LYCEN 9242
Oct. 1992

Some Aspects of q-Boson Calculus[1]

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[1] Contribution to the Symposium "Symmetries in Science VI: From the Rotation Group to Quantum Algebras" held at the Cloister Mehrerau (Bregenz, Austria, 2-7 August 1992) in honor of Professor L.C. Biedenharn on the occasion of his 70th birthday. To appear in the Proceedings of the Symposium, ed. B. Gruber (Plenum Press, New York).

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SOME ASPECTS OF q-BOSON CALCULUS

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1. PRELIMINARIES

The content of this work is concerned with Jordan-Schwinger calculus, using deformed bosons, that has been so largely investigated by Biedenharn. It is thus a real pleasure to dedicate this work to Larry Biedenharn on the occasion of his 70th birthday. The works by Biedenharn and by Biedenharn and his collaborators have been a source of inspiration for both authors of the present paper, especially in connection with (i) the Wigner-Racah algebra of a chain of compact groups (involving finite groups) in view of its applications to nuclear, molecular and condensed matter physics and, more recently, with (ii) quantum algebras.

This work constitutes a first step towards a complete study of the $su_q(2)$ unit tensor. The components $t[q:k\rho\Delta]$ of such a tensor operator are defined by[1,2]

$$< j'm'|t[q:k\rho\Delta]|jm >= \delta(j',j+\Delta)\delta(m',m+\rho)(-1)^{2k}[2j'+1]^{-\frac{1}{2}}(jkm\rho|j'm')_q \quad (1)$$

where $(jkm\rho|j'm')_q$ is a Clebsch-Gordan coefficient (CGc) for $su_q(2)$ as defined, for example, in ref. 3. (In eq. (1) and in the following, we employ the usual notation $[a] = (q^a - q^{-a})/(q - q^{-1})$ with $q \in \mathbb{R}$.) The operator $t[q : k\rho\Delta]$ constitutes a q-deformation of the operator $t_{kq\alpha} \equiv t[1 : kq\alpha]$ worked out in ref. 2. Our program is to find a realization of $t[q : k\rho\Delta]$ in terms of q-bosons. We address here the first part of this program by defining in section 2 the (q-deformed) Schwinger algebra relative to $su_q(2)$ and by giving in section 3 an algorithm for producing recurrent relations (RR's) between CGc's for $su_q(2)$.

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2. THE q-DEFORMED SCHWINGER ALGEBRA

It is well known that the quantum algebra $su_q(2)$, which is characterized (in the Kulish-Reshetikhin-Drinfeld-Jimbo realization) by

$$[J_3, J_-] = -J_- \qquad [J_3, J_+] = +J_+ \qquad [J_+, J_-] = [2J_3] \tag{2}$$

(plus other axioms relative to its Hopf algebraic structure), may be realized by means of two commuting sets of q-bosons. In fact, the generators J_-, J_3 and J_+ of $su_q(2)$ can be written as

$$J_- = a_-^+ a_+ \qquad J_3 = \frac{1}{2}(N_1 - N_2) \qquad J_+ = a_+^+ a_- \tag{3}$$

in terms of the q-deformed annihilation (a_+ and a_-), creation ($a_+^+ \equiv (a_+)^\dagger$ and $a_-^+ \equiv (a_-)^\dagger$) and number ($N_1 = (N_1)^\dagger$ and $N_2 = (N_2)^\dagger$) operators. These operators satisfy[4,5]

$$\begin{gathered}
[a_+, a_-] = [a_+^+, a_-^+] = [a_+, a_-^+] = [a_+^+, a_-] = 0 \\
[N_1, a_+^+] = a_+^+ \quad [N_1, a_+] = -a_+ \quad [N_2, a_-^+] = a_-^+ \quad [N_2, a_-] = -a_- \\
a_+^+ a_+ = [N_1] \quad a_+ a_+^+ = [N_1 + 1] \quad a_-^+ a_- = [N_2] \quad a_- a_-^+ = [N_2 + 1]
\end{gathered} \tag{4}$$

In a (two-particle) Fock space $\mathcal{F}_1 \otimes \mathcal{F}_2$, with the basis vectors

$$|n_1 n_2> = \frac{1}{\sqrt{[n_1]![n_2]!}} (a_+^+)^{n_1} (a_-^+)^{n_2} \, |00> \tag{5}$$

(where $[n]!$ stands for the q-deformed factorial), the a's and N's act in the following way

$$\begin{aligned}
a_+ \, |n_1 n_2> &= \sqrt{[n_1]} \, |n_1 - 1, n_2> \\
a_+^+ \, |n_1 n_2> &= \sqrt{[n_1 + 1]} \, |n_1 + 1, n_2> \\
a_- \, |n_1 n_2> &= \sqrt{[n_2]} \, |n_1, n_2 - 1> \\
a_-^+ \, |n_1 n_2> &= \sqrt{[n_2 + 1]} \, |n_1, n_2 + 1> \\
N_i |n_1 n_2> &= n_i |n_1 n_2> \qquad (i = 1, 2)
\end{aligned} \tag{6}$$

from which it is clear that $a_+^+ = (a_+)^\dagger$ and $a_-^+ = (a_-)^\dagger$ when $q \in \mathbb{R}$ (as supposed in this paper) or $q \in S^1$.

If the $su(2)$ notations are introduced for the basis vectors (5), namely,

$$|jm> \equiv |n_1 n_2> \qquad j = \frac{1}{2}(n_1 + n_2) \qquad m = \frac{1}{2}(n_1 - n_2) \tag{7}$$

then, the operators J_-, J_3 and J_+ (to be considered, in physical applications, as q-analogues of spherical angular momentum operators) act on $\mathcal{F}_1 \otimes \mathcal{F}_2$ through

$$\begin{aligned}
J_- \, |jm> &= \sqrt{[j + m] \, [j - m + 1]} \, |j, m - 1> \\
J_3 \, |jm> &= m \, |jm> \\
J_+ \, |jm> &= \sqrt{[j - m] \, [j + m + 1]} \, |j, m + 1>
\end{aligned} \tag{8}$$

showing that the vectors (5) with a fixed value $2j$ ($2j \in \mathbb{N}$) of $n_1 + n_2$ span the irrep (j) of the quantum algebra $su_q(2)$.

