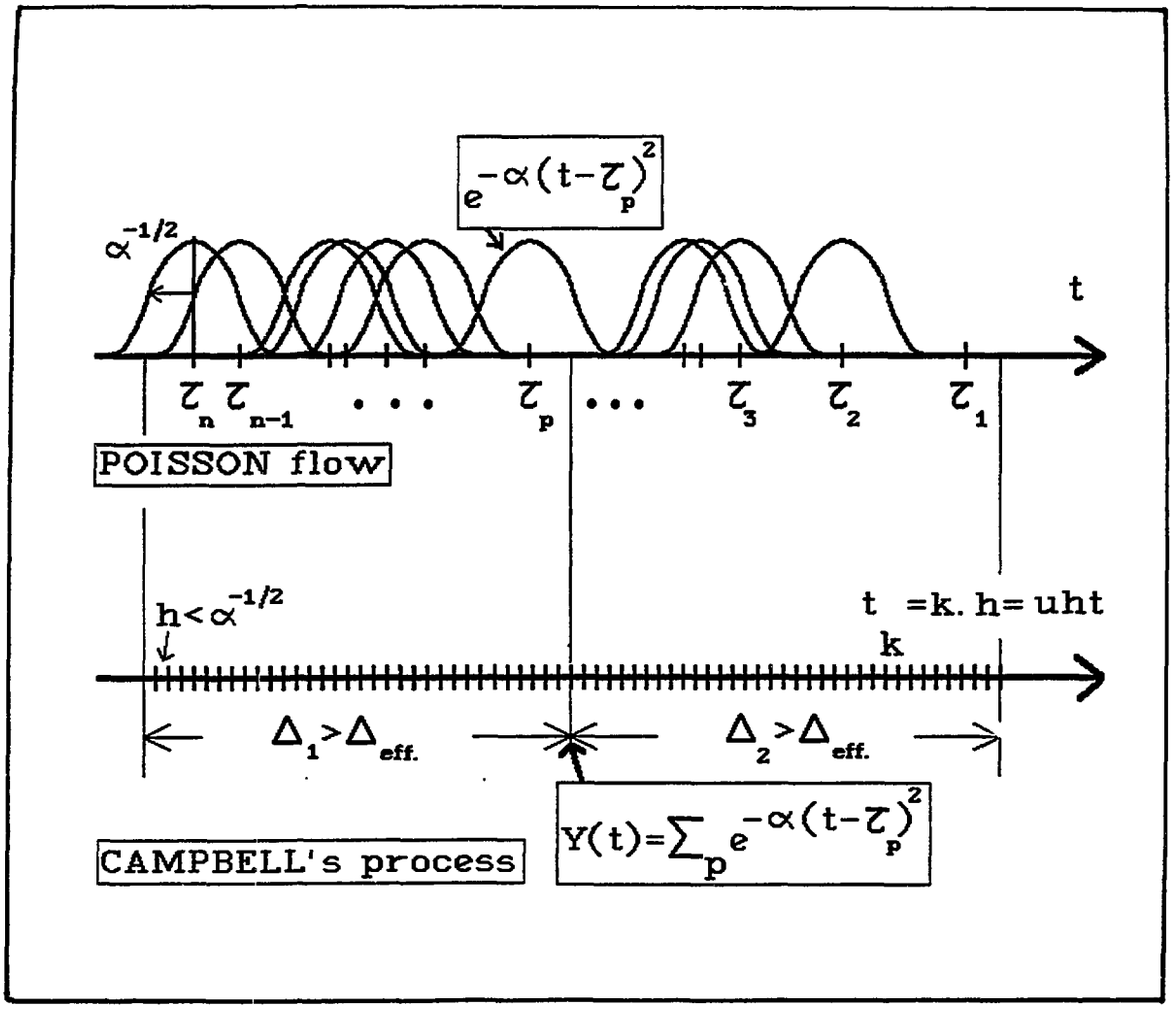


Table II: *Time sequence of the (discrete) Poisson flow, and time steps for the computation of the (continuous) Campbell process.*

5.E. Numerical tests.

