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## INTRODUCTION TO QUANTUM ALGEBRAS\*

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### Abstract

The concept of a quantum algebra is made easy through the investigation of the prototype algebras  $u_{qp}(2)$ ,  $su_q(2)$  and  $u_{qp}(1,1)$ . The latter quantum algebras are introduced as deformations of the corresponding Lie algebras ; this is achieved in a simple way by means of  $qp$ -bosons. The Hopf algebraic structure of  $u_{qp}(2)$  is also discussed. The basic ingredients for the representation theory of  $u_{qp}(2)$  are given. Finally, in connection with the quantum algebra  $u_{qp}(2)$ , we discuss the  $qp$ -analogues of the harmonic oscillator and of the (spherical and hyperbolic) angular momenta.

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## Abstract

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## 1. Where Does It Come From ?

The notion of deformation is very familiar to the physicist. In this connection, quantum mechanics may be considered as a deformation (the deformation parameter being  $\hbar$ ) of classical mechanics and relativistic mechanics is, to a certain extent, another deformation (with  $1/c$  as deformation parameter) of classical mechanics. Although a sharp distinction should be established between *deformations* and *quantized universal enveloping algebras* or *quantum algebras*, the concept of a quantum algebra is more easily introduced in the parlance of deformations. Along this vein, the idea of deformed bosons, introduced as early as in the seventies [1,2,5], plays an important role.

The concept of a quantum algebra (or quantum group) goes back to the end of the seventies. It was introduced, under different names, by Kulish, Reshetikhin,

Sklyanin, Drinfeld (from the Faddeev school) and Jimbo [3,4,6,7] in terms of a *quantized universal enveloping algebra* or an *Hopf bi-algebra* and, independently, by Woronowicz [8] in terms of a *compact matrix pseudo-group*.

Among the various motivations that led to the concept of a quantum group, we have to mention the quantum inverse scattering technique, the solution of the quantum Yang-Baxter equation and, more generally, the study of exactly solvable models in statistical mechanics. More recent applications of quantum algebras concern : conformal field theories in two dimensions ; (quantum) dynamical systems ; quantum optics ; molecular, atomic and nuclear spectroscopies ; condensed matter physics ; knot theory, theory of link invariants (Jones polynomials) and braid groups ; and so on. In addition, the concept of a quantum group constitutes a basic tool in non-commutative geometry. Thus, quantum groups are of paramount importance not only in physics and quantum chemistry but in pure mathematics equally well.

It is the aim of this series of lectures to give a primer on quantum algebras. The lectures are organized as follows. Some  $qp$ -deformed bosons and a  $qp$ -deformed harmonic oscillator are introduced in §2 and 3. In §4, representations other than the Fock representation are given for the deformed boson algebra. Section 5 deals with the quantum algebras  $u_{qp}(2)$ ,  $su_q(2)$  and  $u_{qp}(1,1)$  while the representation theory and the Hopf algebraic structure of  $u_{qp}(2)$  are considered in §6. Section 7 is devoted to an (incomplete) classification of the possible applications of quantum algebras. Finally, a sketch of bibliography is given in §8.

## 2. Introducing $qp$ -Bosons

We start with the usual (one-particle) Fock space

$$\mathcal{F} = \{|n\rangle : n \in \mathbb{N}\} \quad (1)$$

which is very familiar to the physicist. The state vectors  $|n\rangle$  are the eigenstates for an ordinary harmonic oscillator in one dimension. In the terminology to be used below,  $\mathcal{F}$  is a *non-deformed* Fock space.

**Definition 1.** Let us define the linear operators  $a^+$ ,  $a$  and  $N$  on the vector space  $\mathcal{F}$  by the relations

$$a^+ |n\rangle = \sqrt{[n+1]} |n+1\rangle \quad a |n\rangle = \sqrt{[n]} |n-1\rangle \quad N |n\rangle = n |n\rangle \quad (2)$$

with  $a |0\rangle = 0$ , where we use the notation

$$[c] \equiv [c]_{qp} = \frac{q^c - p^c}{q - p} \quad (c \in \mathbb{C}), \quad (3)$$

the two parameters  $p$  and  $q$  being fixed parameters taken (a priori) in the field of complex numbers  $\mathbb{C}$ .

It is to be observed that in the limiting case  $p = q^{-1} \rightarrow 1$ , we have simply  $[c] = c$  so that  $a^+$ ,  $a$  and  $N$  are (respectively) in this case the ordinary creation, annihilation and number operators occurring in various areas of theoretical physics. In the other cases, the operators  $a^+$  and  $a$  defined by equations (1-3) are called  $qp$ -deformed creation and annihilation operators, respectively, and they are collectively referred to as  $qp$ -deformed bosons or simply  $qp$ -bosons. Observe that the operator  $N$  is a non-deformed operator that coincides with the usual number operator.

The (complex) number  $[c]$  defined by (3) is a  $qp$ -deformed number. It can be rewritten as

$$[c] = \frac{\sinh(c \frac{s-r}{2})}{\sinh(\frac{s-r}{2})} \exp \left[ (c-1) \frac{s+r}{2} \right] \quad (4)$$

where

$$s = \ln q \quad r = \ln p \quad (5)$$

(Some algebraic relations satisfied by such  $qp$ -deformed numbers are listed in the Appendix.) Two particular situations are of special interest, viz.,  $p = q^{-1}$  ( $r = -s$ ) and  $p = 1$  ( $r = 0$ ). The situation  $p = q^{-1}$ , for which

$$[c] = \frac{q^c - q^{-c}}{q - q^{-1}} = \frac{\sinh(c \ln q)}{\sinh(\ln q)}, \quad (6)$$

is mainly encountered in the physical literature while the situation  $p = 1$ , for which

$$[c] = \frac{q^c - 1}{q - 1} = q^{\frac{c-1}{2}} \frac{\sinh(\frac{c}{2} \ln q)}{\sinh(\frac{1}{2} \ln q)}, \quad (7)$$

comes from the mathematical literature. Observe that for  $c \in \mathbb{R}$ , the numbers  $[c]$  given by (6) are real for  $q \in \mathbb{R}$  or  $q \in S^1$ .

Note that a simple iteration of equation (2) yields

$$|n\rangle = \frac{1}{\sqrt{[n]!}} (a^+)^n |0\rangle \quad (8)$$

where the  $qp$ -deformed factorial  $[n]!$  is defined by

$$[n]! = [n][n-1] \cdots [1] \quad (9)$$

for  $n \in \mathbb{N}$  with  $[0]! = 1$ .

Property 1. As a trivial property, we have

$$(a)^\dagger = a^+ \quad (N)^\dagger = N \quad [N, a^+] = a^+ \quad [N, a] = -a \quad (10)$$

where  $(X)^\dagger$  denotes the adjoint of the operator  $X$  and  $[X, Y] \equiv [X, Y]_- = XY - YX$  the commutator of  $X$  and  $Y$ . Equation  $(a)^\dagger = a^+$  is valid under the condition (which is supposed to hold in this paper) that the  $qp$ -deformed numbers  $[n]$ ,  $n \in \mathbb{N}$ , are real ; this is certainly the case if  $q \in \mathbb{R}$  and  $p \in \mathbb{R}$  or if  $q = p^{-1} \in S^1$ .\*

Property 2. As a basic property, we can check that

$$aa^+ = [N + 1] \quad a^+a = [N] \quad (11)$$

where we use the abbreviation

$$[X] \equiv [X]_{qp} = \frac{q^X - p^X}{q - p} \quad X \in \mathcal{F} \quad (12)$$

which parallels for operators the defining relation (3) for numbers.