Like for $su_q(2)$, the quantum algebra $su_q(1,1) \simeq sp_q(2,\mathbb{R})$ can be realized in terms of the two commuting sets $\{a_+, a_+^+\}$ and $\{a_-, a_-^+\}$. This algebra is generated by the operators

$$K_- = a_+ a_- \qquad K_3 = \frac{1}{2}(N_1 + N_2 + 1) \qquad K_+ = a_+^+ a_-^+ \tag{9}$$

which satisfy the commutation relations

$$[K_3, K_-] = -K_- \qquad [K_3, K_+] = +K_+ \qquad [K_+, K_-] = -[2K_3] \tag{10}$$

that are typical of $su_q(1,1)$. The generators K_-, K_3 and K_+ act on $\mathcal{F}_1 \otimes \mathcal{F}_2$ according to

$$\begin{aligned} K_- \ |jm> &= \sqrt{[j-m]\,[j+m]}\ |j-1,m> \\ K_3 \ |jm> &= (j+\frac{1}{2})\ |jm> \\ K_+ \ |jm> &= \sqrt{[j-m+1]\,[j+m+1]}\ |j+1,m> \end{aligned} \tag{11}$$

and may thus be considered as q-analogues of hyperbolical angular momentum operators.

The algebras $su_q(2)$ and $su_q(1,1)$ do not commute. The four nonvanishing commutators of the J's and K's may serve to define new bilinear forms of the a's. Indeed, by introducing the operators

$$k_+^+ = -a_+^+ a_+^+ \qquad k_-^+ = a_-^+ a_-^+ \qquad k_-^- = -a_+ a_+ \qquad k_+^- = a_- a_- \tag{12}$$

we can put the nonvanishing commutators of type $[J, K]$ in the form

$$\begin{aligned} [J_+, K_+] &= k_+^+([K_3 - J_3 - \tfrac{1}{2}] - [K_3 - J_3 + \tfrac{1}{2}]) \\ [J_+, K_-] &= k_+^-([K_3 + J_3 - \tfrac{1}{2}] - [K_3 + J_3 + \tfrac{1}{2}]) \\ [J_-, K_+] &= k_-^+([K_3 + J_3 + \tfrac{1}{2}] - [K_3 + J_3 - \tfrac{1}{2}]) \\ [J_-, K_-] &= k_-^-([K_3 - J_3 + \tfrac{1}{2}] - [K_3 - J_3 - \tfrac{1}{2}]) \end{aligned} \tag{13}$$

which go to $-k_+^+$, $-k_+^-$, $+k_-^+$, $+k_-^-$, respectively, in the limiting case $q = 1$. The k's are step operators in the space $\mathcal{F}_1 \otimes \mathcal{F}_2$ since we have

$$\begin{aligned} k_+^+ \ |jm> &= -\sqrt{[j+m+1]\,[j+m+2]}\ |j+1,m+1> \\ k_-^+ \ |jm> &= +\sqrt{[j-m+1]\,[j-m+2]}\ |j+1,m-1> \\ k_-^- \ |jm> &= -\sqrt{[j+m-1]\,[j+m]}\ |j-1,m-1> \\ k_+^- \ |jm> &= +\sqrt{[j-m-1]\,[j-m]}\ |j-1,m+1> \end{aligned} \tag{14}$$

It is clear that the operators $\{J, K, k\}$ close under commutation in the limiting case $q = 1$. In this case, they generates the 10-dimensional noncompact Lie algebra $sp(4,\mathbb{R}) \simeq so(3,2)$. In the case $q \neq 1$, the operators $\{J, K, k\}$ span the quantized universal enveloping algebra $U_q(so(3,2)) \equiv so_q(3,2) \simeq sp_q(4,\mathbb{R})$ that we shall refer to as the q-deformed Schwinger algebra. (This terminology follows from the angular momentum context of ref. 2. In another context, $so_q(3,2)$ may be called a q-deformed de Sitter algebra.) The nonvanishing commutators for $so_q(3,2)$, besides (2), (10) and (13), are given in ref. 6 and reproduced in the appendix. Note that a realization of the algebra $so_q(3,2)$ in the Bargmann-Fock space may be found by making the replacements

$$\begin{aligned} a_+^+ &\mapsto z_1 \qquad a_+ \mapsto D_{z_1} \qquad N_1 \mapsto z_1 \frac{\partial}{\partial z_1} \\ a_-^+ &\mapsto z_2 \qquad a_- \mapsto D_{z_2} \qquad N_2 \mapsto z_2 \frac{\partial}{\partial z_2} \end{aligned} \tag{15}$$

where the finite difference operator D_x defined via

$$D_x f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x} \tag{16}$$

is the Jackson derivative.

At this point, we should discuss the Hopf algebraic structure of $su_q(2)$, $su_q(1,1)$ and $so_q(3,2)$. This is well documented for $su_q(2)$ and $su_q(1,1)$ (see, for instance, refs. 4 and 7). For the algebra $so_q(3,2)$, a coproduct, a counit and an antipode should be defined in order to endow this algebra with an Hopf algebraic structure. Indeed, the generators of the quantized de Sitter algebra $so_q(3,2)$ have been given, in a Cartan-Weyl basis, and the Hopf algebraic structure of $so_q(3,2)$ has been discussed explicitly in ref. 8. The passage formulas, which shall be reported elsewhere, between the generators of $so_q(3,2)$ in a Cartan-Weyl basis and the operators J's, K's and k's allows us to consider the q-deformed Schwinger algebra as an Hopf algebra.

The noncommutativity of $su_q(2)$ and $su_q(1,1)$ seems to prevent the irrep's of both algebras to be fixed simultaneously. However, the second order invariant operators

$$C_2(su_q(2)) = J_- J_+ + [J_3][J_3 + 1] \tag{17}$$

and

$$C_2(su_q(1,1)) = -K_+ K_- + [K_3][K_3 - 1] \tag{18}$$

can be diagonalized simultaneously. Indeed, eqs. (17) and (18) can be expressed as

$$C_2(su_q(2)) = [K_3]^2 - [\frac{1}{2}]^2 \equiv [C_2(u_q(1))]^2 - [\frac{1}{2}]^2 \tag{19}$$

and

$$C_2(su_q(1,1)) = [J_3]^2 - [\frac{1}{2}]^2 \equiv [C_2(so_q(2))]^2 - [\frac{1}{2}]^2 \tag{20}$$