The numerical generator for the Campbell process with Gaussian correlations is tested by computing the main statistical parameters of the random function $Y(t)$, and comparing them with the analytical results. The following parameters are considered:

- the average $K_1 = \langle Y(t) \rangle$;
- the variance of $Y(t)$: K_2;
- the correlation function $K(\tau)$;
- the distribution function $P(Y)$.

The stationarity of the average and of the variance are also tested.

The averages contained in K_1, K_2 and $K(\tau)$ are computed as sums of the values of Y at given time(s), taken on each realization, divided by N_R. Denoting by $Y_i(t_k)$ the value of $Y(t_k)$ at the discrete time t_k in the realization i, the average is computed by using

$$K_1(t_k) = \frac{1}{N_R} \sum_{0<i\leq N_R} Y_i(t_k) \tag{44}$$

the correlation from

$$K(t_k - t_{k'}) = \frac{1}{N_R} \sum_{0<i\leq N_R} Y_i(t_k)\, Y_i(t_{k'}) - K_1(t_k)\, K_1(t_{k'}) \tag{45}$$

and the variance is $K(0)$. The distribution function is obtained by counting the values of Y in consecutive intervals δY.

A study of the dependence of these statistical parameters on the number of realizations is performed for various values of the average density of dots ρ and of the width of the response function. Typical examples of the results are presented in Figs.4 and 5 which contain, on the same graph, the variance calculated as function of time, the correlation function $K(\tau)$ and the probability distribution function $P(Y)$. Since we intend to show the accuracy in reproducing the theoretical results, arbitrary units have been used on both axes. Table III presents the parameters considered in Figs.4 and 5, as well as the precision of the results, and the duration of computations made on a *DEC 5000* workstation.

	Fig. 4			Fig. 5		
	a	b	c	a	b	c
$c = \rho\sqrt{\frac{\pi}{\alpha}}$	3	3	3	7	7	7
ℓ_r	40	40	40	40	40	40
N_R	$2^{11} \simeq 2.10^3$	$2^{12} \simeq 4.10^3$	$2^{20} \simeq 10^6$	$2^{11} \simeq 2.10^3$	$2^{12} \simeq 4.10^3$	$2^{15} \simeq 3.10^4$
$\delta\rho$	8	8	8	20	20	20
$\Delta K_1/K_1$	$3.1\,10^{-2}$	$1.5\,10^{-2}$		$1.1\,10^{-2}$	$1.2\,10^{-2}$	$1.7\,10^{-3}$
$\Delta K_2/K_2$	$1.2\,10^{-1}$	$7.7\,10^{-2}$		$7.3\,10^{-2}$	$4.3\,10^{-2}$	$1.9\,10^{-2}$
CPU time (s.)	39	76	19394	88	177	1417

Table III: *Parameters and results of the numerical tests displayed in Figs. 4 and 5.*

We see that this subroutine generates roughly 2.10^3 values of $Y(t)$ per second on this *DEC 5000* workstation, including tests on K_1, K_2 and $K(\tau)$, and graphics.

The values of the average K_1 are very close to the theoretical prediction (20), even with a small number of realizations (see Table III). Also the stationarity is well verified. Since the representations are very close to a constant, the averages (at a given time) are not plotted in Figs. 4 and 5.

The variance is computed for each mesh point, and is represented by curve 1 in Figs. 4 and 5. It can be seen that it is the most sensible parameter to the number of realizations. The relative error is of the order of 0.1 for $N_R \simeq 1\,000$, but the precision can be improved by increasing the number of realizations (see Table III and Figs. 4 and 5).

The numerically computed probability distribution function is represented by curve 3 in Figs. 4 and 5, and the theoretical one by curve 2. A rather good accuracy results even for a small number of realizations.