As a corollary of Property 2, we can derive the following expression for the  $Q$ -mutator  $[a, a^+]_Q$  :

$$[a, a^+]_Q \equiv aa^+ - Qa^+a = \frac{1}{q - p} [q^N(q - Q) - p^N(p - Q)] \quad (13)$$

where  $Q$  may be any complex number. (The  $Q$ -mutator reduces to the ordinary commutator when  $Q = 1$ .) Three special cases are of importance. First, for  $p = 1$  we have

$$[a, a^+]_q = 1 \quad [a, a^+]_1 = q^N \quad (14)$$

that correspond to the deformed bosons encountered mainly in the mathematical literature. (In passing, note that the relation  $[a, a^+]_q = 1$ , which interpolates between fermions and bosons for  $-1 \leq q \leq 1$ , may be of interest in anyonic statistics.) Second, for  $p = q^{-1}$  we have

$$[a, a^+]_q = q^{-N} \quad [a, a^+]_{q^{-1}} = q^N \quad [a, a^+]_1 = \frac{1}{q - q^{-1}} [q^N(q - 1) - q^{-N}(q^{-1} - 1)] \quad (15)$$

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\* Note that for  $q = p^{-1} = \exp(i\varphi)$ , with  $\varphi \in \mathbb{R}$ , the  $q$ -deformed number  $[n + 1]$ , with  $n \in \mathbb{N}$ , is nothing but the Chebyshev polynomial of the second kind  $U_n(\cos \varphi)$ .

that correspond to the deformed bosons introduced in the physical literature. Third, for generic  $p$  and  $q$  we have

$$[a, a^+]_q = p^N \quad [a, a^+]_p = q^N \quad [a, a^+]_1 = \frac{1}{q-p} [q^N(q-1) - p^N(p-1)] \quad (16)$$

that subsume (14) and (15) and that lead to the usual commutation relations for ordinary bosons (corresponding to  $p = q^{-1} \rightarrow 1$ ).

At this point, it is worth to note that any set  $\{a, a^+\}$  of  $qp$ -bosons acting on the Fock space  $\mathcal{F}$  can be connected to any set  $\{b, b^+\}$  of  $hf$ -bosons acting on the same space  $\mathcal{F}$ . Indeed, we have

$$a = s(N+1) b = b s(N) \quad a^+ = b^+ s(N+1) = s(N) b^+ \quad (17)$$

where the (operator-valued) scaling factor  $s$  is given by

$$s(N) = \left( \frac{[N]_{qp}}{[N]_{hf}} \right)^{\frac{1}{2}} \quad (18)$$

Of course, the passage expressions (17-18) require that none of the involved  $[N]$ -operators be vanishing. This excludes, in particular, that  $q$  be a root of unity when  $p = q^{-1}$ . As a particular case, any set  $\{a, a^+\}$  of  $qp$ -bosons can be related through (17-18) to a set  $\{b, b^+\}$  of ordinary bosons corresponding to the limiting case  $f = h^{-1} \rightarrow 1$ . As another particular case, the set of  $qp$ -bosons  $\{a, a^+\}$  with  $p = q^{-1}$  can be connected to the set of  $hf$ -bosons  $\{b, b^+\}$  with  $f = 1$  and  $h = q^2$  through the relations

$$b = q^{\frac{N}{2}} a = a q^{\frac{N-1}{2}} \quad b^+ = a^+ q^{\frac{N}{2}} = q^{\frac{N-1}{2}} a^+ \quad (19)$$

which follow from (17-18); equation (19) thus allows us to pass from the deformed bosons satisfying

$$aa^+ - qa^+a = q^{-N} \quad aa^+ - q^{-1}a^+a = q^N \quad (20)$$

to the deformed bosons satisfying

$$bb^+ - q^2b^+b = 1 \quad (21)$$

Finally, note that equations (17-18) make it possible to connect the set (of ordinary bosons)  $\{b, b^+\}$  with  $f = h^{-1} \rightarrow 1$  satisfying  $bb^+ - b^+b = 1$  to the set (of fermions)  $\{a, a^+\}$  with  $p = -q \rightarrow 1$  satisfying  $aa^+ + a^+a = 1$ .

Going back to the more general  $qp$ -bosons, we give a few elements useful for constructing  $qp$ -deformed coherent states. First, let us make the replacements

$$|n\rangle \mapsto \frac{1}{\sqrt{[n]!}} z^n \quad a^+ \mapsto z \quad a \mapsto D_{qp} \quad N \mapsto z \frac{\partial}{\partial z} \quad (22)$$

where  $z \in \mathbb{C}$  and  $D_{qp}$  is the finite difference operator defined via its action on any function  $f(x)$  as

$$D_{qp} f(x) = \frac{f(qx) - f(px)}{(q-p)x} \quad (23)$$

(For evident reasons, the operator  $D_{qp}$  may be called  $qp$ -derivative. Some information on  $D_{qp}$  may be found in the Appendix.) It is easy to check that equation (22) defines a realization (a  $qp$ -deformed Bargmann realization, indeed) of the basic relations (2) defining the number operator  $N$  and the  $qp$ -bosons  $a$  and  $a^+$ . Second, let us introduce in  $\mathcal{F}$  the state vector

$$|z\rangle = \sum_{n=0}^{+\infty} \frac{1}{\sqrt{[n]!}} z^n |n\rangle \quad (24)$$

It can be checked that the eigenvalue equation

$$a |z\rangle = z |z\rangle \quad (25)$$

holds in the space  $\mathcal{F}$ . Therefore, the state  $|z\rangle$  can be considered as a (non-normalized)  $qp$ -deformed coherent state. The normalized  $qp$ -deformed coherent state reads

$$|z\rangle = (\exp_{qp}(|z|^2))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{1}{\sqrt{[n]!}} z^n |n\rangle \quad (26)$$

where the  $qp$ -deformed exponential  $\exp_{qp}$  is defined by

$$\exp_{qp}(x) = \sum_{n=0}^{+\infty} \frac{x^n}{[n]!} \quad (27)$$

(See the Appendix for a property of the  $qp$ -exponential.) We stop here our discussion around  $qp$ -deformed coherent states. (A complete study requires the introduction of a measure on the  $qp$ -deformed Bargmann space  $\{z^n / \sqrt{[n]!} : n \in \mathbb{N}\}$ .)

### 3. A $qp$ -Analogue of the Harmonic Oscillator

We are now in a position to introduce a  $qp$ -deformed harmonic oscillator. The literature on this subject is now abundant and the reader may consult, for

example, Refs. [5,10,11] for further details, especially for the particular situations where  $p = q^{-1}$  and  $p = 1$ .