(with evident definitions of $u_q(1)$ and $so_q(2)$), respectively. Equations (19) and (20) are valid as far as matrix elements, on the Fock space $\mathcal{F}_1 \otimes \mathcal{F}_2$, are concerned. The invariants of $su_q(2)$ and $u_q(1)$, on one hand, and of $su_q(1,1)$ and $so_q(2)$, on the other hand, are thus connected. Furthermore, J_3 commutes with the three generators of $su_q(1,1)$:

$$[J_3, K_\nu] = 0 \qquad \nu = -, 3, + \tag{21}$$

while K_3 commutes with the three generators of $su_q(2)$:

$$[K_3, J_\nu] = 0 \qquad \nu = -, 3, + \tag{22}$$

According to the definition given in ref. 9, from eqs. (19)-(22) we deduce that the algebra $su_q(1,1)$, generated by the set $\{K_-, K_3, K_+\}$, and the algebra $so_q(2)$, generated by the operator J_3, are complementary in the frame of some definite representation of the host algebra $sp_q(4,\mathbb{R})$. Similarly, the algebra $su_q(2)$, generated by the set $\{J_-, J_3, J_+\}$, and the algebra $u_q(1)$, generated by the operator K_3, are complementary within some representation of $sp_q(4,\mathbb{R})$. Indeed, two chains are relevant here, viz.,

$$\begin{aligned} sp_q(4,\mathbb{R}) \simeq so_q(3,2) \supset sp_q(2,\mathbb{R}) \simeq su_q^k(1,1) \supset u_q^\kappa(1) \\ sp_q(4,\mathbb{R}) \simeq so_q(3,2) \supset su_q^j(2) \supset so_q^m(2) \end{aligned} \tag{23}$$

for which we have two pairs of complementary algebras : $(su_q^k(1,1), so_q^m(2))$ and $(su_q^j(2), u_q^\kappa(1))$. The symbols j, m, k and κ labelling the irrep's of the algebras $su_q(2)$,

$so_q(2)$, $su_q(1,1)$ and $u_q(1)$, respectively, are put as superscripts. We conclude that the vector

$$|n_1 n_2> \equiv |jm> \equiv |k\kappa> \tag{24}$$

can be considered simultaneously as (i) a basis vector for the irrep (j) of $su_q(2)$, in an $su_q(2) \supset so_q(2)$ basis, with spherical angular momentum

$$j = \frac{1}{2}(n_1 + n_2) \tag{25}$$

and 3-axis projection (eigenvalue of J_3)

$$m = \frac{1}{2}(n_1 - n_2) \tag{26}$$

and as (ii) a basis vector for the irrep $(k+)$, belonging to the positive discrete series of $su_q(1,1)$, in an $su_q(1,1) \supset u_q(1)$ basis, with hyperbolical angular momentum

$$k = \frac{1}{2}(n_1 - n_2 - 1) = m - \frac{1}{2} \tag{27}$$

and 3-axis projection (eigenvalue of K_3)

$$\kappa = \frac{1}{2}(n_1 + n_2 + 1) = j + \frac{1}{2} \tag{28}$$

Note that, in the $su_q(1,1)$ notations, eq. (11) can be rewritten in the useful form

$$\begin{aligned} K_- \; |k\kappa> &= \sqrt{[\kappa + k]\,[\kappa - k - 1]}\; |k, \kappa - 1> \\ K_3 \; |k\kappa> &= \kappa \; |k\kappa> \\ K_+ \; |k\kappa> &= \sqrt{[\kappa - k]\,[\kappa + k + 1]}\; |k, \kappa + 1> \end{aligned} \tag{29}$$

The vectors $|k\kappa>$ and $|jm>$ are eigenvectors of the invariant operators $C_2(su_q(1,1))$ and $C_2(su_q(2))$ with the eigenvalues $[k][k+1]$ and $[j][j+1]$, respectively. For the host algebra $sp_q(4,\mathbb{R}) \simeq so_q(3,2)$, only two infinite-dimensional irrep's, namely, $[\dot{0}]$ and $[\dot{1}]$, belonging to the positive discrete series, are realized in the Fock space $\mathcal{F}_1 \otimes \mathcal{F}_2$ (relative to a two-dimensional q-oscillator). The basis vectors (5) with $n_1 + n_2$ even belong to $[\dot{0}]$ and the ones with $n_1 + n_2$ odd to $[\dot{1}]$.

To close this section, let us briefly discuss the irreducible tensor character of the q-bosons a_+, a_+^+, a_- and a_-^+, a problem also addressed by Biedenharn and Tarlini,[10] Nomura[11,12] and Quesne.[13] First, we note that eq. (6) yields

$$\begin{aligned} a_+ \; |jm> &= \sqrt{[j+m]}\; |j - \frac{1}{2}, m - \frac{1}{2}> \\ a_+^+ \; |jm> &= \sqrt{[j+m+1]}\; |j + \frac{1}{2}, m + \frac{1}{2}> \\ a_- \; |jm> &= \sqrt{[j-m]}\; |j - \frac{1}{2}, m + \frac{1}{2}> \\ a_-^+ \; |jm> &= \sqrt{[j-m+1]}\; |j + \frac{1}{2}, m - \frac{1}{2}> \end{aligned} \tag{30}$$

in the $su_q(2)$ notations. Equation (30) shows that the sets $\{a_+^+, a_-^+\}$ and $\{a_-, a_+\}$ are connected to the $su_q(2)$ irreducible tensorial sets

$$\left\{t[q : \tfrac{1}{2}, \rho, \tfrac{1}{2}] : \rho = \pm\tfrac{1}{2}\right\} \quad \text{and} \quad \left\{t[q : \tfrac{1}{2}, \rho, -\tfrac{1}{2}] : \rho = \pm\tfrac{1}{2}\right\},$$

respectively. Both sets transform as the unitary irrep $(\frac{1}{2})$ of $su_q(2)$. Second, in the $su_q(1,1)$ notations, we have

$$
\begin{aligned}
a_+ \ |k\kappa> &= \sqrt{[\kappa+k]}\ |k-\frac{1}{2},\kappa-\frac{1}{2}> \\
a_+^+ \ |k\kappa> &= \sqrt{[\kappa+k+1]}\ |k+\frac{1}{2},\kappa+\frac{1}{2}> \\
a_- \ |k\kappa> &= \sqrt{[\kappa-k-1]}\ |k+\frac{1}{2},\kappa-\frac{1}{2}> \\
a_-^+ \ |k\kappa> &= \sqrt{[\kappa-k]}\ |k-\frac{1}{2},\kappa+\frac{1}{2}>
\end{aligned}
\tag{31}
$$

which indicate that the sets $\{a_+^+, a_-\}$ and $\{a_-^+, a_+\}$ are connected to the $su_q(1,1)$ irreducible tensorial sets

$$\left\{\tau[q:\tfrac{1}{2},\rho,\tfrac{1}{2}] : \rho = \pm\tfrac{1}{2}\right\} \quad \text{and} \quad \left\{\tau[q:\tfrac{1}{2},\rho,-\tfrac{1}{2}] : \rho = \pm\tfrac{1}{2}\right\},$$