A good accuracy is obtained for the correlation function (curve 5 in Figs. 4 and 5) which reproduces very well the theoretical Gaussian correlation (curve 4). The small errors in the tail actually appear on the *theoretical* curve, due to the truncation of the infinite summations in the numerical calculation of (33), (34) and (34).

Thus, we can conclude that the statistical characteristics of the generator reproduce rather well the theoretical results if the number of realizations is of the order of 1 000. The errors depend on the initial seed, but become negligible as soon as $N_R \gtrsim 30\,000$.

6. CONCLUSIONS

We have shown how the Campbell process can be used for building a stochastic process with a Gaussian correlation function. An algorithm for generating numerically such a process has been explicitly constructed. Many numerical tests have demonstrated that the statistical characteristics of the generated stochastic process reproduce the theoretical results with a very good accuracy. This algorithm provides a *colored noise* with a Gaussian correlation function.

The physical parameters which represent the input of the numerical generator are the width of the Gaussian correlation function $\sqrt{\frac{2}{\alpha}}$, and the average amplitude c.

In practical applications, the procedure and the subroutine given here can be used to generate the random "force" appearing, for instance, in a Langevin stochastic equation describing the motion of a test particle in stochastic fields (see e.g. /7/).

An extension of the algorithm for other prescribed shapes of the correlation function can easily be done (see Appendix 1 for tent-like correlations).

Acknowledgements.

M.V. and F.S. acknowledge the warm hospitality of the members of the "Département de Recherches sur la Fusion Contrôlée" of the C.E. - Cadarache, where this work has been performed. The authors want to thank Pr. Radu Balescu for stimulating discussions on this subject. F.S. wishes to thank the "Ministère de la Recherche et de l'Espace" for having financially supported his stay in France.

APPENDIX 1: TENT-LIKE CORRELATIONS.

Let us consider another example which could be useful in applications. It is the Campbell's process with a constant Ψ function :

$$\Psi(t) = \begin{cases} h & t \in [0, \Delta] \\ 0 & t \notin [0, \Delta] \end{cases} \tag{A.1}$$

The cumulants are in this case $K_m = \rho \Delta h^m$ and the characteristic function can be obtained explicitely:

$$\log G(k) = \rho \Delta \left(e^{ikh} - 1 \right) \tag{A.2}$$

The probability distribution function for the values of Y is:

$$P(Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} exp \left[\rho \Delta \left(e^{ikh} - 1 \right) - ikY \right] dk =$$

$$= \frac{1}{2\pi} e^{-\rho\Delta} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(\rho\Delta)^n}{n!} e^{inkh - ikY} dk =$$

$$= e^{-\rho\Delta} \sum_{n=0}^{\infty} \frac{(\rho\Delta)^n}{n!} \delta(Y - nh) \tag{A.3}$$

Thus, only the positive integer multiples of h ($Y = kh$) have a nonzero probability which is:

$$P(kh) = \frac{(\rho\Delta)^k}{k!} e^{-\rho\Delta} \tag{A.4}$$

i.e. a Poisson distribution (with the average $\rho\Delta$, representing the average number of dots in the time interval Δ). The correlation is a tent-like function:

$$K(\tau) = \begin{cases} \rho h^2 (\Delta - |\tau|) & |\tau| < \Delta \\ 0 & |\tau| \geq \Delta \end{cases} \tag{A.5}$$