**Definition 2.** From the  $qp$ -deformed creation and annihilation operators  $a$  and  $a^+$ , let us define the operators

$$p_x = i \sqrt{\frac{\hbar\mu\omega}{2}} (a^+ - a) \quad x = \sqrt{\frac{\hbar}{2\mu\omega}} (a^+ + a) \quad (28)$$

acting on  $\mathcal{F}$ , where  $\hbar$ ,  $\mu$  and  $\omega$  have their usual meaning in the context of the (ordinary) harmonic oscillator.

Equation (28) defines  $qp$ -deformed momentum and position operators  $p_x$  and  $x$ , respectively, and bears the same form as for the ordinary creation and annihilation operators corresponding to the limiting case  $p = q^{-1} \rightarrow 1$ .

**Property 3.** The commutator of the  $qp$ -deformed operators  $x$  and  $p_x$  is

$$[x, p_x] = i\hbar [a, a^+] = i\hbar ([N+1] - [N]) = i\hbar \frac{1}{q-p} [q^n(q-1) - p^n(p-1)] \quad (29)$$

which reduces to the ordinary value  $i\hbar$  in the limiting case  $p = q^{-1} \rightarrow 1$ .

Thus, we may think of a  $qp$ -deformed ( $N$ -dependent) uncertainty principle. In particular for  $p = q^{-1}$ , equation (29) can be specialized as

$$[x, p_x] = i\hbar \frac{\cosh[(n + \frac{1}{2}) \ln q]}{\cosh(\frac{1}{2} \ln q)} \quad (30)$$

in terms of eigenvalues. The right-hand side of (30) increases with  $n$  (i.e., with the energy, see equation (33) below) and is minimum as well as  $n$ -independent in the limiting case  $q = 1$  [11].

**Definition 3.** We define the self-adjoint operator  $H$  on  $\mathcal{F}$  by

$$H = \frac{1}{2\mu} p_x^2 + \frac{1}{2} \mu \omega^2 x^2 = \frac{1}{2} (a^+ a + a a^+) \hbar \omega = \frac{1}{2} ([N] + [N+1]) \hbar \omega \quad (31)$$

in terms of the  $qp$ -deformed operators previously defined.

In the limiting case  $p = q^{-1} \rightarrow 1$ , the operator  $H$  is nothing but the Hamiltonian for a one-dimensional harmonic oscillator. We take equation (31) as the defining relation for a  $qp$ -deformed one-dimensional harmonic oscillator. The case of a  $qp$ -deformed  $d$ -dimensional, with  $d \geq 2$ , (isotropic or anisotropic) harmonic oscillator can be handled from a superposition of one-dimensional  $qp$ -deformed oscillators.

Property 4. The spectrum of  $H$  is given by

$$E \equiv E_n = \frac{1}{2} ([n] + [n+1]) \hbar \omega = \frac{1}{2} \frac{q^n(q+1) - p^n(p+1)}{q-p} \hbar \omega \quad n \in \mathbb{N} \quad (32)$$

and is discrete.

This spectrum turns out to be a deformation of the one for the ordinary one-dimensional harmonic oscillator corresponding to the limiting case  $p = q^{-1} \rightarrow 1$ . The levels are shifted (except the ground level) when we pass from the ordinary harmonic oscillator to the  $qp$ -deformed harmonic oscillator : the levels are not uniformly spaced.

In the special case  $p = q^{-1}$ , equation (32) yields

$$E_n = \frac{1}{2} \frac{\sinh[(n + \frac{1}{2}) \ln q]}{\sinh(\frac{1}{2} \ln q)} \hbar \omega \quad (33)$$

and, as expected,  $E_n$  is real for  $q \in S^1$  ( $q = \exp(i\varphi)$  with  $\varphi \in \mathbb{R}$ ) or  $q \in \mathbb{R}$ . In this case, we have

$$E_{n+1} - E_n = \hbar \omega \cosh[(n+1) \ln q] \quad (34)$$

which is  $n$ -independent only for  $q = 1$ .

At this place, it is interesting to mention that we may think to obtain the  $qp$ -deformed spectrum (32) from the  $qp$ -deformed Schrödinger equation (by using the  $qp$ -derivative  $D_{qp}$ ). This has been done by some authors in the special case where  $p = 1$  and  $q$  arbitrary.

#### 4. The Algebra $W_{qpQ}$

We have seen that starting from (1-3), we arrive at

$$[N, a^+]_1 = a^+ \quad [N, a]_1 = -a \quad [a, a^+]_Q = \frac{1}{q-p} [q^N(q-Q) - p^N(p-Q)] \quad (35)$$

(cf. (10) and (13)). Reciprocally, we may ask the question : are there other hilbertean representations, besides the usual Fock representation characterized by equations (1-3), of the *oscillator algebra*  $W_{qpQ}$  generated by the operators  $a$ ,  $a^+ = (a)^\dagger$  and  $N = (N)^\dagger$  satisfying (35) with the assumption that the spectrum of  $N$  is discrete and non-degenerate ? Rideau [47] has given a positive answer to this question in the special case where  $p = q^{-1}$  and  $Q = q$ .

We shall not discuss these matters in detail. It is enough to give the key equation (equation (36) below) that permits to extend to  $W_{qpQ}$  the results valid

for  $W_q \equiv W_{qq^{-1}q}$ . Indeed, it can be shown that the relations

$$\begin{aligned}
a^+ |n\rangle &= \left( Q^{n+1} \lambda_0 + p^{\nu_0} \frac{Q^{n+1} - p^{n+1}}{q-p} + q^{\nu_0} \frac{q^{n+1} - Q^{n+1}}{q-p} \right)^{\frac{1}{2}} |n+1\rangle \\
a |n\rangle &= \left( Q^n \lambda_0 + p^{\nu_0} \frac{Q^n - p^n}{q-p} + q^{\nu_0} \frac{q^n - Q^n}{q-p} \right)^{\frac{1}{2}} |n-1\rangle \\
N |n\rangle &= (\nu_0 + n) |n\rangle
\end{aligned} \tag{36}$$

provide us with a *formal* representation of  $W_{qpQ}$ . Equation (36) leads to three types of representations :  $T(\nu'_0)$  with  $n \in \mathbb{N}$ ,  $T(\lambda_0, \nu_0)$  with  $n \in \mathbb{Z}$  and  $T(\nu_0)$  with  $n \in \mathbb{Z}$ . (As a particular case, the Fock representation (1-3) is obtained from (36) by taking  $\lambda_0 = \nu_0 = 0$  and  $n \in \mathbb{N}$ .) These three types of representations coincide with the ones found by Rideau [47] for  $q = p^{-1} = Q$ .

### 5. From $qp$ -Analogues of Angular Momenta to $u_{qp}(2)$

We now continue with the Hilbert space

$$\mathcal{E} = \{|jm\rangle : 2j \in \mathbb{N}, m = -j(1)j\} \tag{37}$$

spanned by the common eigenvectors of the  $z$ -component and the square of a generalized angular momentum. Here, again, we work with non-deformed state vectors.