respectively. (The operator $\tau[q : k\rho\Delta]$ for $su_q(1,1)$ parallels the operator $t[q : k\rho\Delta]$ for $su_q(2)$.) The latter sets have well defined transformation properties with respect to the nonunitary irrep $(\frac{1}{2})$ of $su_q(1,1)$. As a conclusion, the four q-boson operators a_+, a_+^+, a_- and a_-^+ can be united in a double unit tensor operator $w(q)^{\frac{1}{2}\frac{1}{2}}$, with components $w(q)^{\frac{1}{2}\frac{1}{2}}_{\rho\sigma}$ where $\rho = \pm\frac{1}{2}$ labels its components with respect to $su_q(2)$ and $\sigma = \pm\frac{1}{2}$ does the same with respect to $su_q(1,1)$. By applying the Wigner-Eckart theorem for both algebras, we have

$$
\begin{aligned}
< j'm'|w(q)^{\frac{1}{2}\frac{1}{2}}_{\rho\sigma}|jm > &= a(j)\ (j\tfrac{1}{2}m\rho|j'm')_q \\
< k'\kappa'|w(q)^{\frac{1}{2}\frac{1}{2}}_{\rho\sigma}|k\kappa > &= b(k)\ (k\tfrac{1}{2}\kappa\sigma|k'\kappa')_q
\end{aligned}
\tag{32}
$$

Equation (32) leads to some relations, to be discussed elsewhere, connecting $su_q(2)$ CGc's of type $(j\frac{1}{2}m\rho|j'm')_q$ and $su_q(1,1)$ CGc's of type $(k\frac{1}{2}\kappa\sigma|k'\kappa')_q$. The extension of these relations through the use of double tensors of higher rank would be interesting.

3. TOWARDS RECURRENT RELATIONS FOR $su_q(2)$

3.1. The Case $q = 1$

The main ingredient of the method developed in ref. 2 is the use of four commuting pairs $\{a_+, a_+^+\}$, $\{a_-, a_-^+\}$, $\{b_+, b_+^+\}$ and $\{b_-, b_-^+\}$ of ordinary bosons. The a's and b's serve for constructing two copies of $su^j(2)$ (say, $su^{j_1}(2)$ and $su^{j_2}(2)$), respectively. (The Lie algebra $su^j(2)$ is defined by (2) with $q = 1$.) In other words, the two-dimensional harmonic oscillator (with Fock space $\mathcal{F}_1 \otimes \mathcal{F}_2$ or $\mathcal{F}_3 \otimes \mathcal{F}_4$) of section 2 is replaced by a four-dimensional harmonic oscillator (with Fock space $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3 \otimes \mathcal{F}_4$). Therefore, eq. (5) with $q = 1$ yields

$$|j_1m_1 >= \frac{1}{\sqrt{(j_1+m_1)!(j_1-m_1)!}}\ (a_+^+)^{j_1+m_1}(a_-^+)^{j_1-m_1}\ |00> \tag{33}$$

and

$$|j_2m_2 >= \frac{1}{\sqrt{(j_2+m_2)!(j_2-m_2)!}}\ (b_+^+)^{j_2+m_2}(b_-^+)^{j_2-m_2}\ |00> \tag{34}$$

in $su^{j_1}(2)$ (with Fock space $\mathcal{F}_1 \otimes \mathcal{F}_2$) and $su^{j_2}(2)$ (with Fock space $\mathcal{F}_3 \otimes \mathcal{F}_4$) notations, respectively.

Three commuting Lie algebras, denoted here as $su^{\mathcal{J}}(2)$, $su^{\Lambda}(2)$ and $su^{\mathcal{K}}(1,1)$, come to play an important role in ref. 2. The generators of $su^{\mathcal{J}}(2)$ are

$$\begin{aligned} \mathcal{J}_3 &= \frac{1}{2}(N_1 + N_2 - N_3 - N_4) \\ \mathcal{J}_+ &= a_+^+ b_+ + a_-^+ b_- \\ \mathcal{J}_- &= b_+^+ a_+ + b_-^+ a_- \end{aligned} \tag{35}$$

where $N_1 = a_+^+ a_+$, $N_2 = a_-^+ a_-$, $N_3 = b_+^+ b_+$ and $N_4 = b_-^+ b_-$. The algebra $su^{\Lambda}(2)$ corresponds to the sum of the spherical angular momenta associated to $su^{j_1}(2)$ and $su^{j_2}(2)$ in the sense that $su^{\Lambda}(2)$ is generated by

$$\begin{aligned} \Lambda_3 &= \frac{1}{2}(N_1 - N_2 + N_3 - N_4) \equiv (J_1)_3 + (J_2)_3 \\ \Lambda_+ &= a_+^+ a_- + b_+^+ b_- \equiv (J_1)_+ + (J_2)_+ \\ \Lambda_- &= a_-^+ a_+ + b_-^+ b_+ \equiv (J_1)_- + (J_2)_- \end{aligned} \tag{36}$$

Finally, the algebra $su^{\mathcal{K}}(1,1)$ is spanned by the operators

$$\begin{aligned} \mathcal{K}_3 &= \frac{1}{2}(N_1 + N_2 + N_3 + N_4) + 1 \\ \mathcal{K}_+ &= a_+^+ b_-^+ - a_-^+ b_+^+ \\ \mathcal{K}_- &= a_+ b_- - a_- b_+ \end{aligned} \tag{37}$$

It can be easily verified that the three algebras $su^{\mathcal{J}}(2)$, $su^{\Lambda}(2)$ and $su^{\mathcal{K}}(1,1)$ commute. Another important point arises from the fact that the eigenvalues of the Casimir operators $C_2(su^{\mathcal{J}}(2))$, $C_2(su^{\Lambda}(2))$ and $C_2(su^{\mathcal{K}}(1,1))$ are all equal, say to $j(j+1)$; therefore, the irrep's of the three algebras may be labelled by a common (quantum) number j. (This number refers to a spherical, or compact, angular momentum for $su^{\mathcal{J}}(2)$ and $su^{\Lambda}(2)$ and to an hyperbolical, or noncompact, angular momentum for $su^{\mathcal{K}}(1,1)$.) The commutativity of the algebras $su^{\mathcal{J}}(2)$, $su^{\Lambda}(2)$ and $su^{\mathcal{K}}(1,1)$ together with the coincidence of the spectra of their Casimir operators constitute an evidence for complementarity relations, in the sense of Moshinsky and Quesne,[9] between these algebras.