APPENDIX 2: LISTING OF THE SUBROUTINE IN PASCAL.

```
{      -------     SIMULATION D'UN PROCESSUS DE CAMPBELL     ----------

Partie principale du programme en PASCAL  (Pascal for RISC sur DECstation)

      hh : pas d'echantillonnage choisi(=h), hs2=hh/2
     delt: intervalle temporel aleatoire poissonnien (de moyenne 1/rho)
     Lrh : nombre d'echantillons par realisation
     kfin:=kin3+NRea*Lrh-1; tfin:=kfin*hh;
    Mrec : moyenne obtenue par recurrence,  srec :variance ...
    inca(j) ==>  j=j+1
================================================================================

          ### Initialisation  }
 begin  INIT; duref:=sqrT(20/alpha);  durph:=duref+hh;     uht:=hs2;
                  t:=0;       k:=0;        st:=0;{=nb evenements}
               Mrec:=0;    srec:=0;         s:=0;

                  write('GERME  = ');readln(germe);

REPEAT { ### Boucle principale ==> un evenement poissonnien au temps t }
    delt:=POISSON(germe);      t:=t+delt;       if t<tfin then inca(st);

       { ### Numerotation  des temps TAU des evenements  }
n:=0;repeat if n>=st-1 then break;if uht-TAU[n+1]>duref then break;inca(n)
     until false; { ###  narg judicieusement grand }
                    for i:=narg downto 2 do TAU[i]:=TAU[i-1]; TAU[1]:=t;

  WHILE{ ### Boucle interne ==> instants d'echantillonnage uht }
  k*hh+durph < t DO BEGin inca(k);      uht:=k*hh+hs2;

       { ### Signal de Campbell au temps uht:formule  (43)  }
  Yuht:=0; for i:=1 to n do Yuht:=Yuht+exp(-alpha*sqr(uht-TAU[i]));


       { ### Pour verification et details :formules par recurrence  }
    IF k>kin2 then BEgin kbis:=k-kin2;
    Mrec:=((kbis-1)*Mrec+Yuht)/kbis;srec:=srec+sqr(Yuht-Mrec);s:=s+Yuht;
       { ### Totaux pour la moyenne et l'autocorrelation }
     IF k=kin3 then INIAK3; { regle certains details connaissant Mrec }
     IF k>=kin3 then Begin j:=(k-kin3)mod Lrh +1; { ### donc j=1..Lrh }
       X[j] :=Yuht;          MODI[j] :=MODI[j]+Yuht;
       for i:=j downto 1 do COR[j,i]:=COR[j,i]+X[j]*X[i];


      { ### Sortie commune aux IF et  aux boucles }
IF k=kfin then BREAK;End{if kin3} ENd{if kin2} END{while} UNTIL k=kfin;


      { ###   ensuite diviser par NRea=nombre de realisations
   MODI[i] :=MODI[i] /NRea;      ==> moyennes ( i=1..Lrh )
   COR[i,j]:=COR[i,j]/NRea-MODI[i]*MODI[j]    ==>  autocorrelation }
      { ### moyenne globale,           variance globale}
           Mfincal:=s/kbis;           srec:=srec/kbis;
```

```
{  FUNCTIONS   GENERANT  DES  SUITES  PSEUDO  ALEATOIRES

                    listing apparaissant dans /5/ :
   "Four-byte integers are essential for this routine" (valeurs ses CONST)
   "Set idum to any negative value to initialize the sequence "
    ral1, ral2, ral3:integer; et RMA=array[0..nbc]of double; sont globales
    nbc=97 (arbitraire) }

 function RAN1(var idum:integer):double;
{##############}
CONST    m1=259200; ia1=7141; ic1=54773; rm1=3.8580247E-6;
         m2=134456; ia2=8121; ic2=28411; rm2=7.4373773E-6;
         m3=243000; ia3=4561; ic3=51349;      var j:integer;  begin
IF idum<0 then BEGin
  ral1:=(ic1-idum)mod m1;
  ral1:=(ia1*ral1+ic1)mod m1;
  ral2:=ral1 mod m2;
  ral1:=(ia1*ral1+ic1)mod m1;
  ral3:=ral1 mod m3;
for j:=1 to nbc do Begin
         ral1:=(ia1*ral1+ic1)mod m1;
         ral2:=(ia2*ral2+ic2)mod m2;
       RMA[j]:=(ral1+ral2*rm2)*rm1 End; idum:=1 END;
  ral1:=(ia1*ral1+ic1)mod m1;
  ral2:=(ia2*ral2+ic2)mod m2;
  ral3:=(ia3*ral3+ic3)mod m3;
     j:=1 + (nbc*ral3)div m3;
if(j>nbc)or(j<1)then begin write('ANOMALIE GRAVE dans RAN1');readln end;
  RAN1:=RMA[j] ;          RMA[j]:=(ral1+ral2*rm2)*rm1 end;

{=============================================================================}



     { Utilisation de (39) avec delta TAU moyen = detamo = 1/rho }

 function POISSON(var idum:integer):double;         var adu:double;
{################}
begin Repeat adu:=RAN1(idum);Until adu<>0; POISSON:= - detamo * Ln(adu) end;

{=============================================================================}
```

REFERENCES.