**Definition 4.** We define the operators  $a_+$ ,  $a_+^\pm$ ,  $a_-$  and  $a_-^\pm$  on the vector space  $\mathcal{E}$  by the relations

$$\begin{aligned}
a_+ |jm\rangle &= \sqrt{[j+m]} |j - \frac{1}{2}, m - \frac{1}{2}\rangle \\
a_+^\pm |jm\rangle &= \sqrt{[j+m+1]} |j + \frac{1}{2}, m + \frac{1}{2}\rangle \\
a_- |jm\rangle &= \sqrt{[j-m]} |j - \frac{1}{2}, m + \frac{1}{2}\rangle \\
a_-^\pm |jm\rangle &= \sqrt{[j-m+1]} |j + \frac{1}{2}, m - \frac{1}{2}\rangle
\end{aligned} \tag{38}$$

where the  $qp$ -numbers of the type  $[c]$  are given by (3).

In the limiting case  $p = q^{-1} \rightarrow 1$ , equation (38) gives back the defining relations used by Schwinger in his (Jordan-Schwinger) approach to angular momentum (see also Refs. [21,36]). By introducing

$$n_1 = j + m \quad n_2 = j - m \quad n_1 \in \mathbb{N} \quad n_2 \in \mathbb{N} \tag{39}$$

and

$$|jm\rangle \equiv |j+m, j-m\rangle = |n_1 n_2\rangle \in \mathcal{F}_1 \otimes \mathcal{F}_2 \quad (40)$$

equation (38) can be rewritten in the form

$$\begin{aligned} a_+ |n_1 n_2\rangle &= \sqrt{[n_1]} |n_1 - 1, n_2\rangle \\ a_+^\dagger |n_1 n_2\rangle &= \sqrt{[n_1 + 1]} |n_1 + 1, n_2\rangle \\ a_- |n_1 n_2\rangle &= \sqrt{[n_2]} |n_1, n_2 - 1\rangle \\ a_-^\dagger |n_1 n_2\rangle &= \sqrt{[n_2 + 1]} |n_1, n_2 + 1\rangle \end{aligned} \quad (41)$$

Observe that

$$|jm\rangle = \frac{1}{\sqrt{[j+m]![j-m]!}} (a_+^\dagger)^{j+m} (a_-^\dagger)^{j-m} |00\rangle \quad (42)$$

and, therefore, it is possible to generate any state vector  $|jm\rangle$  from the state vector  $|00\rangle$  (cf. equation (8)).

The sets  $\{a_+, a_+^\dagger\}$  and  $\{a_-, a_-^\dagger\}$  are two commuting sets of  $qp$ -bosons. More precisely, we have

$$\begin{aligned} a_+ a_+^\dagger - q a_+^\dagger a_+ &= p^{N_1} & a_+ a_+^\dagger - p a_+^\dagger a_+ &= q^{N_1} \\ a_- a_-^\dagger - q a_-^\dagger a_- &= p^{N_2} & a_- a_-^\dagger - p a_-^\dagger a_- &= q^{N_2} \end{aligned} \quad (43)$$

and

$$[a_+, a_-] = [a_+^\dagger, a_-^\dagger] = [a_+, a_-^\dagger] = [a_+^\dagger, a_-] = 0 \quad (44)$$

with

$$N_1 |n_1 n_2\rangle = n_1 |n_1 n_2\rangle \quad N_2 |n_1 n_2\rangle = n_2 |n_1 n_2\rangle \quad (45)$$

defining the number operators  $N_1$  and  $N_2$ . We also have

$$a_+ a_+^\dagger = [N_1 + 1] \quad a_+^\dagger a_+ = [N_1] \quad a_- a_-^\dagger = [N_2 + 1] \quad a_-^\dagger a_- = [N_2] \quad (46)$$

to be compared with equation (11).

**Definition 5.** Let us consider the operators

$$J_- = a_-^\dagger a_+ \quad J_3 = \frac{1}{2}(N_1 - N_2) \quad J = \frac{1}{2}(N_1 + N_2) \quad J_+ = a_+^\dagger a_- \quad (47)$$

defined in terms of  $qp$ -bosons.

Property 5. The action of the linear operators  $J_-$ ,  $J_3$ ,  $J$  and  $J_+$  on the space  $\mathcal{E}$  is described by

$$\begin{aligned} J_- |jm\rangle &= \sqrt{[j+m][j-m+1]} |j, m-1\rangle \\ J_3 |jm\rangle &= m |jm\rangle \quad J |jm\rangle = j |jm\rangle \\ J_+ |jm\rangle &= \sqrt{[j-m][j+m+1]} |j, m+1\rangle \end{aligned} \quad (48)$$

a result that follows from (38) and (47).

The operators  $J_-$  and  $J_+$  are clearly shift operators for the quantum number  $m$ . Repeated application of (48) leads to

$$\begin{aligned} |jm\rangle &= \left( \frac{[j+m]!}{[2j]![j-m]!} \right)^{\frac{1}{2}} (J_-)^{j-m} |j, +j\rangle \\ |jm\rangle &= \left( \frac{[j-m]!}{[2j]![j+m]!} \right)^{\frac{1}{2}} (J_+)^{j+m} |j, -j\rangle \end{aligned} \quad (49)$$

Furthermore, we have the hermitean conjugation properties :  $J = (J)^\dagger$ ,  $J_3 = (J_3)^\dagger$  and  $J_+ = (J_-)^\dagger$ . Note that  $J_-$ ,  $J_3$  and  $J_+$  reduce to ordinary spherical angular momentum operators in the limiting case  $p = q^{-1} \rightarrow 1$ . The latter assertion is evident from (48) or even directly from (47).

At this stage, the quantum algebra  $u_{qp}(2)$  can be introduced, in a pedestrian way, from equations (47) and (48) as a deformation of the ordinary Lie algebra of the unitary group  $U(2)$ . In this regard, we have the following property.

Property 6. The commutators of the  $qp$ -deformed operators  $J_-$ ,  $J_3$ ,  $J$  and  $J_+$  are

$$\begin{aligned} [J, J_3] &= 0 \quad [J, J_+] = 0 \quad [J, J_-] = 0 \\ [J_3, J_-] &= -J_- \quad [J_3, J_+] = +J_+ \quad [J_+, J_-] = (qp)^{J-J_3} [2J_3] \end{aligned} \quad (50)$$

which reduce to the familiar expressions, known in the angular momentum theory, in the limiting case  $p = q^{-1} \rightarrow 1$ .

Note that changing  $q$  into  $p$  and vice versa does not change the commutation relations (50). Note also that in the basis

$$\{J, J_x = (J_+ + J_-)/2, J_y = (J_+ - J_-)/(2i), J_z = J_3\} \quad (51)$$

the last three relations in (50) can be rewritten as

$$[J_x, J_y] = i \frac{1}{2} (qp)^{J-J_z} [2J_z] \quad [J_y, J_z] = iJ_x \quad [J_z, J_x] = iJ_y \quad (52)$$

which exhibit an anisotropy (or symmetry breaking) when compared to the corresponding relations for the limiting case  $p = q^{-1} \rightarrow 1$ .

Equation (50) is at the root of the definition of the quantum algebra  $u_{qp}(2)$ . Loosely speaking, this algebra is spanned by any set  $J_-, J_3, J, J_+$  of four operators satisfying (50) where we recognize familiar commutators except for the last one.