In order to better understand these complementarity relations, let us consider the more general case of mn pairs of boson operators corresponding to an mn-dimensional harmonic oscillator. Two chains of Lie groups (or Lie algebras) may be exhibited in this case :

$$\begin{aligned} &Sp(2mn,\mathbb{R}) \supset Sp(2m,\mathbb{R}) \supset U(m) \supset \cdots \\ &Sp(2mn,\mathbb{R}) \supset U(n) \supset SO(n) \supset \cdots \end{aligned} \tag{38}$$

for which there are two pairs of complementary groups, namely, $(Sp(2m,\mathbb{R}), SO(n))$ and $(U(m), U(n))$. The invariant operators of the groups in a given pair are connected in a simple manner. We deal here with a four-dimensional oscillator and the situation $mn = 4$ presents some specificities. In this situation, the chains (38) may be specialized as

$$\begin{aligned} &Sp(8,\mathbb{R}) \supset Sp(2,\mathbb{R}) \simeq SU^{\mathcal{K}}(1,1) \supset U^{\kappa}(1) \\ &Sp(8,\mathbb{R}) \supset U(4) \supset SO(4) \simeq SU^{\mathcal{J}}(2) \otimes SU^{\Lambda}(2) \supset SO^{M_{\mathcal{J}}}(2) \otimes SO^{M_{\Lambda}}(2) \end{aligned} \tag{39}$$

for which the two relevant pairs of complementary groups are $(SU^{\mathcal{K}}(1,1), SO(4))$ and $(U(1), U(4))$. The breaking of $SO(4)$ into $SU^{\mathcal{J}}(2) \otimes SU^{\Lambda}(2)$ leads indeed to the lucky situation for which we have three complementary groups : $SU^{\mathcal{J}}(2)$, $SU^{\Lambda}(2)$ and $SU^{\mathcal{K}}(1,1)$. The latter three groups correspond to the three complementary Lie algebras defined by eqs. (35)-(37).

To go further within the just decribed complementarity relations, some precisions are in order. We know that only the symmetric irrep's $< n >$ of $u(4)$ can be realized in the Fock space $\mathcal{F}$ of a four-dimensional oscillator ($< n >$ denotes the Young diagram associated to the total number $n = n_1 + n_2 + n_3 + n_4$ of quanta). Furthermore, only the irrep's $(\omega, 0)$, of class I in the terminology of Vilenkin, of the algebra $so(4)$ are realized in the space $\mathcal{F}$. Let us write the two second-order invariants of $so(4)$ in the form

$$C_2(so(4)) = 2(J^2 + \Lambda^2) \qquad C_2(so(4))' = 2(J^2 - \Lambda^2) \tag{40}$$

where $J^2 = C_2(su^{\mathcal{J}}(2))$ and $\Lambda^2 = C_2(su^{\Lambda}(2))$. The two commuting algebras $su^{\mathcal{J}}(2)$ and $su^{\Lambda}(2)$ are complementary in the framework of the irrep's $(\omega, 0)$ of class I of the algebra $so(4)$. The Casimir operators J^2 and Λ^2 have the same eigenvalues $j(j+1)$. To go from the $so(4)$ to the $su^{\mathcal{J}}(2) \oplus su^{\Lambda}(2)$ notations, we have to use (cf. ref. 14)

$$\mathcal{J} = \Lambda \equiv j = \frac{1}{2}\omega \tag{41}$$

so that the eigenvalues of $C_2(so(4))$ and $C_2(so(4))'$ are $\omega(\omega+2)$ and 0, respectively. The complementarity of $so(4)$ and $sp(2,\mathrm{R}) \simeq su^{\mathcal{K}}(1,1)$ manifests itself by the commutativity of $su^{\mathcal{K}}(1,1)$ with both $su^{\mathcal{J}}(2)$ and $su^{\Lambda}(2)$ and by the fact that the Casimir operator $C_2(su^{\mathcal{K}}(1,1))$ has the eigenvalues $j(j+1)$.

As a result, we can construct basis vectors $|n\omega M_{\mathcal{J}} M_{\Lambda}>$ associated to the chain $u(4) \supset so(4) \simeq su^{\mathcal{J}}(2) \oplus su^{\Lambda}(2) \supset so^{M_{\mathcal{J}}}(2) \oplus so^{M_{\Lambda}}(2)$ and having well defined transformation properties with respect to the three algebras $su^{\mathcal{J}}(2)$, $su^{\Lambda}(2)$ and $su^{\mathcal{K}}(1,1)$. It is appropriate to introduce the notation

$$|n\omega M_{\mathcal{J}} M_{\Lambda} > \equiv |j : \mu m \kappa > \tag{42}$$

where μ, m and κ are the "projections" of the "angular momentum" $j = \frac{1}{2}\omega$ onto the "directions" $so^{M_{\mathcal{J}}}(2)$, $so^{M_{\Lambda}}(2)$ and $u^{\kappa}(1)$, respectively. More precisely, we shall have

$$\begin{aligned} \mu &= M_{\mathcal{J}} = j_1 - j_2 = \frac{1}{2}(n_1 + n_2 - n_3 - n_4) \\ m &= M_{\Lambda} = m_1 + m_2 = \frac{1}{2}(n_1 - n_2 + n_3 - n_4) \\ \kappa &= j_1 + j_2 + 1 = \frac{1}{2}(n_1 + n_2 + n_3 + n_4) + 1 \end{aligned} \tag{43}$$

as eigenvalues of the operators $\mathcal{J}_3$, Λ_3 and $\mathcal{K}_3$.

The vector (42) is obtained from a linear combination of the vectors $|n_1 n_2 n_3 n_4 >$. In turn, the vector $|n_1 n_2 n_3 n_4 >$ in the Fock space $\mathcal{F}$ can be identified to $|j_1 m_1 > \otimes |j_2 m_2 >$ when considered as basis vector for the tensor product $(j_1) \otimes (j_2)$, in $su^{\mathcal{J}}(2)$, of the irrep's (j_1) and (j_2) with

$$2j_1 = n_1 + n_2 \quad 2m_1 = n_1 - n_2 \quad 2j_2 = n_3 + n_4 \quad 2m_2 = n_3 - n_4 \tag{44}$$

In a similar way, $|n_1 n_2 n_3 n_4 >$ can be simultaneously considered as a basis vector $|\lambda_1 \mu_1 > \otimes |\lambda_2 \mu_2 >$ for the tensor product $(\lambda_1) \otimes (\lambda_2)$, in $su^{\Lambda}(2)$, of the irrep's (λ_1) and (λ_2) with

$$2\lambda_1 = n_1 + n_3 \quad 2\mu_1 = n_1 - n_3 \quad 2\lambda_2 = n_2 + n_4 \quad 2\mu_2 = n_2 - n_4 \tag{45}$$