/1/. R.H. CAMPBELL, *Proc. Camb. Phylos. Soc.* **15**, (1909) 117; S.O. RICE, *Bell System Technical Journal* 23 (1944) 282, **24** (1945) 46.

/2/. H.G. VAN KAMPEN *"Stochastic Processes in Physics and Chemistry"* , North Holland, Amsterdam (1981).

/3/. V.A. SVETLICKIJ *"Vibrations aléatoires des systèmes mécaniques"* Technique et Documentation, Paris 1976.

/4/. D.E. KNUTH: *"Seminumerical Algorithms"* Vol.2 of *"The Art of Computer Programming"* , 2d. ed., Addison-Westley, Reading (1981).

/5/. W.H. PRESS, B.P. FLANNERY, S.A. TEUKOLSKY, W.T. VETTERLING: *"Numerical Recipes in Pascal"*, Cambridge University Press, Cambridge (1989).

/6/. Y. ROZANOV: *"Processus Aléatoire"* (in French), Editions Mir, Moscou (1975).

/7/. J.H. MISGUICH, R. BALESCU, R. NAKACH, p.521 in *"Plasma Physics and Controlled Nuclear Fusion Research 1988"* , Proc. 12^{th} I.A.E.A. Int. Conf., Nice 12-19 Oct 1988, I.A.E.A., Vienna (1989); M. PETTINI, A. VULPIANI, J.H. MISGUICH, M. DE LEENER, J. ORBAN, R. BALESCU, *Phys. Rev. A* **38** (1988) 344.

Figure Captions

Fig. 1. The theoretical distribution function $P(Y)$ for different values of c, the average of Y.

Fig. 2. The Gamma distribution function corresponding to the same average and variance as $P(Y)$ in Fig. 1.

Fig. 3. Tests of the numerical generator for $c = 3$: the variance as a function of time (curve [1]); the distribution function $P(Y)$: numerical result (curve [3]) and expression (33) (curve [2]); the correlation function $K(\tau)$ (computed at the final time t_{ℓ_r} of each realization, and averaged over the ensemble): numerical results (curve [5]) and expression (21), (27) (curve [4]). The parameters for this test are presented in Table III.

Fig. 4. The same as in Fig. 3 but for $c = 7$.