The case  $p = q^{-1}$  deserves a special attention. In this case, we have

$$[J_3, J_-] = -J_- \quad [J_3, J_+] = +J_+ \quad [J_+, J_-] = [2J_3] \quad (53)$$

where  $[2J_3]$  stands here for  $(q^{2J_3} - q^{-2J_3})/(q - q^{-1})$ . Equations (53) define the quantum algebra  $su_q(2)$  addressed by many authors. It is to be mentioned that there are several other deformations of  $su(2)$  besides the Kulish-Reshetikhin-Drinfeld-Jimbo deformation [3,6,7] characterized by equation (53). Let us simply quote, among others, the (one-parameter) deformation by Woronowicz [8] and the (one- and two-parameter) deformations by Witten [14] and by Fairlie [16].

In the limiting case  $p = q^{-1} \rightarrow 1$ , the (infinite dimensional) quantum algebras  $su_q(2)$  and  $u_{qp}(2)$  reduce to the (finite dimensional) ordinary Lie algebras  $su(2)$  and  $u(2)$ , respectively. We note that the composition law (i.e., the commutator law) for the quantum algebras under consideration is anti-symmetrical and satisfies the Jacobi identity. However, quantities of the type  $[2J_3]$  are not in the vector space generated by  $J_-, J_3$  and  $J_+$ . They rather belong to the universal enveloping algebra of  $su(2)$  so that neither  $su_q(2)$  nor  $u_{qp}(2)$  are Lie algebras. (This is at the root of another terminology : a *quantum algebra* is also referred to as a *quantized universal enveloping algebra* ; in this respect, the  $q$ -quantized and  $qp$ -quantized universal enveloping algebras  $su_q(2)$  and  $u_{qp}(2)$  may be denoted as  $U_q(su(2))$  and  $U_{qp}(u(2))$ , respectively.)

Because the algebras  $su_q(2)$  and  $u_{qp}(2)$  are not Lie algebras, it is not possible to get Lie groups from them by using the usual "exponentiation" procedure. (It is possible, however, to generate a quantum algebra from what is called a compact matrix pseudo-group by making use of an unusual "derivation" procedure [8]. In the pioneer work by Woronowicz [8], the entries of the matrix of a pseudo-group are non-commutative objects but usual matrix multiplication is preserved.) We shall briefly see in section 6 that the quantum algebras  $su_q(2)$  and  $u_{qp}(2)$  can be endowed with the structure of a (quasi-triangular) Hopf algebra.

The notion of invariant operator also exists for quantum algebras. In this connection, we can verify that the operator

$$J^2 = \frac{1}{2} (J_+ J_- + J_- J_+) + \frac{[2]}{2} (qp)^{J-J_3} [J_3]^2 \quad (54)$$

is a Casimir operator in the sense that it commutes with each of the generators  $J_-$ ,  $J_3$ ,  $J$  and  $J_+$  of the quantum algebra  $u_{qp}(2)$ . Alternatively, we have

$$J^2 = J_+ J_- + (qp)^{J-J_3+1} [J_3][J_3 - 1] = J_- J_+ + (qp)^{J-J_3} [J_3][J_3 + 1] \quad (55)$$

It can be proved that the eigenvalues of the hermitean operator  $J^2$  are

$$\frac{q^{2j+1} - q^j p^{j+1} - q^{j+1} p^j + p^{2j+1}}{(q-p)^2} \equiv [j][j+1] \quad (56)$$

with  $2j \in \mathbb{N}$ , a result compatible with the well-known one corresponding to the limiting case  $p = q^{-1} \rightarrow 1$ .

**Definition 6.** We now introduce the operators

$$K_- = a_+ a_- \quad K_3 = \frac{1}{2} (N_1 + N_2 + 1) \equiv J + \frac{1}{2} \quad K_+ = a_+^\dagger a_-^\dagger \quad (57)$$

which are indeed  $qp$ -deformed hyperbolic angular momentum operators.

**Property 7.** The action of the operators  $K_-$ ,  $K_3$  and  $K_+$  on the space  $\mathcal{E}$  is described by

$$\begin{aligned} K_- |jm\rangle &= \sqrt{[j-m][j+m]} |j-1, m\rangle \\ K_3 |jm\rangle &= (j + \frac{1}{2}) |jm\rangle \\ K_+ |jm\rangle &= \sqrt{[j-m+1][j+m+1]} |j+1, m\rangle \end{aligned} \quad (58)$$

a result to be compared with (48).

The operators  $K_-$  and  $K_+$  behave like shift operators for the quantum number  $j$ . The operators  $K_-$ ,  $K_3 = (K_3)^\dagger$  and  $K_+ = (K_-)^\dagger$  reduce to ordinary hyperbolic angular momentum operators in the limiting case  $p = q^{-1} \rightarrow 1$ . From equation (58), we expect that they generate an algebra that reduces to  $su(1,1)$  when  $p = q^{-1} \rightarrow 1$ .

**Property 8.** The commutators of the  $qp$ -deformed operators  $K_-$ ,  $K_3$ ,  $J_3$  and  $K_+$  are

$$\begin{aligned} [J_3, K_3] &= 0 & [J_3, K_+] &= 0 & [J_3, K_-] &= 0 \\ [K_3, K_-] &= -K_- & [K_3, K_+] &= +K_+ \\ [K_+, K_-] &= -[2K_3] + (1-qp)[K_3 + J_3 - \frac{1}{2}][K_3 - J_3 - \frac{1}{2}] \end{aligned} \quad (59)$$

which lead to familiar expressions in the limiting case  $p = q^{-1} \rightarrow 1$ .

Indeed, equation (59) shows that the operators  $K_-$ ,  $K_3$ ,  $J_3$  and  $K_+$  span the ordinary Lie algebra  $u(1,1)$  when  $p = q^{-1} \rightarrow 1$ . When  $p = q^{-1}$ , with  $q$  arbitrary, equation (59) yields

$$[K_3, K_-] = -K_- \quad [K_3, K_+] = +K_+ \quad [K_+, K_-] = -[2K_3] \quad (60)$$

with  $[2K_3] = (q^{2K_3} - q^{-2K_3})/(q - q^{-1})$ , so that  $K_-$ ,  $K_3$  and  $K_+$  generate the quantum algebra  $su_q(1,1)$  or  $U_q(su(1,1))$  worked out by many authors. When  $p \neq q^{-1}$ , equation (59) may serve to define the quantum algebra  $u_{qp}(1,1)$  or  $U_{qp}(u(1,1))$ .

To close this section, let us mention that the  $J$ 's and the  $K$ 's do not close under commutation. However, by introducing four other bilinear forms in the  $a$ 's (viz.,  $k_+^+ = -(a_+^+)^2$ ,  $k_-^+ = (a_-^+)^2$ ,  $k_+^- = (a_+^-)^2$  and  $k_-^- = -(a_-^-)^2$ ), we end up with a  $qp$ -deformed algebra that reduces to  $so(3,2) \simeq sp(4, \mathbb{R})$  in the limiting case  $p = q^{-1} \rightarrow 1$  (see Refs. [36,50]). This fact might be a good starting point for studying the quantum algebra  $so_{qp}(3,2)$  or  $U_{qp}(so(3,2))$ .

## 6. Representation Theory of $u_{qp}(2)$

We now focus our attention on the algebra  $u_{qp}(2)$ . We shall examine in turn the (irreducible) representations of  $u_{qp}(2)$  and describe with some details its Hopf algebraic structure.