Finally, the vector $|n_1n_2n_3n_4>$ can be also considered as a basis vector $|k_1\kappa_1> \otimes |k_2\kappa_2>$ for the tensor product $(k_1+)\otimes(k_2+)$, in $su^{\mathcal{K}}(1,1)$, of the irrep's (k_1+) and (k_2+) with

$$2k_1 = n_1 - n_2 - 1 \quad 2\kappa_1 = n_1 + n_2 + 1 \quad 2k_2 = n_3 - n_4 - 1 \quad 2\kappa_2 = n_3 + n_4 + 1 \qquad (46)$$

The coupled basis vector

$$|j_1j_2jm> = \sum_{m_1m_2} (j_1j_2m_1m_2|jm)\, |j_1m_1> \otimes |j_2m_2> \qquad (47)$$

for the decomposition of $(j_1)\otimes(j_2)$ into $\bigoplus_{|j_1-j_2|}^{j_1+j_2}(j)$ should coincide with the vector $|n\omega M_{\mathcal{J}} M_{\Lambda}>$ because both of them are labelled unambiguously by four numbers

$$\begin{aligned} n &= 2(j_1+j_2) = n_1+n_2+n_3+n_4 \quad \omega = 2j \\ M_{\mathcal{J}} &= \mu = j_1 - j_2 = \frac{1}{2}(n_1+n_2-n_3-n_4) \\ M_{\Lambda} &= m = m_1 + m_2 = \frac{1}{2}(n_1-n_2+n_3-n_4) \end{aligned} \qquad (48)$$

Therefore, eq. (42) can be reinterpreted as

$$|j_1j_2jm> \equiv |j:\mu m\kappa> \qquad (49)$$

with the projections $\mu = j_1 - j_2$, m and $\kappa = j_1 + j_2 + 1$. Similarly, the coupled basis vector $|\lambda_1\lambda_2 j\mu>$ for the decomposition of $(\lambda_1)\otimes(\lambda_2)$ into $\bigoplus_{|\lambda_1-\lambda_2|}^{\lambda_1+\lambda_2}(j)$ can be taken as

$$|\lambda_1\lambda_2 j\mu> \equiv |j:\mu m\kappa> \qquad (50)$$

with the projections μ, $m = \lambda_1 - \lambda_2$ and $\kappa = \lambda_1 + \lambda_2 + 1$.

The passage for the vectors $|j:\mu m\kappa>$ from the $su^{\mathcal{J}}(2)$ notation (49) to the $su^{\Lambda}(2)$ notation (50) makes it possible to generate several families of useful relations between the CGc's for the group $SU(2)$. As a matter of fact, the inner scalar product $< n_1n_2n_3n_4|j:\mu m\kappa>$ may be tackled in two ways :

$$< n_1n_2n_3n_4|j:\mu m\kappa> = (j_1j_2m_1m_2|jm) \;\text{ or }\; (\lambda_1\lambda_2\mu_1\mu_2|j\mu) \qquad (51)$$

which provide the key of an algorithm for generating various relations between $SU(2)$ CGc's.

As a first example, by taking eqs. (44) and (45) into account, the identity (51) between $(j_1j_2m_1m_2|jm)$ and $(\lambda_1\lambda_2\mu_1\mu_2|j\mu)$ yields

$$\begin{aligned} &(j_1j_2m_1m_2|jm) = \\ &\qquad (\frac{j_1+m_1+j_2+m_2}{2}\,\frac{j_1-m_1+j_2-m_2}{2} \\ &\qquad \frac{j_1+m_1-j_2-m_2}{2}\,\frac{j_1-m_1-j_2+m_2}{2}|jj_1-j_2) \end{aligned} \qquad (52)$$

which is nothing but a Regge symmetry property.

A second family of relations concerns the derivation of three- and four-term RR's. The possibility to replace the scalar product $< n_1n_2n_3n_4|j:\mu m\kappa>$ by one of the two CGc's of (51) leads to the method of ref. 2 for deriving various RR's for the $SU(2)$

CGc's. The algorithm of the method can be described as follows. The starting point is to consider the matrix element

$$x = < n_1 n_2 n_3 n_4 | X | j : \mu m \kappa > \tag{53}$$

where $X = \mathcal{J}_\pm$, $\Lambda_\pm$ or $\mathcal{K}_\pm$. The action of the operator X on the vector $|j : \mu m\kappa >$ is controlled by

$$\begin{aligned} X|j : \mu m\kappa > = (&\sqrt{(j \mp \mu)(j \pm \mu + 1)} \quad \text{or} \\ &\sqrt{(j \mp m)(j \pm m + 1)} \quad \text{or} \\ &\sqrt{(\kappa \mp j)(\kappa \pm j + 1)}) \; |j : \mu' m' \kappa' > \end{aligned} \tag{54}$$

where $(\mu' m' \kappa') = (\mu \pm 1 m\kappa)$ or $(\mu m \pm 1\kappa)$ or $(\mu m\kappa \pm 1)$ according to whether as $X = \mathcal{J}_\pm$ or $\Lambda_\pm$ or $\mathcal{K}_\pm$. The resulting scalar product can then be transformed into a CGc in the $su^{\mathcal{J}}(2)$ or $su^{\Lambda}(2)$ notation thanks to (51). This provides us with a first expression of x. On the other side, we can calculate x starting from $< n_1 n_2 n_3 n_4 | X$, by using the boson realization of X and by making use of (53) and (51). This leads to a new expression of x involving two $SU(2)$ CGc's. By equating the two expressions obtained for x, we get a three-term RR for the $SU(2)$ CGc's. Along the same vein, other RR's can be derived by replacing X by a more involved operator (see ref. 2).

3.2. The Case $q \neq 1$

We now go to a quantum algebra context by replacing the a's and b's by q-bosons. We are thus led to a q-oscillator in four dimensions. For the sake of generality, it would be interesting to consider the case of a q-oscillator in mn dimensions. The Lie groups in the chains (38) would then be replaced by the corresponding quantum algebras. The existence of complementary algebras within the so obtained chains of quantum algebras should be useful. In this direction, it was proved in ref. 15 that a q-analogue $(u_q(m), u_q(n))$ exists, i.e., the quantum algebras $u_q(m)$ and $u_q(n)$ are complementary in the Fock space associated to mn pairs of q-boson operators. The situation is less evident for the couple $(sp_q(2m, \mathrm{R}), so_q(n))$. Fortunately, we are interested here with the case of a q-oscillator in $mn = 4$ dimensions. In this case, eq. (39) can be extended in a quantum algebra context with three complementary algebras, viz., $su_q^{\mathcal{J}}(2)$, $su_q^{\Lambda}(2)$ and $su_q^{\mathcal{K}}(1,1)$.