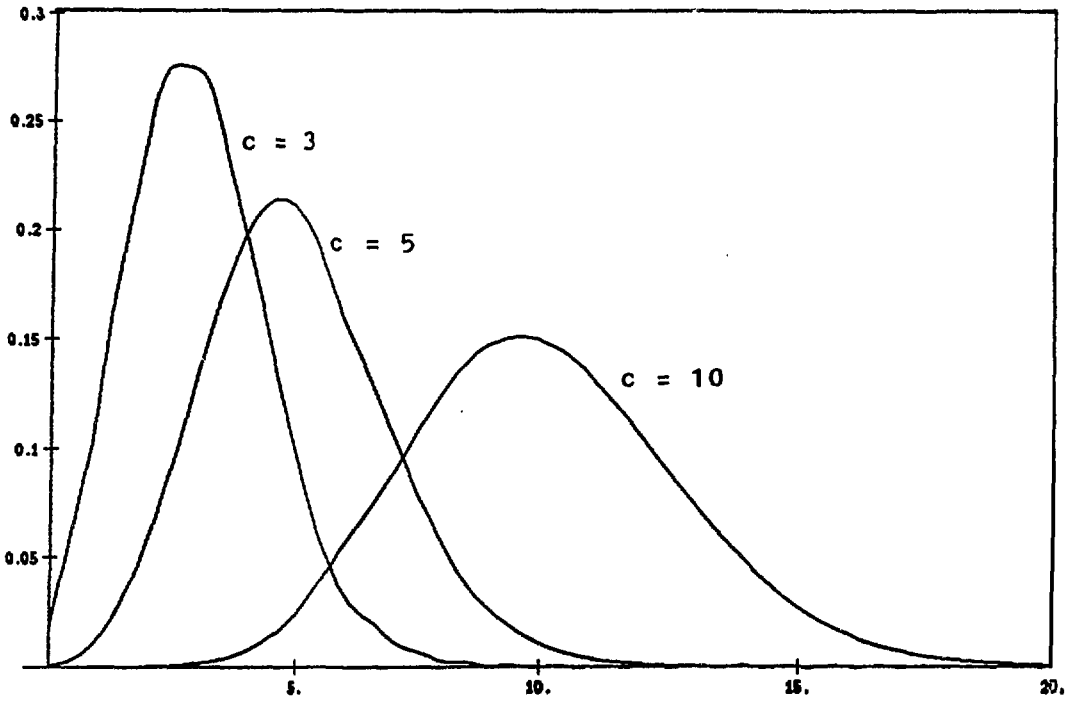


FIGURE 1