The matrix elements of the ( $qp$ -deformed) generators  $J_-$ ,  $J_3$ ,  $J$  and  $J_+$  on the subspace  $\mathcal{E}(j) = \{|jm\rangle : m = -j(1)j\}$  of  $\mathcal{E}$  follow from equation (48). The corresponding matrices clearly define a representation, noted  $(j)$ , of the algebra  $u_{qp}(2)$ . As a result, this representation is irreducible since there is no invariant subspace exactly as in the limiting case where  $p = q^{-1} \rightarrow 1$ . Of course, this result is valid only when none of the  $qp$ -deformed numbers occurring in (48) is vanishing.

Alternatively, the things can be presented as follows. The matrix elements of the  $qp$ -deformed generators  $J_-$ ,  $J_3$ ,  $J$  and  $J_+$  of  $u_{qp}(2)$  can be connected to those of some  $hf$ -deformed generators  $A_-$ ,  $A_3$ ,  $A$  and  $A_+$  acting also on the subspace  $\mathcal{E}(j)$  and spanning the algebra  $u_{hf}(2)$ . As a matter of fact, we have

$$\begin{aligned} J_- &= \sigma(J_3, J) A_- = A_- \sigma(J_3 - 1, J) \\ J_3 &= A_3 \quad J = A \\ J_+ &= A_+ \sigma(J_3, J) = \sigma(J_3 - 1, J) A_+ \end{aligned} \quad (61)$$

where the scaling factor  $\sigma$ , that parallels the scaling factor  $s$  for deformed bosons, is given by

$$\sigma(J_3, J) = ([J - J_3]_{qp}[J + J_3 + 1]_{qp})^{\frac{1}{2}} ([J - J_3]_{hf}[J + J_3 + 1]_{hf})^{-\frac{1}{2}} \quad (62)$$

Therefore, the  $qp$ -deformed generators of the quantum algebra  $u_{qp}(2)$  are simply related to the ordinary generators of the Lie algebra  $u(2)$  which correspond to  $f = \hbar^{-1} \rightarrow 1$ . In this limiting case, the matrices on the subspace  $\mathcal{E}(j)$  of the  $qp$ -deformed generators  $J_-, J_3, J$  and  $J_+$  can be deduced from the well-known ones of the non-deformed generators  $A_-, A_3, A$  and  $A_+$ . In this respect, any  $qp$ -deformation of  $u(2)$  is more or less equivalent to  $u(2)$ .

As a net result, the representations of  $u_{qp}(2)$  may be constructed in a simple manner: To each irreducible representation ( $j$ ) of  $u(2)$  corresponds an irreducible representation, denoted by ( $j$ ) too, of  $u_{qp}(2)$  (and reciprocally) of the same dimension and with the same weight spectrum. Note that the  $(0)$  and  $(\frac{1}{2})$  irreducible representations have the same matrices in  $u(2)$  and  $u_{qp}(2)$ . (For  $j = \frac{1}{2}$ , the matrices of the  $qp$ -deformed generators  $2J_-, 2J_3$  and  $2J_+$  coincide with the Pauli matrices  $\sigma_- = \sigma_1 - i\sigma_2, \sigma_3$  and  $\sigma_+ = \sigma_1 + i\sigma_2$ , respectively.)

The latter result requires that none of the  $qp$ -deformed numbers  $[n]$ , with  $n \in \mathbb{N}$ , involved with (61) and (62), be vanishing. Otherwise, we easily understand that a given irreducible representation of  $u(2)$  could become reducible when passing from  $u(2)$  to  $u_{qp}(2)$ . This may happen, for instance, when  $q$  is a root of unity in the case where  $p = q^{-1}$ . As a trivial example, the reader may check that for  $p^{-1} = q = \exp(i\frac{2\pi}{3})$ , the representation  $(\frac{3}{2})$  of the corresponding quantum algebra  $u_{qp}(2)$  coincides with the direct sum  $(0) \oplus (\frac{1}{2}) \oplus (0)$  of the irreducible representations  $(0), (\frac{1}{2})$  and  $(0)$  of  $su(2)$ . (By denoting the representations by their dimensions, we have, in a more suggestive way,  $4 = 1 \oplus 2 \oplus 1$ .) This example illustrates the fact that we have some symmetry breaking: the irreducible representation  $(\frac{3}{2})$  of  $u_{qp}(2)$ , with generic  $p$  and  $q$ , gives rise to a (completely) reducible representation in the limiting situation where  $p^{-1} = q = \exp(i\frac{2\pi}{3})$ .

Since the algebras  $su_q(2)$  and  $u_{qp}(2)$  are not Lie algebras, we foresee that the composition (or product) of two representations of  $su_q(2)$  or  $u_{qp}(2)$  differs from the one of two representations of  $su(2)$  or  $u(2)$ , respectively. As an illustration, it is easy to check that, if the sets  $\{J_\alpha(i) : \alpha = -, 3, +\}$  for  $i = 1, 2$  span  $su_q(2)$ , then the set

$$\{\Delta(J_\alpha) = J_\alpha(1) \otimes I(2) + I(1) \otimes J_\alpha(2) : \alpha = -, 3, +\} \quad (63)$$

does not span  $su_q(2)$  except for  $q = 1$ . (We use  $I$  to denote the identity operator.) In other words, the operation  $\Delta$  is not  $su_q(2)$  co-variant for  $q \neq 1$ . In the terminology of angular momentum theory, it is not possible to construct a resulting  $q$ -deformed angular momentum  $\mathbf{J}$  from the ordinary sum  $\mathbf{J}(1) + \mathbf{J}(2)$  of two  $q$ -deformed angular momenta  $\mathbf{J}(1)$  and  $\mathbf{J}(2)$  when  $q \neq 1$ .

A correct way of doing the product of representations (or of coupling deformed angular momenta) is to introduce the notion of co-product. In the case of the quantum algebra  $u_{qp}(2)$ , this may be achieved as follows.

Definition 7. Let  $\Delta_{qp}$  be the algebra homomorphism defined, for  $u_{qp}(2)$ , by

$$\begin{aligned}\Delta_{qp}(J_3) &= J_3 \otimes I + I \otimes J_3 & \Delta_{qp}(J) &= J \otimes I + I \otimes J \\ \Delta_{qp}(J_{\pm}) &= J_{\pm} \otimes (qp)^{\frac{1}{2}J} (qp^{-1})^{+\frac{1}{2}J_3} + (qp)^{\frac{1}{2}J} (qp^{-1})^{-\frac{1}{2}J_3} \otimes J_{\pm}\end{aligned}\quad (64)$$

where we have abbreviated  $X(1) \otimes Y(2)$  by  $X \otimes Y$ .

Property 9. The  $\Delta_{qp}()$ 's defined by (64) satisfy the commutation relations (50) and thus span  $u_{qp}(2)$ .

The latter property constitutes an important result. Equation (64) indicates how to couple (or co-multiply) two representations of  $u_{qp}(2)$ . For example, in order of co-multiply two irreducible representations ( $j_1$ ) and ( $j_2$ ) of  $u_{qp}(2)$ , we have to replace, in the various  $X \otimes Y$  occurring in (64),  $X$  and  $Y$  by their matrices taken the ( $j_1$ ) and ( $j_2$ ) representations, respectively. From a physical viewpoint,  $\Delta_{qp}$  provides us with a  $qp$ -analogue for the composition of two  $qp$ -deformed angular momenta. In mathematical terms,  $\Delta_{qp}$  is called a co-product. It is co-associative but not co-commutative.