Indeed, it was shown in ref. 15 that two complementary algebras of the $su_q(2)$ type exist : there are the q-analogues of $su^{\mathcal{J}}(2)$ and $su^{\Lambda}(2)$. More precisely, the operators

$$\begin{aligned} \mathcal{J}_3 &= \frac{1}{2}(N_1 + N_2 - N_3 - N_4) \\ \mathcal{J}_+ &= a_+^+ b_+ q^{\frac{1}{2}(N_2 - N_4)} + a_-^+ b_- q^{-\frac{1}{2}(N_1 - N_3)} \\ \mathcal{J}_- &= b_+^+ a_+ q^{\frac{1}{2}(N_2 - N_4)} + b_-^+ a_- q^{-\frac{1}{2}(N_1 - N_3)} \end{aligned} \tag{55}$$

and

$$\begin{aligned} \Lambda_3 &= \frac{1}{2}(N_1 - N_2 + N_3 - N_4) \\ \Lambda_+ &= a_+^+ a_- q^{\frac{1}{2}(N_3 - N_4)} + b_+^+ b_- q^{-\frac{1}{2}(N_1 - N_2)} \\ \Lambda_- &= a_-^+ a_+ q^{\frac{1}{2}(N_3 - N_4)} + b_-^+ b_+ q^{-\frac{1}{2}(N_1 - N_2)} \end{aligned} \tag{56}$$

generate the quantum algebras $su_q^{\mathcal{J}}(2)$ and $su_q^{\Lambda}(2)$, respectively. Each of these algebras can be completed to an algebra of type $u_q(2)$ owing to the operator $N = N_1 + N_2 +$

$N_3 + N_4$. In addition to N, a more interesting invariant of $su_q^{\mathcal{J}}(2)$ and $su_q^{\Lambda}(2)$ can be found : the operator

$$[a_+^+ b_-^+ q^{\frac{1}{2}(N_2+N_3)} - q^{-1} a_-^+ b_+^+ q^{-\frac{1}{2}(N_1+N_4)}] q^{\frac{1}{2}}$$

is invariant with respect to $su_q^{\mathcal{J}}(2)$ and $su_q^{\Lambda}(2)$. The latter expression plus eq. (37) suggest that $su_q^{\mathcal{K}}(1,1)$ is spanned by

$$\begin{aligned} \mathcal{K}_3 &= \frac{1}{2}(N_1 + N_2 + N_3 + N_4) + 1 \\ \mathcal{K}_+ &= [a_+^+ b_-^+ q^{\frac{1}{2}(N_2+N_3)} - q^{-1} a_-^+ b_+^+ q^{-\frac{1}{2}(N_1+N_4)}] q^{\frac{1}{2}} \\ \mathcal{K}_- &= [a_+ b_- q^{\frac{1}{2}(N_2+N_3)} - q^{-1} a_- b_+ q^{-\frac{1}{2}(N_1+N_4)}] q^{\frac{1}{2}} \end{aligned} \tag{57}$$

It can be effectively checked that the algebras $su_q^{\mathcal{J}}(2)$, $su_q^{\Lambda}(2)$ and $su_q^{\mathcal{K}}(1,1)$ commute. Furthermore, the invariants

$$\begin{aligned} C_2(su_q^{\mathcal{J}}(2)) &= \mathcal{J}_- \mathcal{J}_+ + [\mathcal{J}_3][\mathcal{J}_3 + 1] \\ C_2(su_q^{\Lambda}(2)) &= \Lambda_- \Lambda_+ + [\Lambda_3][\Lambda_3 + 1] \\ C_2(su_q^{\mathcal{K}}(1,1)) &= -\mathcal{K}_+ \mathcal{K}_- + [\mathcal{K}_3][\mathcal{K}_3 - 1] \end{aligned} \tag{58}$$

have the same eigenvalues $[j][j+1]$. Hence, the quantum algebras $su_q^{\mathcal{J}}(2)$, $su_q^{\Lambda}(2)$ and $su_q^{\mathcal{K}}(1,1)$ are complementary and their irrep's can be labelled by the common number j.

As a conclusion, the complementarity relations for the ordinary oscillator in four dimensions can be extended to its q-analogue so that the algorithm, described in section 3.1, for producing relations between CGc's of the group $SU(2)$ can be extended to the quantum algebra $su_q(2)$.

Of course, in order to apply the algorithm, we have to be careful with the similarities and differences between the cases $q = 1$ and $q \neq 1$. In this respect, the following prescriptions should be taken into account. Equations (33) and (34) have to be replaced by[4]

$$\begin{aligned} |j_1 m_1> &= \frac{1}{\sqrt{[j_1 + m_1]![j_1 - m_1]!}} (a_+^+)^{j_1+m_1} (a_-^+)^{j_1-m_1} \, |00> \\ |j_2 m_2> &= \frac{1}{\sqrt{[j_2 + m_2]![j_2 - m_2]!}} (b_+^+)^{j_2+m_2} (b_-^+)^{j_2-m_2} \, |00> \end{aligned} \tag{59}$$

Equations (44)-(46) and (49)-(51) conserve their sense when $q \neq 1$. Equation (47) must be changed into

$$|j_1 j_2 j m>_q = \sum_{m_1 m_2} (j_1 j_2 m_1 m_2 | jm)_q \, |j_1 m_1> \otimes |j_2 m_2> \tag{60}$$

Equation (54) should be modified according to eqs. (8) and (29).

As an example, eq. (52) can be reinterpreted as a Regge symmetry property for $su_q(2)$. As another example, we can find the q-analogues, to be reported elsewhere, of the three-term RR's of ref. 2. Similar results, were obtained by Nomura[11,12] and Kachurik and Klimyk[16] by following different routes. It should be noted that our algorithm is more powerful since repeated action of the q-deformed operators $\mathcal{J}_\pm$, $\Lambda_\pm$ and $\mathcal{K}_\pm$ allows us to obtain more complicated RR's (as, for instance, four-term RR's).

In addition, the q-deformation of the inner product (51) can be interpreted also as a CGc for $su_q^{\mathcal{K}}(1,1)$. Therefore, our algorithm permits to derive relations connecting CGc's for $su_q(2)$ and $su_q(1,1)$.