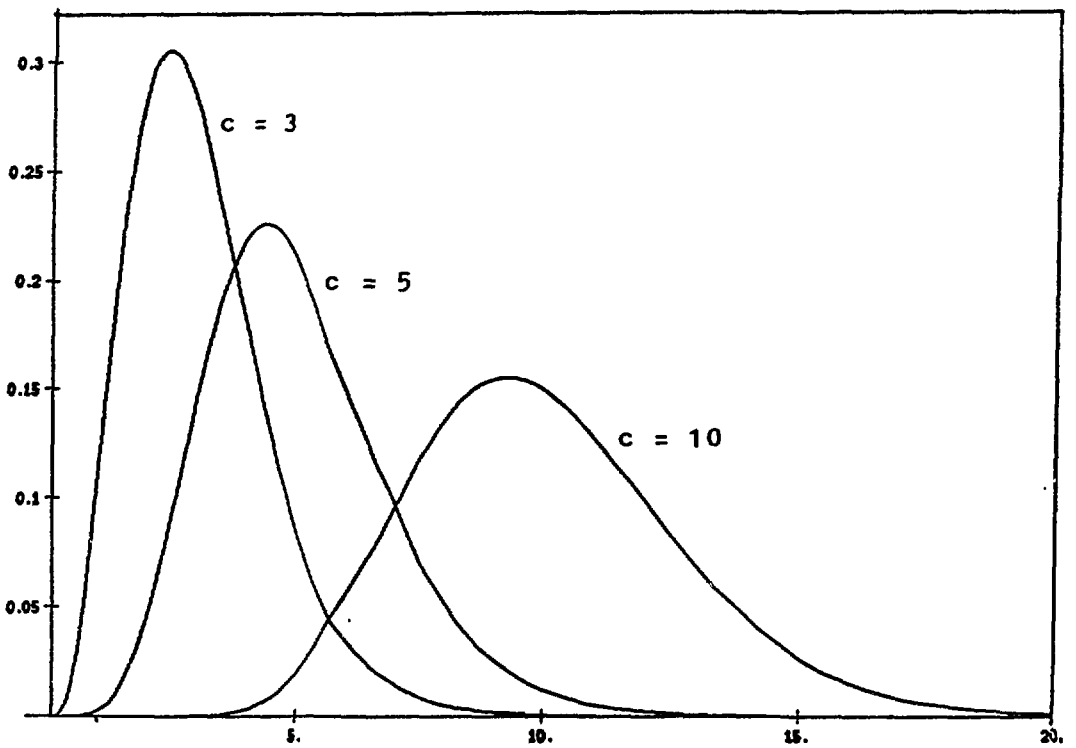


FIGURE 2

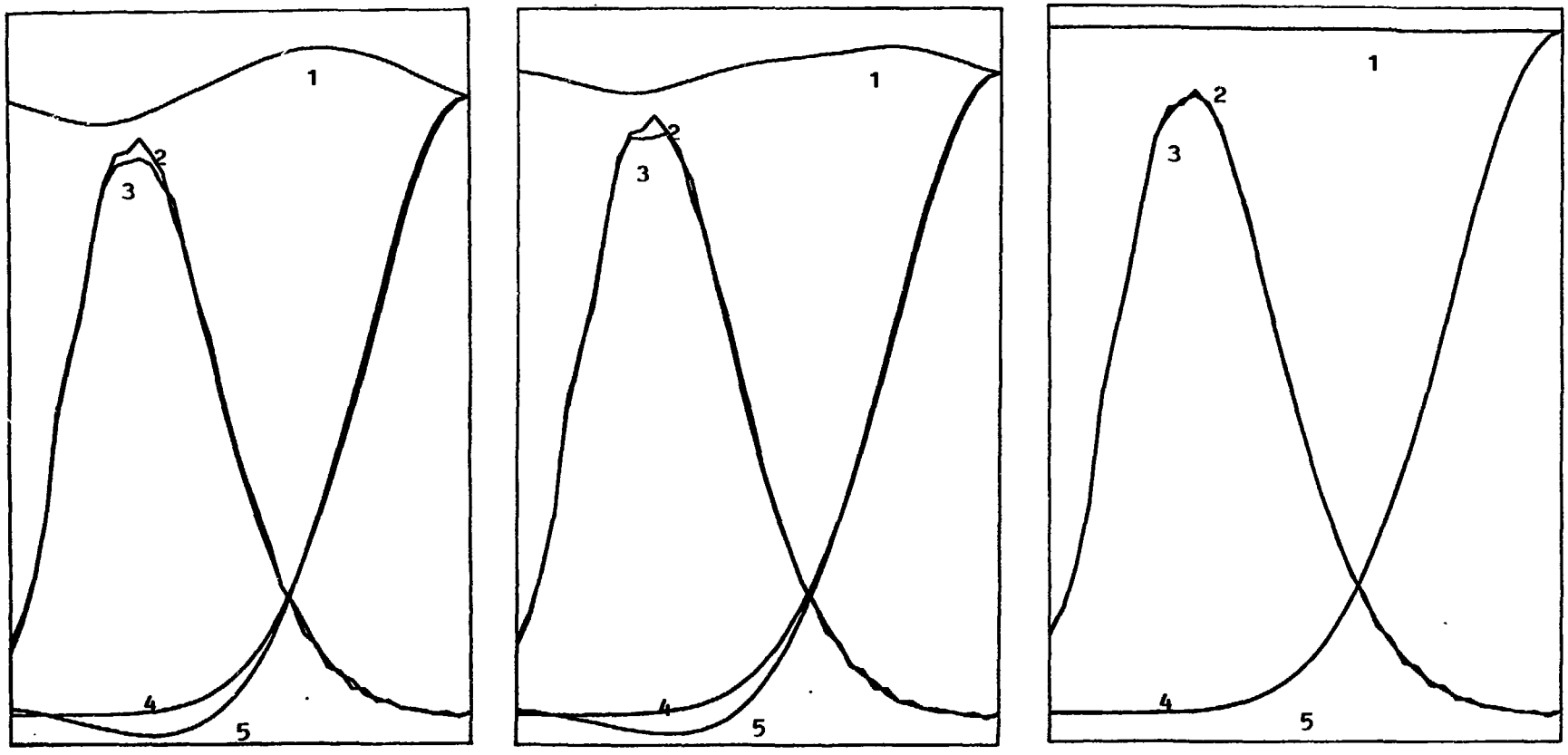