We have the permutational property

$$\Sigma(\Delta_{qp}) = \Delta_{pq} \quad (65)$$

where  $\Sigma$  is the (transposition) operator defined by  $\Sigma(X(1) \otimes Y(2)) = Y(1) \otimes X(2)$ . We note in passing that  $\Delta_{qp}$  and its "double"  $\Delta_{pq}$  are connected through a matrix  $R_{qp}$  in the following way

$$\Delta_{qp} = R_{qp} \Delta_{pq} (R_{qp})^{-1} \quad (66)$$

In the case  $p = q^{-1}$ ,  $R_{qp}$  is known as the universal  $R$ -matrix of Drinfeld. (The  $R$ -matrix leads to solutions of the quantum Yang-Baxter equation.)

Besides the co-product  $\Delta_{qp}$ , it is necessary to introduce a co-unit (an homomorphism)  $\epsilon$  and an anti-pode (an anti-homomorphism)  $S$  in order that the algebras  $su_q(2)$  and  $u_{qp}(2)$  become Hopf algebras. We shall not insist on these two further notions. In the case of  $u_{qp}(2)$ , it is sufficient to know that the homomorphism  $\epsilon$  maps the  $J_{\alpha}$ 's onto 0 or  $I$  and the anti-homomorphism  $S$  is defined by

$$S(J_+) = -(qp^{-1})^{\frac{1}{2}} J_+ \quad S(J_3) = -J_3 \quad S(J) = +J \quad S(J_-) = -(qp^{-1})^{-\frac{1}{2}} J_- \quad (67)$$

To be complete, we should examine the various compatibility relations which have to be satisfied as a consequence of the introduction of the co-product, co-unit and anti-pode (or co-inverse). We leave this point to the reader.

Let us close with some general remarks about the Wigner-Racah algebra of  $u_{qp}(2)$ . In the case where we couple (via the co-product  $\Delta_{qp}$ ) two irreducible representations ( $j_1$ ) and ( $j_2$ ) of  $u_{qp}(2)$ , the resulting representation is in general a (completely) reducible representation of  $u_{qp}(2)$ . The reduction is accomplished by means of  $qp$ -deformed coupling matrices, the elements of which are  $qp$ -deformed Clebsch-Gordan coefficients from which we can define symmetrized  $(3 - jm)_{qp}$  symbols. These coupling coefficients ( $qp$ -deformed Clebsch-Gordan coefficients or  $(3 - jm)_{qp}$  symbols) may then be combined to produce  $qp$ -deformed recoupling coefficients (e.g., in the form of  $(6 - j)_{qp}$ ,  $(9 - j)_{qp}$ ,  $\dots$  symbols). The algebraic relations involving  $qp$ -deformed coupling and recoupling coefficients of  $u_{qp}(2)$  furnish the basis for developing an  $u_{qp}(2)$  Wigner-Racah calculus. Indeed, the  $qp$ -analogue of an irreducible tensor operator can be defined for  $u_{qp}(2)$ . This naturally leads to a  $qp$ -analogue of the Wigner-Eckart theorem. Even further, the concept of an  $u_{qp}(2)$  unit tensor can be developed exactly as in the limiting case  $p = q^{-1} \rightarrow 1$  (see Ref. [50]).

## 7. Towards Applications

Rather than dealing with a specific application, as done during the lectures, we shall concentrate in this section on the general philosophy inherent to *some* applications of quantum algebras to physics. As a preliminary, a few remarks are in order.

(i) One may first ask the question : is there anything new with  $q$ - or  $qp$ -deformed objects ? Equations of type (17) and (61) incline one to think that deformed objects are more or less equivalent to the corresponding non-deformed objects. On the other hand, an operator developed in terms of deformed bosons (e.g.,  $aa^+ + a^+a$ ) generally exhibits a spectrum that differs from the one corresponding to non-deformed bosons. The new representations (36) of  $W_{qpQ}$  go in the sense of a positive answer too.

(ii) To every quantum dynamical system, one can associate a (non-unique)  $qp$ -deformed partner or  $qp$ -analogue. This may be achieved indeed in several ways. For example, one may  $qp$ -deform the dynamical invariance algebra of the system or solve the  $qp$ -deformed Schrödinger (or Dirac) equation of the system ; there is no reason to obtain the same result for the energy spectrum in the two approaches.

(iii) The Lie algebra  $su(2)$  enters many fields of theoretical physics. Therefore, if the quantized universal enveloping algebra  $su_q(2)$  comes to play a role, it is hardly conceivable to have an *universal significance* for the deformation parameter  $q$ . In this connection, one may expect, for example, that  $q$  be a function of the fine structure constant  $\alpha$  in quantum electrodynamics and a function of the so-called Weinberg angle  $\theta_W$  in the theory of electroweak interactions.

The reader has understood from (i)-(iii) that a wind of *pessimism* sometimes accompanies the quantum group invasion. On the other side, there are plenty of avenues of investigation based on the concept of quantum algebra. In this direction, the following four series of applications should constitute an encouragement towards a certain *optimism*.

1. A first series of applications merge from equations (2), (32) and (56). More precisely, in any problem involving ordinary bosons or ordinary harmonic oscillators or ordinary angular momenta (any kind of angular momentum), one may think of replacing them by the corresponding  $qp$ -deformed objects. If the limiting case where  $p = q^{-1} \rightarrow 1$  describes the problem in a reasonable way, one may expect that the case where  $p$  and  $q$  are close to 1 can describe some fine structure effects. In this approach, the (dimensionless) parameters  $p$  and  $q$  are two further fitting parameters describing additional degrees of freedom ; the problem in this approach is to find a physical interpretation of the (fine structure or anisotropy or curvature) parameters  $p$  and  $q$ . Along this first series, we have the following items.

(i) Application of  $q$ -deformed (and, more generally,  $qp$ -deformed) harmonic oscillators and of  $su_q(1,1)$  (and, more generally,  $u_{qp}(1,1)$ ) to vibrational spectroscopy of molecules and nuclei.

(ii) Application of the algebras  $su_q(2)$  and  $u_q(2)$  (and, more generally,  $u_{qp}(2)$ ) to (vibrational-)rotational spectroscopy of molecules and nuclei.

(iii) Let us also suggest that  $qp$ -bosons might be of interest for investigating the interaction between radiation and matter.

Most of the applications (i) and (ii) have been concerned up to now with only one parameter (say  $q$ ). The introduction of a second parameter (say  $p$ ) should permit more flexibility ; this is especially appealing for rotational spectroscopy of nuclei that involves two parameters in the VMI (variable moment of inertia) model.