4. PERSPECTIVE

To close this paper, we would like to point out that developments similar to the ones in this work are presently under study, by the authors, for the two-parameter quantum algebra $u_{qp}(2)$. Such an algebra is spanned by the four operators J_-, J_3, J and J_+ satisfying the commutation relations[17]

$$\begin{aligned} &[J,J_3] = 0 \qquad [J,J_+] = 0 \qquad [J,J_-] = 0 \\ &[J_3,J_-] = -J_- \qquad [J_3,J_+] = +J_+ \qquad [J_+,J_-] = (qp)^{J-J_3}\,[2J_3] \end{aligned} \tag{61}$$

where $[X]$ is now given by

$$[X] = \frac{q^X - p^X}{p-q} \tag{62}$$

The algebra $u_{qp}(2)$ admits the invariant

$$J^2 = \frac{1}{2}\,(J_+J_- + J_-J_+) + \frac{[2]}{2}\,(qp)^{J-J_3}\,[J_3]^2 \tag{63}$$

and may be endowed with an Hopf algebraic structure. We forsee from eq. (63) that $u_{qp}(2)$ presents more flexibility than $su_q(2)$ for physical applications. In particular, the quantum algebra $u_{qp}(2)$ should be of interest in two-parameter models for rotational spectroscopy of super-deformed nuclei.[18]

5. APPENDIX : THE ALGEBRA $so_q(3,2)$

The nonvanishing commutators of type $[J,J]$, $[K,K]$ and $[J,K]$ are given by eqs. (2), (10) and (13), respectively. The other nonvanishing commutators are as follows.

Commutators $[k,k]$:

$$[k_+^+, k_-^-] = -[2K_3+2J_3-1] - [2K_3+2J_3+1] \to -4(K_3+J_3)$$

$$[k_-^+, k_+^-] = -[2K_3-2J_3-1] - [2K_3-2J_3+1] \to -4(K_3-J_3)$$

Commutators $[J,k]$:

$$[J_3,k_+^+] = k_+^+ \quad [J_3,k_-^+] = -k_-^+ \quad [J_3,k_-^-] = -k_-^- \quad [J_3,k_+^-] = k_+^-$$

$$\begin{aligned} [J_+,k_-^+] &= K_+([K_3-J_3+\tfrac{3}{2}] - [K_3-J_3-\tfrac{1}{2}]) \to +2K_+ \\ [J_+,k_-^-] &= K_-([K_3+J_3+\tfrac{1}{2}] - [K_3+J_3-\tfrac{3}{2}]) \to +2K_- \\ [J_-,k_+^+] &= K_+([K_3+J_3-\tfrac{1}{2}] - [K_3+J_3+\tfrac{3}{2}]) \to -2K_+ \\ [J_-,k_+^-] &= K_-([K_3-J_3-\tfrac{3}{2}] - [K_3-J_3+\tfrac{1}{2}]) \to -2K_- \end{aligned}$$

Commutators $[K,k]$:

$$[K_3,k_+^+] = k_+^+ \quad [K_3,k_-^+] = k_-^+ \quad [K_3,k_-^-] = -k_-^- \quad [K_3,k_+^-] = -k_+^-$$

$$
\begin{array}{lclclcl}
[K_+, k_-^-] & = & J_-([K_3 + J_3 + \frac{1}{2}] & - & [K_3 + J_3 - \frac{3}{2}]) & \rightarrow & +2J_- \\
[K_+, k_+^-] & = & J_+([K_3 - J_3 - \frac{3}{2}] & - & [K_3 - J_3 + \frac{1}{2}]) & \rightarrow & -2J_+ \\
[K_-, k_+^+] & = & J_+([K_3 + J_3 - \frac{1}{2}] & - & [K_3 + J_3 + \frac{3}{2}]) & \rightarrow & -2J_+ \\
[K_-, k_-^+] & = & J_-([K_3 - J_3 + \frac{3}{2}] & - & [K_3 - J_3 - \frac{1}{2}]) & \rightarrow & +2J_-
\end{array}
$$

The arrows $\rightarrow$ indicate limits when q goes to 1.

6. REFERENCES

1. L.C. Biedenharn and J.D. Louck, *in:* Encyclopedia of Mathematics and its Applications, G.-C. Rota, ed., Addison-Wesley, Reading, Massachusetts (1981) vols. 8 and 9.
2. M. Kibler and G. Grenet, On the SU_2 unit tensor, J. Math. Phys. 21:422 (1980).
3. Yu.F. Smirnov, V.N. Tolstoĭ and Yu.I. Kharitonov, Method of projection operators and the q analog of the quantum theory of angular momentum. Clebsch-Gordan coefficients and irreducible tensor operators, Sov. J. Nucl. Phys. 53:593 (1991).
4. L.C. Biedenharn, The quantum group $SU_q(2)$ and a q-analogue of the boson operators, J. Phys. A 22:L873 (1989).
5. A.J. Macfarlane, On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$, J. Phys. A 22:4581 (1989).
6. M. Kibler and T. Négadi, On quantum groups and their potential use in mathematical chemistry, J. Math. Chem. 11:13 (1992).
7. P.P. Kulish and E.V. Damaskinsky, On the q oscillator and the quantum algebra $su_q(1,1)$, J. Phys. A 23:L415 (1990).
8. J. Lukierski, H. Ruegg, A. Nowicki and V.N. Tolstoy, q-deformation of Poincaré algebra, Phys. Lett. B 264:331 (1991).
9. M. Moshinsky and C. Quesne, Noninvariance groups in the second-quantization picture and their applications, J. Math. Phys. 11:1631 (1970).
10. L.C. Biedenharn and M. Tarlini, On q-tensor operators for quantum groups, Lett. Math. Phys. 20:271 (1990).
11. M. Nomura, Recursion relations for the Clebsch-Gordan coefficient of quantum group $SU_q(2)$, J. Phys. Soc. Jpn. 59:1954 (1990).
12. M. Nomura, A Jordan-Schwinger representation of quadratic relations for $SU_q(2)$ operators and of the q-analog Wigner-Eckart theorem, J. Phys. Soc. Jpn. 59:2345 (1990).
13. C. Quesne, q-bosons and irreducible tensors for q-algebras, *in:* Symmetries in Science VI, B. Gruber, ed., Plenum Press, New York (1993).
14. G.F. Filippov, V.I. Ovcharenko and Yu.F. Smirnov, Microscopic Theory of Collective Excitations of Nuclei [in Russian], Naukova Dumka, Kiev (1981).
15. Yu.F. Smirnov and V.N. Tolstoy, *in* : Group Theory and Special Symmetries in Nuclear Physics, J.P. Draayer and J. Jänecke, eds., World Scientific, Singapore (1992) p. 375.
16. I.I. Kachurik and A.U. Klimyk, On Racah coefficients of the quantum algebra $U_q(su_2)$, J. Phys. A 23:2717 (1990).
17. M. Kibler, Introduction to quantum algebras, IPN de Lyon preprint LYCEN 9234 (1992).
18. J. Meyer, Yu.F. Smirnov and M. Kibler, work in progress.