FIGURE 3

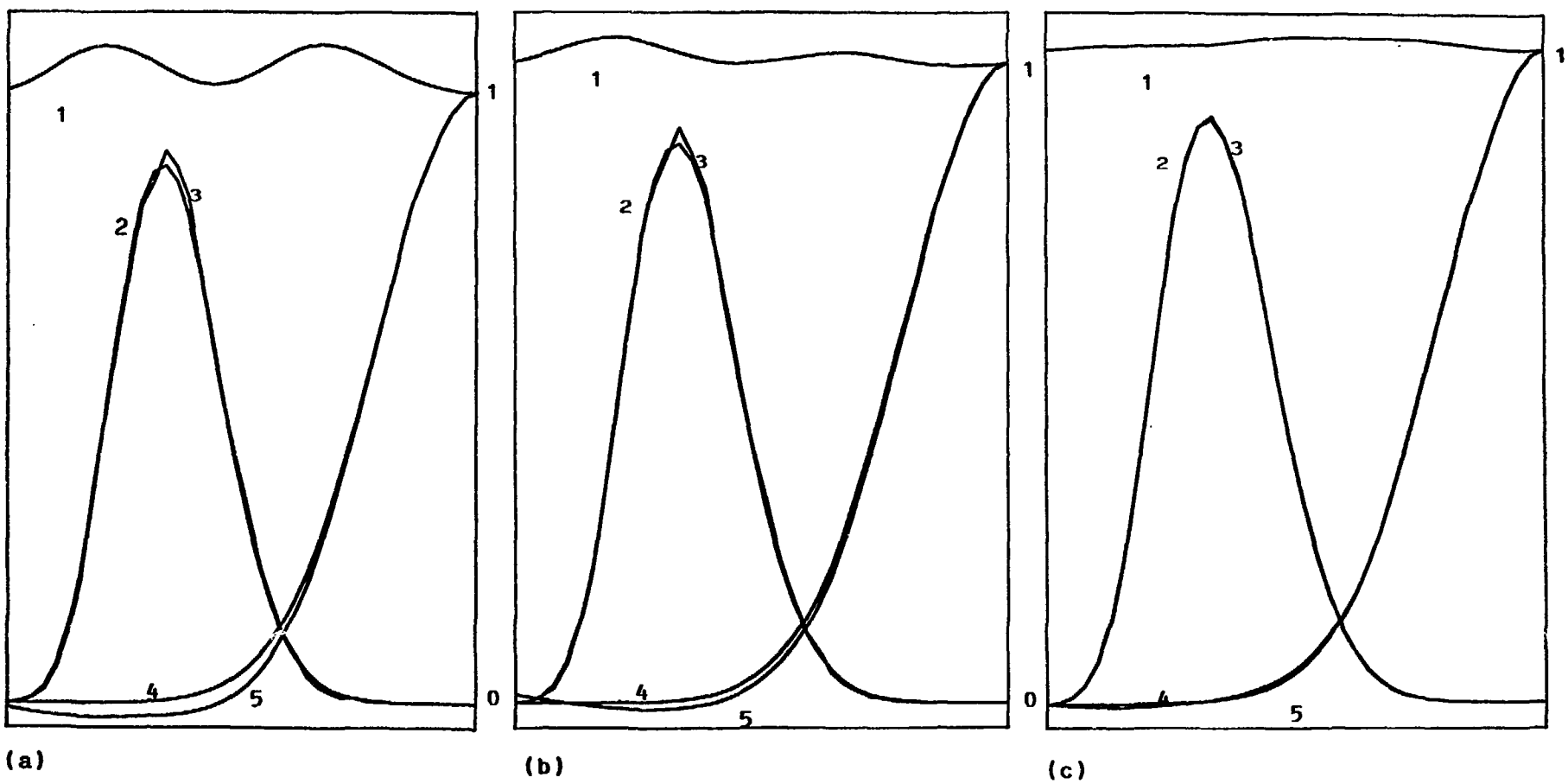


FIGURE 4