2. A second series of applications concerns the more general situation where a physical problem is well described by a given (simple) Lie algebra  $g$ . One may

then consider to associate a (one- or) two-parameter quantized universal enveloping algebra  $U_{qp}(g)$  to the Lie algebra  $g$ . For generic  $p$  and  $q$  (avoiding exotic cases as  $p = q^{-1} = \text{root of unity}$ ), the representation theory of  $U_{qp}(g)$  is connected to the one of  $g$  in a trivial manner since we can associate a representation of  $U_{qp}(g)$  to any representation of  $g$  (cf.  $g = u(2)$ ). Here again, the case where  $p$  and  $q$  are close to 1 may serve to describe fine structure effects. Symmetries described by Lie algebras are thus replaced by quantum algebra symmetries. See for instance the passage from  $g = so(4)$  to  $su_q(2) \oplus su_q(2)$  for the hydrogen atom system [36]. In this example, a  $q$ -deformation of the non-relativistic (discrete) energy spectrum leads to results that parallel those afforded by the Dirac relativistic equation. (Here, the parameter  $q$  can be related to the fine structure constant  $\alpha$ .)

3. A third series arises by allowing the deformation parameters ( $p$  and  $q$ ) not to be restricted to (real) values close to 1. Completely unexpected models may result from this approach. This is the case for instance when  $q = p^{-1}$  is a root of unity for which case the representation theory of a quantum algebra  $U_{qp}(g)$  may be very different from the case of generic  $p$  and  $q$ . This may be also the case when  $p$  and/or  $q$  takes (real) values far from unity.

4. Finally, a fourth series concerns more fundamental applications (more fundamental in the sense not being subjected to fitting procedures). We may mention applications to statistical mechanics, gauge theories, conformal field theories and so on. Also, quantum groups (algebras) might be interesting for a true definition of the quantum space-time. (See Ref. [27] for a pertinent discussion.)

## 8. Sketch of Bibliography

It is not the aim of this section to give an exhaustive list of references. We shall give under the form of a listing some references of interest for the theory of quantum algebras and their applications to physics. Not all the applications are covered by the listed references. The author apologizes for not quoting other important papers (known or unknown to him) about the theory of quantum algebras, the chosen applications and other possible applications. (Any suggestion to the author will be appreciated.) As far as the reader is a beginner on quantum algebras, the references [1-50] should be a good starting point. The fifty references (ordered according to the year of publication) may be roughly classified as follows. (Sorry for the shortcomings in this tentative classification !)

Some basic papers on quantum algebras : [3, 4, 6, 7, 8, 9, 26, 27]. Paradigms of quantum algebras for physicists : [16, 19, 25, 27, 41, 42]. One-parameter deformed

bosons : [1, 2, 5] and many other references. Deformed oscillators : [5, 10, 11, 13, 17, 18, 24]. The  $q$ -oscillator ( $q$ -boson) algebra : [10, 11, 47, 48]. On  $q$ -boson realizations of  $su_q(n)$  and  $su_q(1, 1)$  : [10, 11, 12, 17] and many other references. The quantum algebra  $u_q(3)$  : [40]. On  $qp$ -bosons,  $qp$ -oscillators,  $su_{qp}(2)$  and  $u_{qp}(2)$  : [28, 38, 45, 49, 50]. Wigner-Racah algebra of  $su_q(2)$  and  $u_{qp}(2)$  : [11, 20, 21, 39, 49, 50]. Coherent states : [29, 35]. On the  $q$ -deformed Schrödinger equation : [24, 43]. Nuclear physics : [14, 22, 33]. Atomic physics : [36, 43, 46]. Rotational spectroscopy of (diatomic) molecules [22, 23, 30, 32] and (deformed and super-deformed) nuclei [22, 33]. Vibrational spectroscopy of molecules : [31, 32, 34, 37]. Spin Heisenberg chain : [15]. Condensed matter physics : [44].

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### Appendix

We first begin with some formulas useful for dealing with  $qp$ -deformed numbers  $[c]_{qp}$ . From equation (3), we easily get

$$[c]_{qp} = [c]_{pq} \quad [-c]_{qp} = -(qp)^{-c} [c]_{qp}$$

Furthermore, the following relations (with  $[ ] \equiv [ ]_{qp}$ )

$$\begin{aligned} [a + b] &= [a] q^b + p^a [b] \\ [a + b + 1] &= [a + 1] [b + 1] - qp [a] [b] \\ [a] [b + c] &= [a + c] [b] + (qp)^b [a - b] [c] \\ [a - b] [a + b] &= [a]^2 - (qp)^{a-b} [b]^2 \end{aligned}$$

hold for arbitrary numbers  $a$ ,  $b$  and  $c$ .

In the case where  $n$  is a positive integer, we have

$$[n] = q^{n-1} + q^{n-2} p + q^{n-3} p^2 + \cdots + qp^{n-2} + p^{n-1} \quad n \in \mathbb{N} - \{0\}$$

where the factorial of  $[n] \equiv [n]_{qp}$  is defined by (9). As illustrative examples, we have

$$[0] = 0 \quad [1] = 1 \quad [2] = q + p \quad [3] = q^2 + qp + p^2 \quad [4] = q^3 + q^2 p + qp^2 + p^3$$

and

$$[2][2] = qp[1] + [3] \quad [2][3] = qp[2] + [4] \quad [3][3] = (qp)^2[1] + qp[3] + [5]$$

which is reminiscent of the addition rule for angular momenta.

In the case where  $p = q^{-1}$  with  $q$  being a root of unity, i.e.,

$$p^{-1} = q = \exp(i2\pi \frac{k_1}{k_2}) \quad k_1 \in \mathbb{N} \quad k_2 \in \mathbb{N}$$

we have

$$[c] = \frac{\sin(2\pi \frac{k_1}{k_2} c)}{\sin(2\pi \frac{k_1}{k_2})}$$

For instance,

$$k_1 = 1 \quad k_2 = 4 \quad \Rightarrow \quad q = i = \sqrt{-1} \quad \Rightarrow \quad [0] = [2] = [4] = \dots = 0$$

so that  $[c] = 0$  can occur for  $c \neq 0$ .

We might continue and elaborate on  $qp$ -arithmetics,  $qp$ -combinatorics,  $qp$ -analysis,  $qp$ -polynomials and so on.\* For the purpose of the present paper, it is sufficient to restrict ourselves to a few remarks concerning the  $qp$ -derivative  $D_{qp}$ .

As an elementary property, we have

$$D_{qp} x^k = [k]_{qp} x^{k-1}$$

Consequently,  $\exp_{qp}(x)$  is an eigenfunction of  $D_{qp}$  with the eigenvalue equal to 1. More generally, we obtain

$$D_{qp} \exp_{qp}(\lambda x) = \lambda \exp_{qp}(\lambda x)$$

for  $\lambda \in \mathbb{C}$ . In the same spirit, observe that

$$D_{qp} f(\lambda x) = \lambda (D_{qp} f(y))_{y=\lambda x}$$

Also of interest, we have

$$D_{qp} (f(x)g(x)) = (D_{qp} f(x)) g(px) + f(qx)(D_{qp} g(x))$$

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\* The  $q$ -analysis, which corresponds to  $p = 1$ , goes back to the end of the nineteenth century and was sometimes referred at this time to as the " $q$ -disease" (cf., another infestation, viz., the "Gruppenpest" in the first quarter of the twentieth century).

Finally, let us mention that for  $p = 1$  and  $q$  arbitrary, the derivative  $D_{qp}$  is sometimes called the Jackson derivative.

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